0 Introduction

Dye’s celebrated theorem ([5]) states that any ergodic non-singular action of a countable amenable group is orbit equivalent to a measured odometer action. Hence, a complete classification of these amenable group actions up to orbit equivalence will follow from a classification up to orbit equivalence of measured odometer actions.

There is a well-known classification of ergodic non-singular group actions into classes I, II and III respectively according to whether the measure is concentrated on a single orbit, the measure is equivalent to an invariant measure (finite or \(\sigma\)-finite) or neither of the above holds. The case I is a relatively trivial case. The case II is fairly well understood, so the remaining interesting case (which in some sense is the most prevalent) is case III.

Krieger introduced a subdivision of case III using an invariant which he called the ratio set (see [9] and [10]). The ratio set may be informally defined as the set of limits of ratios \(d\mu \circ \gamma / d\mu\) for \(\gamma\) in the group \(\Gamma\) of transformations and it may be shown to be a closed multiplicative subgroup of \(\mathbb{R} \cup \{0, \infty\}\). There are then three possibilities for the ratio set. It can be \(\{0, 1, \infty\}\), \(\mathbb{R}\) or \(\{\lambda^n : n \in \mathbb{Z}\}\) for a fixed \(\lambda \in (0, 1)\). The corresponding
classes of transformations are known as $\text{III}_0$, $\text{III}_1$ and $\text{III}_\lambda$. It may be shown that any two systems in $\text{III}_1$ are orbit equivalent (see [12]). Similarly for any fixed $\lambda \in (0, 1)$, any two $\text{III}_\lambda$ systems are orbit equivalent. The situation for $\text{III}_0$ is much less well understood and it is this category upon which we shall focus.

We will, as described above, be considering measured odometer actions. A natural class of examples is given by the actions where the measure is a product measure. An action which is orbit equivalent to one of this type is said to be product type. A necessary and sufficient condition for an odometer action to be of product type was introduced by Connes and Woods ([4]) who use a proof based on operator algebras. The condition which they introduced is that the Poincaré flow associated to the action has a property which they call approximately transitive or AT. Hawkins ([8]) showed necessity of Connes and Woods’ condition with a simpler ergodic theoretic proof and Hamachi [7] was able to show sufficiency, by using purely ergodic techniques.

It nevertheless remains a difficult task to give examples of measures of type $\text{III}_0$ or, given a product measure to decide whether it is of type $\text{III}_0$. (Moore’s criterion [15] allows us readily to decide when a product measure is of type I, $\text{II}_1$, $\text{III}_\infty$ or $\text{III}$.)


Brown and Dooley [2] introduced the notion of a $G$–measure. (Their formalism favoured the use of the groups of finite coordinate changes over the odometer; these two actions have the same orbits.) These provide an explicit description of in some senses the most general quasi-invariant measure. In [3], it was shown how to compute the ratio sets of $G$–measures in some cases, and the machinery was applied to product measures. However, there were some unresolved conjectures and some rather sketchy proofs.

The present article aims to refine the techniques of [3], resolve some conjectures therein and give full details of some results on product measures. At the same time, we are able to somewhat sharpen the examples of type $\text{III}_0$ product measures in [10]. From the
perspective of $G$–measures, the next most complicated measures after product measures are Markov measures. A second aim is to consider a class of Markov measures on the infinite product of two point spaces, where the transition probabilities remain constant on long blocks. We are able to explicitly compute the Poincaré flow of such a measure and show that it is AT.

A detailed description of the results follows. Consider the infinite product measure $\mu = \otimes \mu_i$ or $X \prod Z_2$, where $\mu_i(\{0\}) = 1 - a_i$, $\mu_i(\{1\}) = 1 + a_i$ ($-1 < a_i < 1$). In section 2, we give an example of a measure of type III, but not of type III$_0$ with $a_i \uparrow 1$, disproving a conjecture in [3]. We also give a relatively easy example of a family of product measures of type III$_0$ on $X$. These improve upon the examples found in [10] where the size of the factors in the product space was unbounded. Examples are found by taking a suitable sequence $\{a_i\}$ which is constant on blocks of increasing length. In section 3, we give a detailed proof that if $a_i \to 0$ and $\sum a_i^2 = \infty$ then $\mu$ is of type III$_1$, providing full details of a claims made in [3]. The essential technique in these two sections is Lemma 2.1, a generalization of Theorem 3.1 of [3] and a primitive version of Theorem 1 of [9].

The final section considers Markov measures on the infinite product of two point spaces, which have the property that their transition probabilities are constant on long blocks, behaving in the same way as the probabilities in the examples of section 2. We are able to compute the Poincaré flow explicitly as an odometer with parity bit. These flows are AT, and hence the measures are orbit equivalent to product measures (although they are certainly far from being equivalent to products).

More recent work of Dooley and Hamachi [6] finds examples of non-AT Markov measures. These are realized on $\prod Z_{\ell(n)}$ where $\ell(n)$ increases rapidly.

1 Definitions and Notation

We consider transformations of finite or $\sigma$–finite measure spaces. The transformations which we consider will be measurable and invertible, with measurable inverses and be
non-singular: that is a set has measure 0 if and only if its image has measure 0. These transformations will be known as isomorphisms. In the case where the transformation is from a measure space \(X\) to itself, it will be called an automorphism of \(X\). \(\Gamma\) will denote a countable group of automorphisms of \((X, \mathcal{B}, \mu)\). The full group \([\Gamma]\) of \(\Gamma\) consists of those automorphisms \(\theta\) of \(X\) which have the property that for almost every \(x \in X\), \(\theta(x) = \gamma(x)\) for some \(\gamma \in \Gamma\). Note that we use similar notation for the orbit of a point. Namely, \([x]\) is the orbit of the point \(x\) under the group \(\Gamma\) of transformations.

As an example, define \(X\) to be \(\{0, 1\}^{\mathbb{Z}}\) and \(\Gamma\) to be the group generated by the maps \(\gamma_n\) which reverses the \(n\)th coordinate (so \((\gamma_n(x))_i = \delta_{m} + x_i \mod 1\)). Then defining \(\theta\) to be the standard odometer mapping obtained by regarding points of \(x\) as 2-adic integers and adding 1 (with carry) (so \(\theta(\ldots 10110) = \ldots 10111; \theta(\ldots 10111) = \ldots 11000\)), we see that \(\theta \in [\Gamma]\).

Two group actions \(\Gamma\) acting on a measure space \((X_1, \mathcal{B}_1, \mu_1)\) and \(\Gamma'\) acting on \((X_2, \mathcal{B}_2, \mu_2)\) are orbit equivalent (sometimes also called weakly equivalent) if there exists an isomorphism \(\Phi\) from \(X_1\) to \(X_2\) such that for almost every \(x \in X_1\), \(\Phi([x]) = [\Phi(x)]\). In the example above, the actions of \(\Gamma\) on \(X\) and \(\{\theta^n : n \in \mathbb{Z}\}\) on \(X\) are orbit equivalent.

The ratio set \(R_\mu\), as defined in [13], is the set of \(r\) in \([0, \infty]\) such that for each \(\epsilon > 0\) and set \(A\) of positive measure, there exists a subset \(B \subset A\) of positive measure and a \(\theta \in [\Gamma]\) such that \(\theta(B) \subset A\) and \(|d\mu \circ \theta/d\mu - r| < \epsilon\). In our case, where \(\Gamma\) is a countable group, it is equivalent to define \(r \in R\) if and only if for each \(\epsilon > 0\) and set \(A\) of positive measure, there exists a subset \(B\) of positive measure and \(\gamma \in \Gamma\) such that \(\gamma(B) \subset A\) and \(|d\mu \circ \gamma/d\mu - r| < \epsilon\) on \(B\) (that is the automorphism \(\theta\) may be chosen from the group itself, not the full group). We use this latter definition in what follows.

Given an action of a group \(\Gamma\) on a space \(X\), we define an action of \(\Gamma\) on \(X \times \mathbb{R}\). For \(\gamma \in \Gamma\), define \(\hat{\gamma}(x, t) = (\gamma(x), t - \log(d\mu \circ \gamma/d\mu(x)))\). Then form \(Y = X \times \mathbb{R}/\Gamma\), the set of \(\Gamma\)–orbits in \(X \times \mathbb{R}\). There is a natural projection \(\pi\) from \(X \times \mathbb{R}\) to \(Y\). The measure on \(Y\) is taken to be projection of \(\mu \times \nu\) where \(d\nu(x) = \exp x \, d\lambda(x)\). The projection can
be used to give $Y$ a $\sigma-$algebra by defining a subset to be measurable if and only if its inverse image under $\pi$ is a measurable subset of $X$. Since the action of $\mathbb{R}$ on $X \times \mathbb{R}$ given by $\theta_s(x,t) = (x, s + t)$ commutes with the action of $\Gamma$ on $X \times \mathbb{R}$, it follows that the action of $\mathbb{R}$ may be pushed down to an action on $Y$. This is the associated flow (or Poincaré flow) of the action of $\Gamma$ on $X$.

An important property which the associated flow may or may not possess is approximate transitivity (abbreviated to the AT property, so we often say if this property holds that the associated flow is AT). An action of a group $G$ on a measure space $X$ is AT if for all $\epsilon > 0$ and any sequence $f_1, f_2, \ldots, f_n$ of functions in $L^1(X)^+$, the space of positive integrable functions, there exists a function $f \in L^1(X)^+$, finitely many elements $g_{i,j}$ of $G$ and constants $\lambda_{i,j}$ such that

$$\left\| f_i - \sum_j \left( \lambda_{i,j} f \circ g_{i,j} \frac{d\mu \circ g_{i,j}}{d\mu} \right) \right\|_1 < \epsilon$$

for each $i$. We will write $L_g(f)$ for the function defined by

$$L_g(f)(x) = f(g(x)) \frac{d\mu \circ g}{d\mu}(x).$$

Note that for positive functions $f$, $L_g$ is an $L^1$ norm-preserving operator. The AT condition may be re expressed as $\| f_i - \sum_j \lambda_{i,j} L_{g_{i,j}} f \|_1 < \epsilon$.

