ENTROPY ALONG CONVEX SHAPES, RANDOM TILINGS AND SHIFTS OF FINITE TYPE

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ABSTRACT. A well-known formula for the topological entropy of a symbolic system is $h_{\mathrm{top}}(X) = \lim_{n\to\infty} \log N(\Lambda_n)/|\Lambda_n|$, where Λ_n is the box of side n in $\mathbb{Z}^d$ and $N(\Lambda)$ is the number of configurations of the system on the finite subset Λ of $\mathbb{Z}^d$. We investigate the convergence of the above limit for sequences of regions other than Λ_n and show in particular that if Ξ_n is any sequence of finite 'convex' sets in $\mathbb{Z}^d$ whose inradii tend to infinity, then the sequence $\log N(\Xi_n)/|\Xi_n|$ converges to $h_{\mathrm{top}}(X)$.

We apply this to give a concrete proof of a 'strong Variational Principle', that is, the result that for certain higher dimensional systems the topological entropy of the system is the supremum of the measure-theoretic entropies taken over the set of all invariant measures with the Bernoulli property.

1. Introduction

To define a d-dimensional subshift, one starts with a finite alphabet $\mathcal{A}$ and then $\mathcal{A}^{\mathbb{Z}^d}$ is the collection of all d-dimensional square lattice configurations of the symbols from $\mathcal{A}$. We endow this space with the product topology, making it a compact metrizable space. If ξ is a configuration from $\mathcal{A}^{\mathbb{Z}^d}$ and $u \in \mathbb{Z}^d$, we write ξ_u for the symbol occurring in position u. For $v \in \mathbb{Z}^d$, the shift map through v, $\sigma_v \colon \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$ is defined by $(\sigma_v(\xi))_u = \xi_{u+v}$.

A *subshift* is a non-empty closed shift-invariant subset of $\mathcal{A}^{\mathbb{Z}^d}$. A shift of finite type is a subshift in which configurations are required to satisfy local rules. Given a finite set $\Lambda \subset \mathbb{Z}^d$ and a collection $R \subset \mathcal{A}^{\Lambda}$, we define a subset X of $\mathcal{A}^{\mathbb{Z}^d}$ by $X = X_{\Lambda,R} = \{\xi \in \mathcal{A}^{\mathbb{Z}^d} : \sigma_v(\xi)|_\Lambda \in R, \text{ for all } v \in \mathbb{Z}^d\}$, where $\xi|_\Lambda$ is the restriction of the configuration ξ to the coordinate set Λ. If the set X is non-empty, then it is known as a *shift of finite type.* We note that such a set is necessarily closed and shift-invariant.

In the case $d = 1$, much is known about the structure of shifts of finite type and many questions can be answered using elementary properties of matrices.

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In the case $d \geq 2$, it has been shown to be undecidable whether the set $X_{\Lambda,R}$ is even non-empty and little is known about the general structure of d-dimensional shifts of finite type. There is interest in finding broad classes of shifts of finite type in which questions are tractable.

A d-dimensional shift of finite type X is said to be *mixing* if given any two non-empty open subsets U and V of X, it is the case that for all but finitely many $u \in \mathbb{Z}^d$, $\sigma_u(U) \cap V$ is non-empty. A d-dimensional shift of finite type X is said to be *strongly irreducible* if there exists an $r > 0$ such that for any two subsets A and B of $\mathbb{Z}^d$ satisfying $d(A, B) = \min(\{\|a - b\|_\infty \colon a \in A,\ b \in B\}) > r$ and any two points ξ and ζ of X, there exists a point $\eta \in X$ such that $\eta|_A = \xi|_A$ and $\eta|_B = \zeta|_B$. The number r is called the *filling length* of X. The shift X is said to have a *safe symbol* if there is a symbol such that given subsets of $\mathbb{Z}^d$ satisfying $d(A, B) > 1$ and points ξ and ζ as above, then the configuration defined to be equal to ξ on A, ζ on B and the safe symbol elsewhere belongs to X. In particular, if X has a safe symbol, it is strongly irreducible. Since any non-empty open set contains a cylinder set (a set of the form $[\eta]_\Lambda = \{\xi \colon \xi|_\Lambda = \eta_\Lambda\}$), one sees that strong irreducibility implies mixing. In one dimension, the converse holds also: mixing implies strong irreducibility. It is well-known that this fails in two dimensions.