2 **Product measures of type $\text{III}_0$**

**Example 1.** In the notation of [3], we give an example with $a_i \nearrow 1$ such that the ratio set is not a subset of $\{0, 1, \infty\}$. This disproves conjecture 6.3 of [3].

Fix $\rho > 0$. Let $\{\sigma_i\}$ be a sequence of the form

$$\rho, \rho, \ldots, \rho, 2\rho, 2\rho, \ldots, 2\rho, 3\rho, \ldots$$

such that if $n_k$ denotes the number of terms of type $k\rho$, then

$$n_k \geq e^{kp} \quad \text{and} \quad n_{k+1} \geq n_k.$$
The \( \{a_i\} \) are then determined as in [3] by
\[
\frac{1 + a_i}{1 - a_i} = e^{\sigma_i}, \quad \text{so} \quad a_i = \frac{e^{\sigma_i} - 1}{e^{\sigma_i} + 1}/1.
\]

Let \( \mu \) denote the resulting measure. We prove:

**Proposition 2.1** \( e^{-\rho} \in R_{\mu} \).

We need the following sufficient condition. It is a generalization of Theorem (3.2) (i) of [3] to the case of more than one \( \gamma \in \Gamma^n \). In this case, we need to assume the disjointness of the sets of \( u \)'s for these \( \gamma \)'s, and that of their images under the \( \gamma \)'s:

**Lemma 2.1** Let \( r \in (0, \infty) \). Suppose that \( \forall \epsilon > 0, \exists \beta > 0 \) such that \( \forall n, \forall \gamma_0 \in \Gamma^n \) there exist:
\[
\gamma_1, \gamma_2, \ldots, \gamma_k \in \Gamma^n \\
U_1, U_2, \ldots, U_k \subset \gamma_0 X^n
\]
such that:

- the \( \{U_j\} \) are disjoint,
- the \( \{\gamma_j U_j\} \) are disjoint,
- we have \( \mu(\bigcup_{j=1}^k U_j) \geq \beta \mu(\gamma_0 X^n) \),
- \( \forall j \),
\[
u \in U_j \iff \left| \frac{d\mu \circ \gamma_j}{d\mu}(u) - r \right| < \epsilon.
\]

Then \( r \in r(X, \Gamma, \mu) \).

**Proof of Lemma 2.1** The proof is based on the method of proof of theorem (3.2) (i) of [3].

Let \( r > 0 \) and let \( \epsilon \ll r \) (this will be specified). Let \( \beta > 0 \) be fixed as in the statement. Let \( A \) be an arbitrary set of positive \( \mu \)-measure. Then there exists \( n \) and \( \gamma_0 \in \Gamma_n \) such that
\[
\mu(A \cap \gamma_0 X^n) > (1 - c)\mu(\gamma_0 X^n)
\]
where $c$ to be specified.

For this $n$ and $\gamma_0$, there exist \{\gamma_j\} and \{U_j\} as stated. It is given that $\cup U_j$ fills up a proportion $\beta$ of $\gamma_0X^n$. Let us check that $\cup \gamma_j U_j$ also has this property for some constant $\beta'$:

Since $|d\mu_0 \gamma_j(u) - r| < \epsilon$ on $U_j$, it follows that $\forall j \mu(\gamma_j U_j) \geq (r - \epsilon)\mu(U_j)$. So by disjointness,

$$\mu(\cup \gamma_j U_j) \geq (r - \epsilon)\mu(\cup U_j) \geq (r - \epsilon)\beta\mu(\gamma_0X^n)$$

Thus we can take $\beta' = (r - \epsilon)\beta$.

We now claim that for at least one index $j = j_0$, we have

$$\mu(A \cap [\gamma_{j_0}(A \cup U_{j_0})]) > 0.$$  

We have $\mu(A \cap (\cup U_j)) \geq \mu(\cup U_j) - \mu([\gamma_0X^n]\setminus A) \geq (\beta - c)\mu(\gamma_0X^n)$.

Hence, using disjointness:

$$\mu(\cup_j \gamma_j(A \cap U_j)) = \sum_j \mu(\gamma_j(A \cap U_j))$$

$$\geq \sum_j (r - \epsilon)\mu(A \cap U_j)$$

$$= (r - \epsilon)\mu(\cup(A \cap U_j))$$

$$\geq (r - \epsilon)(\beta - c)\mu(\gamma_0X^n)$$

By definition, the set $S = \cup_j \gamma_j(A \cap U_j)$ is contained in $\gamma_0X^n$ and we have shown that

$$\mu(S) \geq (r - \epsilon)(\beta - c)\mu(\gamma_0X^n)$$

Hence

$$\mu(A \cap S) \geq \mu(S) - \mu([\gamma_0X^n]\setminus A) \geq ((r - \epsilon)(\beta - c) - c)\mu(\gamma_0X^n).$$

This is positive if we ensure that $\epsilon < r$ and $c < \min(\beta/2, \ (r - \epsilon)\beta/2)$.

But the statement $\mu(A \cap S) > 0$ gives us:

$$0 < \mu(A \cap \cup_j \gamma_j(A \cap U_j)) = \sum_j \mu(A \cap [\gamma_j(A \cap U_j)]) \text{ by disjointness of } \gamma_j U_j.$$
Hence $\exists j = j_0$ with

$$\mu(A \cap [\gamma_{j_0}(A \cap U_{j_0})]) > 0.$$ 

It follows that letting

$$B = \{a \in A \cap U_{j_0} : \gamma_{j_0}a \in A\} \subset A$$

we get $\gamma_{j_0}B = A \cap [\gamma_{j_0}(A \cap U_{j_0})] \subset A$ and $\mu(\gamma_{j_0}B) > 0$, which also implies $\mu(B) > 0$.

Also, since $B \subset U_j$, we have

$$\left| \frac{d\mu \circ \gamma_{j_0}}{d\mu}(u) - r \right| < \epsilon \quad \forall u \in B.$$ 

Since $\epsilon > 0$ was arbitrary, we have $r \in r(X, \tau, \mu)$. $\Box$

**Proof of Proposition 2.1** We shall in effect take $\epsilon = 0$ and find $a\beta > 0$. Let any $n, \gamma_0 \in \Gamma_n$ be given. Then, in $\{\sigma_i\}_{i=n}^\infty$ there is a block of the form

$$\sigma_i = \rho k, \ldots, \rho k, \rho(k + 1), \rho(k + 1), \ldots$$

for some $k$. Fix such a block and $k$. Since $n_{k+1} \geq n_k$, there are available at least $n_k$ terms of the next constant, $\rho(k + 1)$, to the right of this block.

The following table defines our choices of $\{\gamma_j\}$ and $\{U_j\}$ ($k = n_k$) (below $k = 4$)

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\gamma_1$</th>
<th>$\gamma_2$</th>
<th>$\gamma_3$</th>
<th>$\gamma_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_i$</td>
<td>$\ldots$, $\rho k$, $\rho k$, $\rho k$, $\rho(k + 1)$, $\rho(k + 1)$, $\ldots$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$U_1$</td>
<td>$X$</td>
<td>$1$</td>
<td>$X$</td>
<td>$X$</td>
</tr>
<tr>
<td>$U_2$</td>
<td>$X$</td>
<td>$0$</td>
<td>$1$</td>
<td>$X$</td>
</tr>
<tr>
<td>$U_3$</td>
<td>$X$</td>
<td>$0$</td>
<td>$0$</td>
<td>$1$</td>
</tr>
<tr>
<td>$U_4$</td>
<td>$X$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

Here the convention is that $U_j$ consists of all $u$ having the coordinates shown, and $X$ denotes either 0 or 1. (And we assume all this in $\gamma_0 X^n$ of course).

Clearly, applying the $\gamma_j$'s to the $U_j$'s gives
The disjointness of the \(\{U_j\}_{j=1}^{n_k}\) is guaranteed by the 1’s on the diagonal, preceded by the 0’s. Similarly for the \(\{\gamma_j U_j\}\).

Next, let us verify that \(u \in U_j \implies \frac{d\mu \circ \gamma_j}{d\mu}(u) = e^{-\rho}\).

[We are following the convention \(\mu_i(\{0\}) = \frac{1 + a_i}{2} = \frac{1}{1 + e^{-\sigma_i}} := 1 - p_i := q_i\)]

Thus \(u \in U_j \implies \frac{d\mu \circ \gamma_j}{d\mu}(u) = \frac{\mu(1)\mu(2)\mu(1)\mu(0)}{\mu(1)\mu(2)\mu(0)} = \frac{1}{e^{-\sigma(1)}} = e^{-\sigma(1) - \sigma(2)} = e^{k\rho - (k+1)\rho} = e^{-\rho}\) where \(\sigma(1), \sigma(2)\) denote \(k\rho\) and \((k + 1)\rho\) for convenience.

It remains to estimate the measures \(\mu(U_j)/\mu(\gamma_0 X^n)\). We get

\[
\tilde{\mu}(U_i) = p(1)q(2) \geq p(1)q(1)
\]
\[
\tilde{\mu}(U_2) = q(1)p(1)q(2) \geq p(1)q(3)
\]
\[
\tilde{\mu}(U_3) = q(3)p(1)q(3) \geq p(1)q(5)
\]

(where \(\tilde{\mu}\) denotes \(\mu/\mu(\gamma_0 X^n)\)). Thus, using the notation

\[
p(1) = e^{-k\rho}/(1 + e^{-k\rho}), \quad q(1) = 1 - p(1);
\]

\[
\tilde{\mu}\left(\bigcup_{j=1}^{n_k} U_j\right) \geq p(1)q(1) + p(1)q(3)^2 + p(1)q(5)^3 + \ldots
\]
\[
= p(1)q(1)(1 + q(3)^2 + q(5)^3 + \ldots)
\]
\[
= p(1)q(1) \left( \frac{1 - q(2n_k)}{1 - q(3)} \right)
\]
\[
= \frac{q(1)}{1 + q(1)} \left( 1 - \left( \frac{1}{1 + e^{-k\rho}} \right)^{2n_k} \right)
\]

[Note: it is clear that the infinite series gives \(\frac{q(1)}{1 + q(1)} \sim \frac{1}{2}\). Therefore we could have defined \(n_k\) more conveniently by taking it such that the sum of the first \(n_k\) terms is \(\geq \frac{1}{3} =: \beta\) say.]
Now $n_k \geq e^{k\rho}$, so \( \left( \frac{1}{1 + e^{-k\rho}} \right)^{2n_k} \leq \left( \frac{1}{1 + e^{-k\rho}} \right)^{2e^{k\rho}} \sim \frac{1}{e^2} \) (for $k \to \infty$). Thus we get say

\[
\beta = \frac{1}{3} \left( 1 - \frac{1}{e^2} \right) > 0.
\]

Hence the Lemma applies and the proposition is proved. \( \square \)

**Remark 1** It turns out that Example 1 would have been easier if we could have used Theorem 4.4 \((a) \implies (b)\) of [3]. In particular, this application implies the truth of the above Lemma 1, and moreover the condition that the \( \{ \gamma_j U_j \} \) be disjoint appears unnecessary. Unfortunately, [3] Theorem 4.4 \((a) \implies (b)\) is false. We give a counter example, which was motivated by this observation. It was in fact found by first looking for a situation where the \( \{ \gamma_j U_j \} \) are not disjoint.

We present this counterexample after presenting a method for construction \( \mu \)'s with ratio set contained in \( \{0, 1, \infty\} \). (This method will also be needed for the counterexample.)

**Remark 2** Here is a special case of Example 1.

Let $e^{\rho} = 2$.

Let $\sigma_i = 2^k$ for $i \in [2^k, 2^{k+1})$ (thus $n_k = 2^{k+1} - 2^k = 2^k = e^{k\rho}$).

This gives $a_i = (2^k - 1)/(2^k + 1) = 1 - \left( \frac{2}{2^k + 1} \right)$, $i \in [2^k, 2^{k+1})$.

We have $\frac{1}{2} \in r(X, \Gamma, \mu)$ by the above.

It is easy to verify that any $d\mu \circ \gamma/d\mu$ takes on only the values $2^m$, $m \in \mathbb{Z}$ (or possibly 0, $\infty$).

Thus in fact $\mu$ is type III$_\lambda$, with $\lambda = 1/2$.

**Remark 3** By choosing rationally independent $\rho_1, \rho_2$ and including infinitely many block pairs $(k\rho_1, k\rho_1, (k+1)\rho_1, \ldots (k+1)\rho_1)$ of both types, we clearly get a $\mu$ of type III$_1$.

**Example 2.** We now give a family examples of product measure of type III$_0$ on an infinite product of two-point spaces. The examples of [10] are somewhat more elaborate and are not realized on products of two point spaces.

In the notation of [3], it is clear that everything is determined if the sequence $\{ \sigma_i \}$ is
specified.

Let \{\sigma_i\} be of the form

\[ \ldots, 2^k, 2^k, \ldots, 2^k, 2^{k+1}, 2^{k+1}, \ldots 2^{k+1}, 2^{k+2}, \ldots \]

where the \(n_k\) are chosen large enough, to ensure that:

\[ \sum_{i=1}^{\infty} (1 - a_i) = \infty. \]

For example, since \(1 - a_i = 2/(1 + e^{\sigma_i}) = 2/(1 + e^{2^k})\) on the \(k\)th block, the choice

\[ n_k \geq \frac{1}{2} (e^{2^k} + 1), \quad (n_k \in \mathbb{N}) \]

will do. By Moore’s criterion [15], this ensures \(\mu\) is of type III, for eventually,

\[ \min (\frac{2a_i}{1 - a_i}, 1) = 1 \quad (\text{since } a_i \not\to 1) \]

so \(\sum \min (\frac{2a_i}{1 - a_i}, 1)^2 (1 - a_i) = \sum (1 - a_i) = \infty.\)

**Proposition 2.2** The ratio set of \(\mu\) is contained in \(\{0, 1, \infty\}\).

**Proof** Suppose not. Let \(r \in r(X, \Gamma, \mu), \ r > 2. \ Let \ \epsilon = 1.\)

Choose \(k\) such that \(\{\sigma_i\}_{i=n}^{\infty}\) takes values in \(\{2^k, 2^{k+1}, \ldots\}\). Let \(A = \gamma_0 X^n\) for this \(n\) and any \(\gamma_0\) (say \(\gamma_0 = 0\))

Clearly for any \(u \subseteq A\), and \(\gamma\) such that \(\gamma u \subseteq A\) we get

\[ \frac{d\mu \circ \gamma}{d\mu} (u) = \exp \sum_{i=n}^{m} \alpha_i \sigma_i \]

for some \(m\) depending on \(\gamma\) and \(\alpha_i \in \{-1, 0, 1\}\). Thus

\[ \left| \sum_{i=n}^{m} \alpha_i \sigma_i \right| \in \{0\} \cup 2^k \mathbb{N}, \]

so clearly

\[ \left| \frac{d\mu \circ \gamma}{d\mu} (u) - r \right| > 1 \quad (\text{since } r > 2.) \]
Thus there is no set $B$ as required by the definition of $R_\mu$. □

**Example 3** (Counterexample to [3] Theorem 4.4 $(a) \implies (b)$)

To define $\{\sigma_i\}$, take blocks as in example 2 and insert the value $2^k - 1$ in front of the $k$th block for each $k$. Thus $\{\sigma_i\}$ is a sequence of the form

$$2^{k-1}, 2^{k-1}, \ldots, 2^{k-1}, 2^k - 1, 2^k, 2^k, \ldots, 2^k, 2^{k+1} - 1, 2^{k+1}, 2^{k+1}, \ldots$$

The condition on $n_k$ is the same as before, e.g:

$$n_k \geq \exp(2^k).$$

or, more conveniently, define $n_k$ instead by considering the series

$$1 + q + q^2 + \ldots = \frac{1}{1 - q}$$

where $q = \mu_i(\{0\}) = \frac{1}{(1 + \exp(-2^k))}$ (on the block $k$) and let $n_k$ be the number of terms needed to get a partial sum of say at least

$$\frac{1}{2} \left( \frac{1}{1 - q} \right).$$

The idea in the form of the sequence is that it is essentially like example 2, but with a tiny nick on each big step.

**Proposition 2.3** Let $\mu$ be the resulting measure. Then

(i) The ratio set of $\mu$ is contained in $\{0, 1, \infty\}$.

(ii) $\forall \epsilon > 0, \exists \beta > 0$, such that $\forall n \in \mathbb{N}, \forall \gamma_0 \in \Gamma_n \exists L \geq n$ such that:

$$\mu(\{u \in \gamma_0X^n : \exists v \in \gamma_0X^n, \text{ eventually equal to } u, \text{ such that} \}} \geq \beta \mu(\gamma_0X^n).$$
Remark 1  (ii) is the case \( r = e \) of Theorem (4.4)(a). But by (i), \( e \) is not in the ratio set of \( \mu \), contradicting (4.4) (b).

Remark 2  It will be evident from the proof that if (4.4) (a) is altered to read, instead of \( \forall n \exists L \), to "\( \exists L \forall n \ldots \ell \geq n + L \ldots \)"; then this would not be a counterexample. In fact, the proof given in [3] is valid with this change. Furthermore, this is the version adopted later, in Theorem (5.2) (where it should read \( \sup_n (N(n) - n) < K(\epsilon) \)).

Proof of Proposition 3  (i) Suppose not. Let \( r < 2 \) be in the ratio set. Choose \( k \) such that \( \exp(2^k) > r + 1 \). Now consider the set \( A \) defined as follows. Choose \( n \in \mathbb{N} \) such that \( \{\sigma_i\}_{i=n}^{\infty} \) is the tail starting from block \( k \). Let \( i = i_\ell \) be the indices where the nicks occur, i.e. where
\[
\sigma_i = 2^\ell - 1, \quad \ell = 1, 2, \ldots
\]
Fix a \( \gamma_0 \in \Gamma_n \), define
\[
A = \{ u \in \gamma_0 X^n : 0 = u_{i_k} = u_{i_{k+1}} = \ldots \} = \{ u \in \gamma_0 X^n : u_{i_\ell} = 0 \quad \forall \ell \geq k \}.
\]
Let us first observe that if \( u \in A \) and \( \gamma u \in A \) then clearly
\[
\left| \frac{d\mu \circ \gamma}{d\mu}(u) - r \right| > 1 \quad \text{for all such } u.
\]
Hence for this set \( A \), and for \( \epsilon = 1 \), there is no set \( B \) as required in the definition of \( r \in R_\mu \).
It remains to check, however, that $\mu(A) > 0$. Clearly

$$\mu(A) = \prod_{i \geq n} \mu_i(\{0\}) = \prod_{\ell = k}^{\infty} \mu_i(\{0\}).$$

But $\mu_i(\{0\}) = \frac{1}{1 + \exp(-\sigma_i)} = \frac{1}{1 + \exp(-(2^\ell - 1))}$ since this is the definition of $\sigma_i$ on the “nicks”. This certainly gives a convergent product (since $\sum_{\ell = k}^{\infty} \exp(-2^\ell) < \infty$).