Let Λ_n be the finite grid of n^d points in $\mathbb{Z}^d$ whose coordinates satisfy $0 \leq u_i < n$. It is a standard result that if $N(\Lambda)$ denotes the number of distinct restrictions to Λ of points of a d-dimensional subshift X, then the sequence $(\log N(\Lambda_n))/n^d$ is convergent. The limit is called the *topological entropy* of X and is denoted by $h_{\mathrm{top}}(X)$. Clearly, the quantity $N(\Lambda)$ can be defined for any finite subset Λ of $\mathbb{Z}^d$. The second section of the paper considers more general sequences Ξ_n of regions than Λ_n with the property that $\log N(\Xi_n)/|\Xi_n|$ converges to $h_{\mathrm{top}}(X)$. The following theorem is a corollary of Theorem 2.1, which addresses averages of subadditive functionals on $\mathbb{Z}^d$. In particular, Theorem 2.1 applies equally to measure-theoretic entropy or to the calculation of topological entropy with respect to an open cover. After writing the paper, we discovered that a special case of the following theorem appears in [5].

Theorem A. *Let X be a d-dimensional symbolic dynamical system. Given $\epsilon > 0$, there exists an $R > 0$ such that if C is any bounded convex set in $\mathbb{R}^d$ containing a ball of radius R, then*

$$\left| \frac{\log N(C \cap \mathbb{Z}^d)}{|C \cap \mathbb{Z}^d|} - h_{top}(X) \right| \leq \epsilon.$$

This implies that in order for a sequence of 'convex' subsets Ξ_n of $\mathbb{Z}^d$ to have the property that $\log N(\Xi_n)/|\Xi_n| \to h_{\mathrm{top}}(X)$, it is sufficient that the inradii of the Ξ_n converge to infinity.

In the next section, we apply this theorem to construct some measures with high entropy on an arbitrary strongly irreducible shift of finite type.

We shall need to consider invariant probability measures on X (that is those satisfying $\mu(B) = \mu(\sigma_v(B))$ for any Borel subset B of X and $v \in \mathbb{Z}^d$). If μ is an invariant measure, one can associate to it an entropy $h_\mu(X)$. Given a finite subset $A \subset Z^d$, we shall denote by $\mathcal{P}^A$ the partition of X according to the configuration on A: $\mathcal{P}^A = \{\{x \in X : x|_A = y\} : y \in \mathcal{A}^A\}$. The entropy of a partition with respect to a measure μ is given by $H_\mu(\mathcal{P}^A) = \sum_{B \in \mathcal{P}^A} \phi(\mu(B))$, where ϕ is the concave function $\phi(t) = -t \log t$. The *measure-theoretic entropy* of X with respect to μ is given by $h_\mu(X) = \lim_{N\to\infty} H_\mu(\mathcal{P}^{\Lambda_N})/|\Lambda_N|$, where convergence is guaranteed by a standard argument using the concavity of ϕ to show subadditivity of H_μ.

The Variational Principle [6] states that for any continuous action of $\mathbb{Z}^d$ on a compact metric space, the topological entropy, $h_{\text{top}}(X)$, is equal to $\sup h_\mu(X)$ where the supremum is taken over the set of all invariant probability measures μ on X. It is well-known that in the case of a shift space, the supremum is attained on a non-empty set of measures of maximal entropy.

A measure μ on a shift space X is said to be *Bernoulli* if it has the property that it is the image under a shift-commuting mapping of an independent identically distributed measure on some space $Y = B^{\mathbb{Z}^d}$. That is if there exists a product measure ν on Y and a mapping $\Phi : Y \to X$ such that $\Phi \circ \sigma_u = \sigma_u \circ \Phi$ for all $u \in \mathbb{Z}^d$ and $\mu = \nu \circ \Phi^{-1}$. It can be shown that such a measure is measure-theoretically isomorphic to a product measure (i.e. that the measure is the image of a product measure under a shift-commuting bijection), and thus up to measure-theoretic isomorphism, a Bernoulli measure is 'as random as possible'. There is a hierarchy of ergodic properties of which the Bernoulli property is the strongest. Some weaker properties in order are the K property, the strong-mixing property, the weak-mixing property and ergodicity.

In the one-dimensional case, a strongly irreducible shift of finite type has a unique measure of maximal entropy that is necessarily Bernoulli. In the case of shifts of finite type in higher dimensions, Burton and Steif [1, 2] gave examples of strongly irreducible shifts of finite type for which there is more than one measure of maximal entropy and for which the unique measure of maximal entropy fails to be weak-mixing respectively.

In [9], Robinson and Şahin showed that the Variational Principle holds for uniformly filling shifts of finite type even if the supremum is taken over the set of shift-invariant measures with the K-property. This is achieved by using the machinery of Kakutani equivalences. As a corollary of a result in a second paper [10], they show that, provided that the shift of finite type has dense periodic orbits, the Variational Principle holds if one takes the supremum over measures with the Bernoulli property. Uniform filling is *a priori* a weaker property than strong irreducibility although no example is known of a system that has one property but not the other.