Thus $\mu(A) > 0$ as required. □

**Proof of 3 (ii)** Fix $\epsilon \geq 0$ (it may as well be 0 in fact). Fix $n \in \mathbb{N}$ and $\gamma_0 \in \Gamma_n$. Consider a $k$ such that the $k$th block, including its “nick” occurs inside the tail $\{\sigma_i\}_{i > n}$. We shall do coordinate changes within this block $k$ only, hence we can take the $L$ to be the end of block $k$.

Consider $u \in \gamma_0 X^n$ of the following types:

<table>
<thead>
<tr>
<th>$i_k$</th>
<th>$(i_k + n_k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>X X X 0 1 X X X X</td>
<td>(U_1)</td>
</tr>
<tr>
<td>X X X 0 0 1 X X X</td>
<td>: (U_2)</td>
</tr>
<tr>
<td>X X 0 0 0 1 X X</td>
<td>: : (U_3)</td>
</tr>
<tr>
<td>: : :</td>
<td>: :</td>
</tr>
<tr>
<td>X X 0 0 ... 0 0 0 0 1</td>
<td>... (U_{n_k})</td>
</tr>
</tbody>
</table>

For each $U_j$ consider the $\gamma_j$ shown below:

| 0 0 1 | 1 0 0 0 0 | $\gamma_1$ |
| 0 0 1 | 0 1 0 0 0 | $\gamma_2$ |
| 0 0 1 | 0 0 1 0 0 | ... |
| 0 0 1 | 0 0 0 1 0 | $\gamma_{n_k}$ |

Clearly $u \in U_j \rightarrow \gamma_j u =: v \in \gamma_0 X^n$ since we have agreed that all our $U_j$ are in $\gamma_0 X^n$.

Let us compute $\frac{d\mu \circ \gamma_j}{d\mu}(u)$ for $u \in U_j$. This is just

$$\frac{\mu_i(\{1\}) \mu_*(\{0\})}{\mu_i(\{0\}) \mu_*(\{1\})}$$

where $*$ denotes an irrelevant index between $i_k$ and $L$ (i.e. in the block $k$).

This of course, by the way, coincides with $\prod_{i = n+1}^{L} \frac{g_i(v)}{g_i(u)}$. It reduces to

$$\frac{e^{-\sigma_i}}{1} \frac{1}{e^{-\sigma_*}} = e^{-(2^k - 1)} e^{+(2^k)} = e.$$
In other words, it is true, for all \( u \in \bigcup_{j=1}^{n_k} U_j \), that \( \exists v \) in \( \gamma_0 X^n \), eventually, equal to \( u \), such that \( \ell > L \) implies

\[
\left| \prod_{i=n}^{\ell} \frac{g_i(v)}{g_i(u)} - e \right| < \epsilon \geq \beta \mu(\gamma_0 X^n) \quad \text{holds.}
\]

It remains to check that the measure satisfies

\[
\tilde{\mu}(\bigcup_{j=1}^{n_k} U_j) \geq \beta \quad \text{for some absolute } \beta > 0.
\]

We have

\[
\begin{align*}
\tilde{\mu}(U_1) &= \mu_{i_k}(\{0\}) \mu_*(\{1\}) \\
\tilde{\mu}(U_2) &= \mu_{i_k}(\{0\}) \mu_*(\{0\}) \mu_*(\{1\}) \\
\tilde{\mu}(U_3) &= \mu_{i_k}(\{0\}) \mu_*(\{0\}) \mu_*(\{0\}) \mu_*(\{1\}) \\
&\cdots \\
\end{align*}
\]

and they are all disjoint. Thus, we get a total of

\[
q_{i_k} p(k)(1 + q_k + q_k^2 + \ldots)
\]

where \( q_{i_k} = \mu_{i_k}(\{0\}) \), \( p(k) = \mu_*(\{1\}) \), \( q_k = 1 - p(k) \).

We agreed that \( n_k \) was large enough to give us at least

\[
q_{i_k} p(k) \geq \frac{1}{3}
\]

since \( q_{i_k} = \frac{1}{1 + \exp(-(2^k - 1))} \) is nearly equal to 1.

Hence we can take \( \beta = \frac{1}{3} \). This proves 3 (ii). \( \square \)

**Remark** Notice how this fits in with Lemma 2.1. There it was required also that \( \{\gamma_j U_j\} \) be disjoint. Here we can check directly how badly they fail to be disjoint (of course, they must fail, because otherwise Lemma 2.1 would be contradicted).

The \( \gamma_j U_j \) are:
\[
\begin{array}{ccccc}
1 & 0 & X & X & X \\
1 & 0 & 0 & X & X \\
1 & 0 & 0 & 0 & X \\
1 & 0 & 0 & 0 & 0 \\
\end{array}
\gamma_1 \mathcal{U}_1
\gamma_2 \mathcal{U}_2
::
::
::
etc; clearly all are subsets of the first one, which has \( \tilde{\mu}(\gamma_1 \mathcal{U}_1) \) very small (\( \leq \mu_i(\{1\}) \)).
\]

\section{Proof of a Proposition}

The following proposition was given as Proposition 6.2 in [3]. Unfortunately, its proof used Theorem 4.4, which we have just disproved! Here is a corrected proof, not without interest in its own right.

\textbf{Proposition 3.1} If \( a_i \to 0 \) and \( \sum a_i^2 = \infty \), then the measure \( \mu \) is of type \( \text{III}_1 \). i.e., its ratio set is \([0, \infty]\).

We need the following probabilistic lemma. First let us fix some notation as on pp. 13-14 of [3]. For given \( 0 \leq a_i \leq 1/2, \ i = 1, 2, \ldots \) we associate the sequence
\[
\sigma_i = \log\left\{ \frac{1+a_i}{1-a_i} \right\} \quad (a_i \leq \sigma_i \leq 4a_i)
\]
and vice versa (i.e. if a statement refers to \( \{\sigma_i\} \) first, we assume \( \{a_i\} \) defined in terms of \( \{\sigma_i\} \)).

We also associate with a given \( \{a_i\} \) independent random variables as follows. These are \( \{u_i\} \) and \( \{v_i\} \), where
\[
P(u_i = 0) = P(v_i = 0) = \frac{1+a_i}{2}
\]
\[
P(u_i = 1) = P(v_i = 1) = \frac{1-a_i}{2},
\]
and all are taken to be independent (thus \( \{u_i\} \) are independent and \( \{v_i\} \) is a second independent copy of the sequence \( \{u_i\} \)). Here as usual \( P(*) \) is the probability of the event \(*\).

Now put \( \Delta_i = v_i - u_i \).
Then $\Delta_i \in \{-1, 0, 1\}$ and $E(\Delta_i) = 0$. A simple calculation shows that $E(\Delta_i)^2 = \frac{1 - a_i^2}{2}$.

Also define for $1 \leq n \leq m$, (and for given $\{a_i\}$),

$$S_n^m = \sum_{i=n}^{m} \sigma_i \Delta_i = \sum_{i=n}^{m} \sigma_i (v_i - u_i).$$

**Lemma 3.1** Given $-\infty < a < b < \infty$, there exist $p : p(a, b) > 0$ and $1 \geq \delta := \delta(a, b) > 0$ such that given any $\{a_i\}$ and any $m \in \mathbb{N}$ satisfying: $0 \leq \sigma_i \leq \delta$, $1 \leq i \leq m$, and $2 \leq \sum_{i=1}^{m} \sigma_i^2 \leq 2 + \delta^2$, it follows that:

$$P(a \leq S_1^m \leq b) \geq p.$$ 

**Remarks:**

1. We emphasize that $p$ and $\delta$ depend only on the given interval $[a, b]$. So the conclusion holds “uniformly”, whenever $\{a_i\}_{i=1}^{m}$ has the stated properties.

2. Obviously, if instead of “$1 \leq i \leq m$” we consider “$n \leq i \leq m$”, we get the same result (applying the lemma): If $0 \leq \sigma_i \leq \delta$, $n \leq i \leq m$, and $2 \leq \sum_{i=1}^{m} \sigma_i^2 \leq 2 + \delta^2$, then

$$P(a \leq S_n^m \leq b) \geq p,$$

since the index $i$ plays no role in the statement. (Here $\delta = \delta(a, b)$, $p = p(a, b)$).

3. The $\{\Delta_i\}$ are independent, but not identically distributed, the distribution of $\Delta_i$ is given by $a_i$. However, the distributions are all “comparable” since $a_i \leq 1/2$ (in fact $a_i \leq \sigma_i \leq \delta$ imposes an even stronger uniformity on them if $\delta$ is small.)

4. If $\delta$ is small, then $m$ must be large (at least $1/\delta^2$). Thus $S_1^m$ is an essentially normalized sum of a large number of independent random variables.

**Proof of Lemma**  It follows from an exercise in [5], § 7.1 page 205, problem 5.
This exercise asserts the following: For every $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that whenever $k \in \mathbb{N}$ and $X_1, \ldots, X_k$ are independent random variables with
\[
E(X_i) = 0 \quad \forall i,
\]
\[
\sum_{i=1}^{k} E(X_i^2) = 1,
\]
\[
\sum_{i=1}^{k} E(|X_i|^3) \leq \delta, \quad \text{and} \quad S := \sum_{i=1}^{k} X_i,
\]
Then $\sup_{x \in \mathbb{R}} |P(S \leq x) - \Phi(x)| \leq \epsilon$ where $\Phi(x) = \int_{-\infty}^{x} e^{-t^2/2} dt / \sqrt{2\pi}$ is the standard normal distribution function.

This is simply a quantitative version of Liapunov’s Central Limit Theorem and can be verified by following the steps in the proof of it given in [5] (Theorem 7.1.2).

(A more direct proof can be given for our random variables $\Delta_i$).

Given the above result, the lemma follows by normalizing $S_1^n$ and substituting it for the $S$ in this result: (indeed $S_1^n$ is almost normalized, but not quite, because of the $\delta$).

Given $-\infty < a < b < \infty$, choose $a < a' < b' < b$ (say $a' = \frac{3}{4}a + \frac{1}{4}b, \quad b' = \frac{1}{4}a + \frac{3}{4}b$ to be definite).

Let $p' = \Phi(b') - \Phi(a') = \int_{a'}^{b'} e^{-t^2/2} dt / \sqrt{2\pi}$ and put $\epsilon' = p'/100, \quad \delta' = \delta(\epsilon')$.

To obtain the lemma, choose $\delta_0 < 1$ and small enough to ensure
\[
a \leq a' \sqrt{1 + \delta_0^2} < b' \sqrt{1 + \delta_0^2} \leq b
\]
and put
\[
\delta(a, b) = \frac{1}{1000} \min(\delta_0, \delta').
\]
We shall compute the resulting $p = p(a, b)$ : Assume $\sigma_i \leq \delta(a, b) := \delta$ and
\[
2 \leq \sum_{i=1}^{m} \sigma_i^2 \leq 2 + \delta^2 \quad \text{as in the statement of the lemma}.
\]

To normalize $S_1^n$, we need the variance
\[
E(S_1^n)^2 = \sum_{i=1}^{m} E(\sigma_i^2 \Delta_i^2) = \sum_{i=1}^{m} \sigma_i^2 (1 - a_i^2) / 2
\]
Since $\sigma_i \leq \delta(a, b) =: \delta$, and $2 \leq \sum_{i=1}^{m} \sigma_i^2 \leq 2 + \delta^2$, $a_i \leq \sigma_i^2 \leq \delta(a, b)^2$.

So that
\[
\frac{1}{2} \sum_{i=1}^{m} \sigma_i^2 \geq \sum_{i=1}^{m} \sigma_i^2 \left(1 - \frac{a_i^2}{2}\right) \geq \frac{1}{2} \sum_{i=1}^{m} \sigma_i^2 - S^2 \sum_{i=1}^{m} \sigma_i^2 / 2 \geq 1 - \delta^2.
\]

i.e.
\[
1 - \delta^2 \leq E((S_1^m)^2) \leq 1 + \frac{1}{2} \delta^2.
\]

Putting $X_i = \sigma_i \Delta_i / \sqrt{E((S_1^m)^2)}$
\[
S = \sum_{i=1}^{m} X_i,
\]
we have that
\[
|X_i|^3 = \|X_i\|_\infty \cdot |X_i|^2 \leq \max_i (\sigma_i) \cdot \frac{1}{\sqrt{1 - \delta^2}} |X_i|^2 \leq \frac{\delta}{\sqrt{1 - \delta^2}} X_i^2.
\]

Thus
\[
\sum_{i=1}^{m} E(|X_i|^3) \leq \frac{\delta}{\sqrt{1 - \delta^2}} \leq 2\delta \leq \delta' = \delta(\epsilon).
\]

Hence Chung’s exercise applies with $\epsilon = \epsilon'$. Thus
\[
P(a' < S < b') = P(S < b') - P(S \leq a') \geq (\Phi(b') - \Phi(a')) - 2\epsilon' = p' - 2p'/100 = (0.98)p'.
\]

But now, we show that the inequalities
\[
a' < S < b'
\]

imply
\[
a \leq S_1^m \leq b :
\]

If $S < b'$, then
\[
S_1^m = S\sqrt{E(S_1^m)^2} < b' \sqrt{1 + \frac{1}{2} \delta^2} \leq b' \sqrt{1 \pm \delta_0} \leq b.
\]
If \(a' < S\), then \(a'\sqrt{1-\delta^2} < S_1^m\), so \(a \leq S_1^m\).

We conclude that

\[
P(a \leq S_1^m \leq b) \geq P(a' < S < b') \geq (0.98)p'
\]

and thus we can take

\[
p(a, b) = (0.98)p' = (0.98) \int_{\frac{a}{4} + \frac{b}{4}} e^{-t^2/2} \, dt / \sqrt{2\pi}
\]

and the lemma is proved. \(\square\)

**Proof of Proposition 6.2** It suffices to show the following, which is a version of (2.3) (i) of [3]:

For any \(0 < r < \infty, \rho > 0\), put \(\rho = \log r\). Then for every \(A \subset X\) with \(\mu(A) > 0\), there exists \(\gamma \in \Gamma\) such that

\[
\mu(\{x \in A : \gamma x \in A \text{ and } |\log \frac{d\mu \circ \gamma}{d\mu}(x) - \rho| \leq \epsilon\}) > 0.
\]

Recall that if \(u = (u_i)_{i=1}^{\infty}\), \(\gamma u = ((\gamma u)_i)_{i=1}^{\infty} := (v_i)_{i=1}^{\infty}\) then \(\frac{d\mu \circ \gamma}{d\mu}(u) = \prod_{i=1}^{\infty} \frac{g_i((\gamma u)_i)}{g_i(u_0)} = \prod_{i=1}^{\infty} \left(\frac{1 + a_i}{1 - a_i}\right)^{u_i - (\gamma u)_i}, \text{ so } \log \frac{d\mu \circ \gamma}{d\mu}(u) = - \sum_{i=1}^{\infty} \sigma_i(\gamma(u)_i - u_i).\) Thus, consider the interval, (for given \(\rho, \epsilon\)),

\([-\rho - \epsilon, -\rho + \epsilon] =: [a, b].\)

Let the set \(A \subset X\) be given, \(\mu(A) > 0\). Let

\[
k = \frac{1}{200} p(a, b).
\]

(recall that \(p(a, b)\) is defined in Lemma 3.1). Let \(n_1 \in \mathbb{N}\) be large enough to ensure that

\[
n \geq n_1 \implies \sigma_n \leq \delta(a, b).
\]

(recall : \(\delta(a, b)\) is also defined in the Lemma 3.1).
Since $\mu(A) > 0$, we can find $n \geq n_1$ and $\gamma_0 \in \Gamma_n$ such that the cylinder $\gamma_0 X^n$ satisfies

$$\mu(A \cap \gamma_0 X^n) \geq (1 - \kappa)\mu(\gamma_0 X^n)$$

Put $\tilde{\mu} = \frac{\mu}{\mu(\gamma_0 X^n)}$ restricted to $\gamma_0 X^n$, $A_0 = A \cap \gamma_0 X^n$, $X_0 = \gamma_0 X^n$.

Consider $X_0 \times X_0$ with measure $\tilde{\mu} \times \tilde{\mu}$. Observe:

$$A_0 \times A_0 \subset X_0 \times X_0 \quad \text{and}$$

$$(\tilde{\mu} \times \tilde{\mu})(A_0 \times A_0) \geq (1 - \kappa)^2 \geq 1 - 2\kappa.$$

Let $(u, v) \in X_0 \times X_0$, and identify

$$u = (u_n, u_{n+1}, \ldots)$$
$$v = (v_n, v_{n+1}, \ldots)$$

Then $\{u_i\}_{i=n}^{\infty}$, $\{v_i\}_{i=n}^{\infty}$ are independent random variables on the probability space $(X_0 \times X_0, \tilde{\mu} \times \tilde{\mu})$ satisfying all the conditions of the Lemma (in the same notation). Since

$$\sigma_i \leq \delta(a, b) \quad i \geq n,$$

we can also choose $m > n$ large enough (not too large) so that

$$2 \leq \sum_{i=n}^{m} \sigma_i^2 \leq 2 + \delta(a, b)^2.$$

Since $\sum_{i=n}^{\infty} \sigma_i^2 = \infty$, this is achieved by letting $m$ be the smallest integer such that

$$2 \leq \sum_{i=n}^{m} \sigma_i^2.$$

Thus, applying the lemma to

$$S_n^m = \sum_{i=n}^{m} \sigma_i(v_i - u_i)$$

we have

$$(\tilde{\mu} \times \tilde{\mu})(a \leq S_n^m \leq b) \geq p(a, b).$$
Let \( G^m_n \) denote the “good set”

\[
G^m_n = \{(u, v) \in X_0 \times X_0 : a \leq S^m_n(u, v) \leq b\}
\]

so that

\[
\tilde{\mu} \times \tilde{\mu}(G^m_n) \geq p(a, b).
\]

Since \( S^m_n \) depends only on the coordinates \( n, \ldots, m \), then \( G^m_n \) is a disjoint union of cylinder sets

\[
G^m_n = \bigcup_{\alpha}(U_\alpha \times V_\alpha)
\]

where each \( U_\alpha, V_\alpha \) is of the form

\[
U_\alpha = (u^\alpha_n, u^\alpha_{n+1}, \ldots, u^\alpha_m) \times \{0, 1\} \times \{0, 1\} \times \ldots
\]

\[
V_\alpha = (v^\alpha_n, v^\alpha_{n+1}, v^\alpha_m) \times \{0, 1\} \times \{0, 1\} \times \ldots
\]

(where the \( u^\alpha_i, v^\alpha_i \ i = n, \ldots, m \) are the “good” choices of zeros and ones).

Our problem is to show that there is an \( \alpha = \alpha_0 \) such that the \( \gamma = \gamma_{\alpha_0} \) (determined by sending \( u^\alpha_i \mapsto v^\alpha_i, \ i = n, \ldots, m \)) satisfies property (6).

Put \( P = \tilde{\mu} \times \tilde{\mu} \) for convenience. We have

\[
P(G^m_n \cap (A_0 \times A_0)) = P(G^m_n) - P(G^m_n \setminus (A_0 \times A_0))
\]

\[
\geq P(G^m_n) - P((X_0 \times X_0) \setminus (A_0 \times A_0))
\]

\[
\geq P(G^m_n) - 2\kappa = P(G^m_n) - \frac{1}{100}p(a, b)
\]

\[
\geq P(G^m_n) - \frac{1}{100} P(G^m_n)
\]

\[
= (0.99)P(G^m_n).
\]

i.e. \( A_0 \times A_0 \) covers 99% or more of the good set, \( G^m_n \).

Consequently, \( A_0 \times A_0 \) covers 99% or more of at least one of the \( U_\alpha \times V_\alpha \) (whose disjoint union is \( G^m_n \)).

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Proof. Put $E = A_0 \times A_0$. Suppose on the contrary that $\forall \alpha, P(E \cap (U_\alpha \times V_\alpha)) < (0.99)P(U_\alpha \times V_\alpha)$. Summing over $\alpha$,

$$P(E \cap G_n^m) = \sum_\alpha P(E \cap (U_\alpha \times V_\alpha)) < \sum_\alpha (0.99)P(U_\alpha \times V_\alpha) = (0.99)P(G_n^m),$$

a contradiction.

Thus, there exists $\alpha_0$ such that

$$P((A_0 \times A_0) \cup (U_{\alpha_0} \times V_{\alpha_0})) \geq (0.99)P(U_{\alpha_0} \times V_{\alpha_0})$$

where $U_{\alpha_0} \times V_{\alpha_0} \subset G_n^m$.

Put $U_0 = U_{\alpha_0}$, $V_0 = V_{\alpha_0}$. We have

$$\frac{P((A_0 \cap U_0) \times (A_0 \cap V_0))}{P(U_0 \times V_0)} \geq (0.99) = \frac{\mu(A_0 \cap U_0)\mu(A_0 \cap V_0)}{\mu(U_0)\mu(V_0)} \quad (\because P = \mu \times \tilde{\mu})$$

$$\Rightarrow \frac{\mu(A_0 \cap U_0)}{\mu(U_0)} \geq (0.99) \quad \text{and} \quad \frac{\mu(A_0 \cap V_0)}{\mu(V_0)} \geq (0.99)$$

Defined the proposed $\gamma \in \Gamma$ by the conditions

$(\gamma x)_i = X_i$ if $i < n$ or $i > m$

$(\gamma x)_i = v_i^{\alpha_0}$ if $x_i = U_i^{\alpha_0}$ and $n \leq i \leq m$

Clearly $\gamma(U_0) = V_0$ and

$$\frac{\mu(E)}{\mu(U_0)} = \frac{\mu(\gamma(E))}{\mu(V_0)} \quad \text{whenever} \ E \subset U_0.$$

Thus

$$\frac{\mu(\gamma(A_0 \cap U_0))}{\mu(V_0)} = \frac{\mu(A_0 \cap U_0)}{\mu(U_0)} \geq (0.99).$$

Combining this knowledge with the above fact that $\frac{\mu(A_0 \cap U_0)}{\mu(U_0)} \geq (0.99)$, we get

$$\frac{\mu(\gamma(A_0 \cap U_0) \cap (a_0 \cap V_0))}{\mu(V_0)} \geq (0.99) - (0.01) = 0.98 > 0$$

Put $B = \gamma^{-1}(\gamma(A_0 \cap V_0) \cap (A_0 \cap V_0))$. Then

$$\mu(B) > 0,$$

$B \subset A_0 \cap U_0 \subset A_0 \subset A$,

$$\gamma(B) \subset A_0 \cap V_0 \subset A_0 \subset A,$$

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and $B \times \gamma(B) \subset U_0 \times V_0 \subset G_n^m$, i.e., if $u \in B$, then

$$- \sum_{i=1}^m ((\gamma u)_i - u_i)\sigma_i \in [\log r - \epsilon, \log r + \epsilon]$$

which is the required property. \qed

4 Markov measures of type $\text{III}_0$

We now present a class of examples which are measured odometers, but with a measure which are not a product measures, but rather a Markov measures.

We use the Daniell-Kolmogorov consistency theorem to define a measure on $X = \{0, 1\}^{\mathbb{Z}^+}$ by specifying the measure of each cylinder. To be specific, let $[x_0 x_1 \ldots x_n]$ denote the set of points in $X$ whose first $n + 1$ coordinates are $x_0, x_1, \ldots, x_n$. A set of this kind will be called an $n-$cylinder and will also be denoted $[x]^n$. Then define

$$\mu([x_0 x_1 \ldots x_n]) = \frac{1}{2} P_{x_0 x_1} P_{x_1 x_2} \ldots P_{x_{n-1} x_n},$$

where $P_{ij}^{(k)} = \begin{cases} 1 - q_k & \text{if } i = j; \\ q_k & \text{if } i \neq j. \end{cases}$

This gives a measure on $X$. We then define the transformation group on the space: $\Gamma$ is the group of all finite coordinate rotations of $X$ generated by the $\gamma_n$ as introduced above.

We will now demonstrate briefly an orbit equivalent system which in some ways resembles the more familiar systems. Define a second measure $\nu$ on $X$ which is just a product measure:

$$\nu([x_0 x_1 \ldots x_n]) = \frac{1}{2} q_{x_1}^{(1)} \ldots q_{x_n}^{(n)},$$

where $q_i^{(k)} = \begin{cases} 1 - q_k & \text{if } i = 0; \\ q_k & \text{if } i = 1. \end{cases}$

When then define a second transformation group on the space: $\Gamma'$ is the group of all finite coordinate changes on $X$ which change an even number of coordinates (this is generated
by the $\gamma_m \circ \gamma_n$). Then we can show that the $\Gamma$ action on $(X, \mu)$ is orbit equivalent to the $\Gamma'$ action on $(X, \nu)$. The equivalence is given by a map which is in fact a measure-preserving homeomorphism. Namely, define $\Phi : X \rightarrow X$ by $\Phi(x)_n = x_0 + x_1 + \ldots + x_n \mod 1$. If $x_0, \ldots, x_n$ are given, define $y_0, \ldots, y_n$ by $y_k = x_0 + x_1 + \ldots + x_k \mod 1$. Then a quick check shows that $P_{y_k-1}^{(k)} = q_{x_k}^{(k)}$ so it follows that $\nu([x_0 \ldots x_n]) = \mu([y_0 \ldots y_n])$. This shows that $\Phi$ is measure-preserving as required. We can write down an explicit inverse $\Psi$ as follows: $\Psi(y)_n = y_{n-1} + y_n \mod 1$, thereby showing that $\Phi$ is a homeomorphism (and in particular is invertible). To check that $\Phi$ is an orbit equivalence, from $(X, \nu, \Gamma')$ to $(X, \mu, \Gamma)$, pick an $x \in X$ and note that $\Phi(\gamma_m \circ \gamma_n(x))$ differs from $\Phi(x)$ in all coordinates between the $m$th and the $n-1$st. It is thus clear that the image of a $\Gamma$ orbit under $\Phi$ is exactly a $\Gamma$ orbit as required.

Note that the system $(X, \nu, \Gamma)$ is known to have an associated flow with the AT property by Hawkins’ result. (In fact, more is true: the flow on $X \times \mathbb{R}$ prior to forming the quotient also has the AT property.) But the system which we are considering, $(X, \mu, \Gamma)$, is not orbit equivalent to the above, but rather to $(X, \nu, \Gamma')$ which appears at first sight to be very similar to $(X, \nu, \Gamma)$ ($\Gamma'$ is a subgroup of $\Gamma$ of index 2), but Hawkins’ proof does not seem to work in this situation where there is more dependence.

The system as defined so far has a number of parameters $q_n$. We now show how to choose them in such a way that the system is a III$_0$ system by analogy with the construction in §2. We will construct an increasing sequence of integers $n_i$ and a rapidly decreasing sequence of real numbers $p_i$ and define

$$q_n = \begin{cases} 
\frac{1}{2} & \text{if } n = 0 \\
p_i & \text{if } n_i-1 < n \leq n_i.
\end{cases}$$

Defining $n_0 = 0$ and $m_i = n_i - n_{i-1}$, the sequences are chosen so that

$$(1) \quad \sum_{i=1}^{\infty} m_i p_i = \infty;$$

$$(2) \quad R_j > 1 \quad \text{and} \quad \lim_{j \to \infty} R_j = \infty \quad \text{where}$$

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\[
R_j = \prod_{i<j} \left( \frac{p_i}{1-p_i} \right)^{m_i} \cdot \frac{1-p_i}{p_j} = \infty.
\]

We remark that it is possible to simultaneously satisfy these conditions by an inductive construction. Supposing \(n_1, \ldots, n_k\) and \(p_1, \ldots, p_k\) are chosen. Then \(p_{k+1}\) may be chosen to ensure that \(p_{k+1} > k+1\) and subsequently, \(n_{k+1}\) may be chosen so that \(m_{k+1} p_{k+1} > 1\).

The first condition, (1) is to ensure that the system is not of type I. The condition is that the expected number of transitions (i.e. places at which \(x_i \neq x_{i-1}\)) is infinite). By the Kolmogorov 0-1 law, this ensures that the probability of having a sequence which is eventually all 1s or eventually all 0s is 0 and this is sufficient to guarantee that the system is not of type I.

The second condition, (2) is to ensure that the only ratios occurring in the ratio set are 0, 1 and \(\infty\). Further, it can be checked as in \(\S\)2, that all these occur so that the system really is of type III\(_0\). Given \(\gamma \in \Gamma\) and \(x \in X\), we see that the ratio \(d\mu \circ \gamma / d\mu\) is constant on any \(n_k\)-cylinder about \(x\) where \(\gamma\) only affects coordinates before the \(n_k\)th and so \(d\mu \circ \gamma / d\mu(x) = \mu([\gamma(x)]^{n_k}) / \mu([x]^{n_k})\).

Given \(x \in X\), define its block type as follows: the block type is a sequence of numbers \((a_0, a_1, a_2, \ldots)\) where \(a_0 = x_0\) and \(a_i\) denotes the number of transitions in block \(i\) (that is the number of \(n\) with \(n_{i-1} < n \leq n_i\) such that \(a_n \neq a_{n-1}\), where \(n_0\) is taken to be 0).

The number \(a_0\) is to be interpreted as the number of transitions in the 0th block (i.e. we have a notional initial state of 0 and then \(a_0\) denotes the number of transitions from 0 in the 0th block.) Next, we note that the measure of a cylinder set of the form \([x]^{n_k}\) is determined by its block type. If the block type is \((a_0, a_1, a_2 \ldots)\) then the measure of \([x]^{n_k}\) is \(\prod_{i=1}^{k} p_i^{a_i} (1-p_i)^{m_i-a_i}\). If \(\gamma\) is as above and \(\gamma(x)\) has block type \((b_0, b_1, \ldots, b_k, a_{k+1}, a_{k+2}, \ldots)\) then we see that

\[
\frac{d\mu \circ \gamma}{d\mu}(x) = \frac{\mu([\gamma(x)]^{n_k})}{\mu([x]^{n_k})} = \left( \frac{p_1}{1-p_1} \right)^{b_1-a_1} \cdots \left( \frac{p_k}{1-p_k} \right)^{b_k-a_k}.
\]

We then see that either all of the terms \(b_i - a_i\) are 0 (in which case the ratio is 1) or there is a largest \(i\) for which \(b_i - a_i\) is non-zero. It is then straightforward to check that the
ratio is either larger than $R_k$ or smaller than $1/R_k$ according to whether $b_i - a_i$ is negative or positive. Since in the definition of a ratio set, the ratios are required to be found in any set $A$ of positive measure, letting $A$ be an $n_k-$cylinder, if $B$ is a subset of $A$ and $\gamma(B) \subseteq A$, then the ratio $d\mu \circ \gamma d\mu$ on $B$ is either 1 or larger than $R_k$ or smaller than $1/R_k$.

Conversely, inside any set of positive measure, there are ratios which are arbitrarily close to 0 and $\infty$. This proves that the system is of class $\text{III}_0$).

We use a construction of Hamachi and Osikawa ([8]) to give an explicit description of the associated flow of the system $(X, \nu, \Gamma)$ and show that it still has the AT property. To construct the flow, it is first necessary to get an explicit description of a quotient space which arises in their construction. We now give this description.

Since the functions $d\mu \circ \gamma/d\mu$ are continuous, they are defined on the whole space and not just on sets of measure 1. This means that the following definition makes sense. Given $x$ and $y$ in $X$, write $x \sim y$ if $y = \gamma(x)$ for some $\gamma \in \Gamma$ and $d\mu \circ \gamma/d\mu(x) = 1$. Then let $Y$ denote the collection of equivalence classes $X/\sim$ and let $\Pi$ denote that natural projection from $X$ to $Y$. The $\sigma-$algebra on $Y$ is then given by $\mathcal{F} = \{A \subset Y : \Pi^{-1}A \in \mathcal{B}\}$. Let $\mathcal{G}$ denote the collection of measurable subsets of $X$ which are unions of $\sim-$equivalence classes. Then we see that $\mathcal{F} = \Pi(\mathcal{G})$.

We are then able to identify certain elements of $\mathcal{G}$. We first note that if $x$ has block type $(a_0, a_1, a_2, \ldots)$ and $y$ has block type $(b_0, b_1, b_2, \ldots)$ then

$$\frac{\mu([y]_{n_k})}{\mu([x]_{n_k})} = (\frac{p_1}{1-p_1})^{b_1-a_1} \cdots (\frac{p_k}{1-p_k})^{b_k-a_k}.$$  
In particular, by the remarks made in the section on the $\text{III}_0$ property, we see that $x \sim y$ if and only if $x$ and $y$ lie in the same orbit and have $a_i = b_i$ for all $i \geq 1$. We now show that if $x \sim y$ then $a_0 = b_0$ as well. To see this, suppose $x \sim y$. Then as noted above, $a_i = b_i$ for each $i \geq 1$. This means that $x_{n_i} = x_{n_{i-1}} + a_i \pmod{2}$ and $y_{n_i} = y_{n_{i-1}} + a_i \pmod{2}$ for each $i \geq 1$. In particular it follows that $y_{n_i} - x_{n_i} \pmod{2}$ is independent of $i$ for $i \geq 0$. Since $x$ and $y$ live in the same orbit, we require that $x$ and $y$ differ only in finitely many places, so in particular, $x_{n_i} = y_{n_i}$ for all $i \geq 0$. This implies that $a_0 = b_0$. Denote the block type
of $x$ by $B(x)$. We have shown that $x \sim y \implies B(x) = B(y)$. This allows us to identify certain elements of $\mathcal{G}$ as follows. Set $Z = \{(a_0, a_1, \ldots) : 0 \leq a_0 \leq 1; 0 \leq a_i \leq m_i\}$. Given $b \in Z$, write $C_k(b)$ for $\{x : B(x)_i = b_i, \forall i \leq k\}$. Then $C_k(b)$ is a finite union of cylinder sets of length $n_k$ so is certainly a measurable set. If $x \in C_k(b)$ and $x \sim y$, then since $y$ has the same block type as $x$, we see $y \in C_k(b)$ so $C_k(b)$ is a union of $\sim$-equivalence classes as required. It will be useful to note that by the above arguments, if $x$ and $y$ are members of $C_k(b)$ then $x_{n_i} = y_{n_i}$ for each $i \leq k$.

We have therefore identified a collection of cylinder type sets which belong to $\mathcal{G}$. It is then possible to show that these sets generate $\mathcal{G}$. We demonstrate this by showing that the algebra consisting of finite unions of sets of the form $C_k(b)$ may be used to approximate any element of $\mathcal{G}$.

Let $A$ be any element of $\mathcal{G}$. We will show that $A$ may be arbitrarily closely approximated by taking a union of sets of the form $C_k(b)$. Let $\epsilon > 0$ be given. Then pick $\delta < \min(1, \epsilon/2)$. Then let $\mathcal{A}$ denote the algebra of all finite unions of cylinder sets in $X$. Then since $\mathcal{A}$ generates the $\sigma-$algebra $\mathcal{B}$, any element of $\mathcal{B}$ may be arbitrarily closely approximated by an element of $\mathcal{A}$. In particular, there exists a finite union of cylinders $S$ such that $\mu(A \triangle S) < \delta^2$. Since $S$ consists of a finite union of cylinders, one of these cylinders has a maximum length and in particular, there exists a $k$ such that all of the cylinders forming $S$ have length less than $n_k$. We may then assume that $S$ is formed of cylinders of length exactly $n_k$, say $C_1, \ldots C_r$. A cylinder will be called good if it satisfies $\mu(C \setminus A)/\mu(C) < \delta$ and bad otherwise. We let $\mathcal{G}$ be the union of the good cylinders forming $S$ and $B$ be the union of the bad cylinders forming $S$. Since the cylinders forming $B$ are disjoint and for each, we have $\mu(C) \leq \mu(C \setminus A)/\delta$, it follows that $\mu(B) \leq \mu(B \setminus A)/\delta \leq \mu(S \setminus A)/\delta < \delta$. Now, we have that $G = S \setminus B$ consists of a finite disjoint union of good $n_k-$cylinders. Further, we have $\mu(G \triangle A) \leq \mu(G \triangle S) + \mu(S \triangle A) < \delta + \delta^2 < \epsilon$.

Now if $C$ is an $n-k-$cylinder forming part of $G$, then $C$ is one of the $n_k-$cylinders forming $C_k(b)$ for some $b \in Z$. We now show that any other cylinder making up $C_k(b)$
is also good. To show this, let $D$ be another $n_k$–cylinder which is a subset of $C_k(b)$. Then there exists a $\gamma \in \Gamma$ which only affects coordinates up to the $n_k$–1st such that $\gamma(C) = D$. Further, restricted to $C, \gamma$ is a measure-preserving map. Since we assumed that $S$ consisted of a union of $\sim$–equivalence classes, it follows that $\mu(S \cap D) = \mu(S \cap C)$ from which it follows that $D$ is good as required.

Finally, let $\widetilde{G}$ be the union of those $C_k(b)$ which intersect $G$. Then from the above, it follows that $\mu(\widetilde{G} \cap S)/\mu(\widetilde{G}) > 1 - \delta$. In particular, $\mu(\widetilde{G} \setminus S) < \epsilon$. But we have also that $\mu(S \setminus \widetilde{G}) < \mu(S \setminus G) < \epsilon$ so we see that $\mu(S \Delta \widetilde{G}) < 2\epsilon$, proving the claim that any element of $\mathcal{G}$ may be arbitrarily well approximated by unions of sets of the form $C_k(b)$. From this, it follows that these sets generated the $\sigma$–algebra $\mathcal{G}$.

We now show that the quotient space $Y$ may be identified with $Z$. There is a natural map from $Y$ to $Z$ and the above shows that any measurable subset of $Y$ agrees with the inverse image of a Borel measurable subset of $Z$ up to a set of measure 0. This is sufficient to guarantee the identification of $Y$ and $Z$. We are also able to calculate the quotient measure on $Z$. This is defined by

$$\nu([b]^k) = \frac{1}{2} \prod_{i=1}^k \left( \frac{m_i}{b_i} \right) (1 - p_i)^{m_i - b_i} p_i^{b_i}.$$ 

This is because the inverse under the projection of the cylinder set $[b]^k$ is the union of $\prod_{i=1}^k \left( \frac{m_i}{b_i} \right)$ cylinder sets in $X$ of measure $\frac{1}{2} \prod_{i=1}^k (1 - p_i)^{m_i - b_i} p_i^{b_i}$.

Having identified the quotient space, define a function $\phi$ on $X$ by $\phi(x) = \min\{\log d\mu \circ \gamma/d\mu(x) : \log d\mu \circ \gamma/d\mu(x) > 0\}$. Since the $R_i$ were taken to be greater than 1, we see that this is a strictly positive quantity (and in fact bounded below by $\min R_i$). Further, it is clear that if $x \sim y$, then $\phi(x) = \phi(y)$. This shows that $\phi$ may be regarded as a function on the quotient $Z$.

The final ingredient in the construction of Hamachi and Osikawa is the construction of an automorphism $U$ of $X$ such that $d\mu \circ U/d\mu = \exp \phi(x)$. Again, it is clear that if $x \sim y$ then $U(x) \sim U(y)$ so once again, $U$ may be regarded as a map of $Z$. Clearly from
the construction of the measure on $X$, $U(x)$ should be a point in the orbit of $x$ which has
the transitions modified in such a way that
\[
\frac{\mu([U(x)]^{n_k})}{\mu([x]^{n_k})} = \left(\frac{1 - p_1}{p_1}\right)^{a_1-b_1} \left(\frac{1 - p_2}{p_2}\right)^{a_2-b_2} \cdots \left(\frac{1 - p_k}{p_k}\right)^{a_k-b_k}
\]
is minimal but greater than 1, where $n_k$ is defined such that $U(x)$ only disagrees with $x$
before the $n_k$th terms and $(a_0, a_1, \ldots)$ and $(b_0, b_1, \ldots)$ are the block types of $U(x)$ and $x$.

One can identify the effect of $U$ on the cycle types. Namely, $U$ increases $b_1$ by 1 to $a_1$
unless $b_1$ is already maximal, in which case $a_1$ is set to 0 and $b_2$ is increased (unless $b_2$
should happen to be maximal etc.). This is nothing other than an odometer action where
$b_i$ can range between 0 and $m_i$. This determines $U$ apart from its effect on $b_0$. This is
determined by the requirement that $x$ and $U(x)$ should lie in the same $\Gamma-$orbit which
determines that the total number of transitions up to $n_k$ for $x$ and $U(x)$ should have the
same parity (even or odd). The digit $a_0$ is then a ‘parity bit’ which must be chosen to
ensure that $a_0 + a_1 + \ldots + a_k$ differs from $b_0 + b_1 + \ldots + b_k$ by an even number.

We call the automorphism $U$ of $Z$ an odometer with parity. The ergodicity of such
odometers with parity is not immediately apparent, but they turn out always to be ergodic.
This will in any case follow from results about the associated flow.

Finally, the construction of Hamachi and Osikawa gives an explicit description of the
associated flow. Namely, it is isomorphic to the suspension flow of $U : Z \rightarrow Z$ with ceiling
function $\phi(z)$. To describe this, let $Z_\phi$ denote the space $\{(z, t) : z \in Z, 0 \leq t < \phi(z)\}$.
The flow on this space is given by the maps $T_s$ where $s > 0$,
\[
T_s((z, t)) = \begin{cases} (z, s + t) & \text{if } s + t < \phi(z) \\ (U(z), s + t - \phi(z)) & \text{if } \phi(z) \leq s + t < \phi(z) + \phi(U(z)) \\ \ldots & \ldots \end{cases}
\]
Since the map $U$ is invertible, the flow is also defined for negative time. An alternative
description of the flow is the following. An equivalence relation is defined on the space
$Z \times \mathbb{R}$, namely $\approx$ is the equivalence relation generated by (i.e. the transitive closure of)
$(z, t) \approx (U(z), t - \phi(z))$. Then letting $[(z, t)]$ denote the $\approx$-equivalence class of $(z, t)$, $T_s$
acts on the quotient space $Z \times \mathbb{R}/ \approx$ by $T_s[(s, t)] = [(z, t + s)]$. In particular, we see that if $z$ and $z'$ lie differ in finitely many places, then we may pick $x$ and $x'$ whose block types are respectively $z$ and $z'$ which lie on the same orbit. There is then a $\gamma_0 \in \Gamma$ such that $\gamma_0(x) = x'$. The ratio $d\mu \circ \gamma/d\mu(x)$ is determined by the block types of $x$ and $x'$ (namely $z$ and $z'$) alone so is independent of the particular values of $x$ and $x'$. Forming $\tau = \log d\mu \circ \gamma/d\mu(x)$, we show that $(z, t)$ is independent of the particular values of $x$ and $z$. This follows from the fact that if $z$ and $z'$ differ in finitely many places, then we may pick $x, x'$ such that $(z, t) = (z', t + \tau)$. Note that in doing this, we may assume that $\tau$ is positive. Since the ratios $d\mu \circ \gamma/d\mu(x)$ take values in a discrete set, there can only be finitely many ratios between 0 and $\tau$ (say). Write $\mathcal{R}(x)$ for $\{\log d\mu \circ \gamma/d\mu(x) : \gamma \in \Gamma\}$. Then by the chain rule, we see that $\mathcal{R}(\gamma(x)) = \mathcal{R}(x) - \log d\mu \circ \gamma/d\mu(x)$.

It remains to demonstrate that this flow has the AT property. We will let $\chi_S$ denote the characteristic function of a set $S$ and use the notation $[z]^k$ or $[b_0 \ldots b_k]$ for cylinder sets in $Z$. Fix $k > 0$. Then we will show, taking sufficiently small $\delta$ and letting $f = \chi_{[00 \ldots 0]^k \times [0, \delta]}$, that we can closely approximate arbitrarily closely any given finite collection of functions of the form $\chi_{[b_0 \ldots b_k]^k \times [c, d]}$. Since any positive integrable function may be arbitrarily closely approximated by a finite linear combination of functions of this form, it will follow from this that the flow has the AT property.

First, we observe that letting $\tau = \sum_{i=1}^k a_i K_i$ where $0 \leq a_i \leq m_i$, we have

$$\mathcal{L}_{-\tau}f(z, t) = \chi_{[00 \ldots 0]^k \times [0, \delta]}(T_{-\tau}(z, t)) \frac{d\mu \circ T_{-\tau}}{d\mu}(z, t).$$

But we see that $\chi_{[00 \ldots 0]^k \times [0, \delta]}(T_{-\tau}(z, t))$ is equal to $\chi_{T_{-\tau}([00 \ldots 0]^k \times [0, \delta])}(z, t)$ and it is straightforward to check that $T_{-\tau}([00 \ldots 0]^k \times [0, \delta]) = [c(a)a_1a_2 \ldots a_k]$ where $c(a)$ is such that $c(a) + a_1 + \ldots + a_k = 0 \pmod{2}$. This follows from the fact that if $x$ has block type in $[00 \ldots 0]^k$ and $\gamma \in \Gamma$ affect only the first $n_k$ coordinates, leaving $\gamma(x) \in [c(a)a_1a_2 \ldots a_k]$
then \( \log d\mu \circ \gamma / d\mu(x) = \tau \). This means that \( L_{-\tau} f \) is just \( \chi_{[c; a_1, a_2, ... a_k] \times [0, \delta]} \). From this, it clearly follows that we can approximate arbitrarily closely (by taking small \( \delta \)) any function of the form \( \chi_{[b]^k \times [c, d]} \) provided that \( \sum_{i=0}^{k} b_i = 0 \) (mod 2). It remains to show that we can approximate cylinders where \( \sum_{i=0}^{k} b_i = 1 \) (mod 2).

To this end, pick a very large \( M \) and consider \( L_{-K_M} f \). Then as before, we have \( f(T_{-K_M}(z, t)) = \chi_{T_{K_M}([0 ... 0]^k \times [0, \delta])}(z, t) \). Then we observe that if \((y, s) \in [0 ... 0]^k \times [0, \delta] \) and \( y_M \neq m_M \) then \( T_{K_M}(y, s) = (\hat{y}, s) \) where \( \hat{y}_0 = 1 \) and \( y_i = y_i + \delta_{i, M} \) for \( i \geq 1 \). Setting \( B = \{(z, t) \in [0 ... 0]^k \times [0, \delta] : z_M \neq m_M \} \), and \( B' = ([0 ... 0]^k \times [0, \delta]) \setminus B \), we have that \( f = \chi_B + \chi_{B'} \). Since \( \|\chi_{B'}\|_1 = p_M^m \|f\|_1 \) and \( L_{-K_M} \) is linear and norm-preserving, we have

\[
\|L_{K_M} f - \alpha \chi_{[000...0]^k \times [0, \delta]}\|_1 = 2p_M^m \|f\|_1.
\]

where \( \alpha \) is chosen to ensure that \( \|\alpha \chi_{[000...0]^k \times [0, \delta]}\|_1 = \|f\|_1 \). This shows that for large \( M \), we can get an arbitrarily close approximation to \( \chi_{[000...0]^k \times [0, \delta]} \) and then by a similar argument to that for the even parity cylinders, we see that we can approximate any function of the desired type. This completes the proof.
References