In the second part of the paper we give an alternative proof of the fact that given a strongly irreducible shift of finite type, it supports Bernoulli measures with entropy arbitrarily close to the topological entropy. The main advantage of the method presented here is that it is much more direct and so it gives a clear description of the measures involved. A simple technique is introduced for establishing existence of Bernoulli measures or alternatively for establishing that a given measure is Bernoulli; we hope this technique will find applications elsewhere. The method also has the property that it shows how to exhibit explicitly the measures in question as factors of an independent identically distributed process.

The theorem that we prove is the following.

Theorem B. *Let X be a strongly irreducible shift of finite type. Then, for all $\epsilon > 0$, there is a Bernoulli measure μ on X such that $h_\mu(X) \geq h_{top}(X) - \epsilon$.*

The application above is described in the third section. For the proof, we tile the plane randomly with Voronoi cells and fill in their interiors with randomly chosen configurations. The results from the second section are used to guarantee that there is sufficient entropy in the configurations arising from this process. The proof proceeds by repeatedly 'grouting' together a number of tiles to form larger tiles and ensuring that the resulting process remains Bernoulli.

2. Subadditive functionals

Let $d \geq 1$ and let $\mathcal{F}$ denote the collection of finite subsets of $\mathbb{Z}^d$. Let $a\colon \mathcal{F} \to \mathbb{R}$ be a monotonic translation-invariant subadditive functional, where a functional $a\colon \mathcal{F} \to \mathbb{R}$ is said to be *monotonic* if $\Lambda \subset \Lambda'$ implies $a(\Lambda) \leq a(\Lambda')$; *subadditive* if for disjoint subsets Λ and Λ' of $\mathbb{Z}^d$, $a(\Lambda \cup \Lambda') \leq a(\Lambda) + a(\Lambda')$; and *translation-invariant* if $a(\Lambda + v) = a(\Lambda)$ for all $v \in \mathbb{Z}^d$. We note that these properties force $a(\Lambda)$ to be non-negative for all $\Lambda \in \mathcal{F}$. We write $|\Lambda|$ for the cardinality of Λ. A set $\Lambda \in \mathcal{F}$ is said to *tessellate* $\mathbb{Z}^d$ if there is a sequence $v_1, v_2, \ldots$ in $\mathbb{Z}^d$ such that $\{\Lambda + v_i \colon i \in \mathbb{N}\}$ is a partition of $\mathbb{Z}^d$.

The subadditivity and translation-invariance of a implies that

$$\lim_{n\to\infty} \frac{a(\Lambda_n)}{|\Lambda_n|} = \inf_n \frac{a(\Lambda_n)}{|\Lambda_n|}. \tag{2.1}$$

We shall denote this limit by α. The main result of this section is the following generalization of the above to convex sets.

Theorem 2.1. *Let $a\colon \mathcal{F} \to \mathbb{R}$ be a monotonic translation-invariant subadditive functional. Given $\epsilon > 0$, there exists an R such that if C is any bounded convex set in $\mathbb{R}^d$ containing a ball of radius R, then*

$$\left| \frac{a(C \cap \mathbb{Z}^d)}{|C \cap \mathbb{Z}^d|} - \alpha \right| \leq \epsilon.$$

We note that a proof of this theorem immediately gives a proof of Theorem A since $a(\Lambda) = \log N(\Lambda)$ satisfies the hypotheses.

To prove the theorem, we first show that if a subset Λ of $\mathbb{Z}^d$ tessellates $\mathbb{Z}^d$, then $a(\Lambda)/|\Lambda| \geq \alpha$. We then approximate the convex set as the difference of a parallelepiped and a union of finite boxes. Since there is a bounded ratio between the volumes of the parallelepiped and the original convex set (by a theorem of Chakerian and Stein [3]), this allows us to estimate the value of $a(C \cap \mathbb{Z}^d)/|C \cap \mathbb{Z}^d|$. Throughout this section, we shall assume that a is as in the statement of the theorem.

Lemma 2.2. *If $\Lambda \in \mathcal{F}$ tessellates $\mathbb{Z}^d$, then $a(\Lambda) \geq |\Lambda|\alpha$.*

Proof. By (2.1), we see $a(\Lambda_n) \geq |\Lambda_n|\alpha = n^d\alpha$ for all n. Write ℓ for the $\|\cdot\|_\infty$-diameter of Λ. Let $\{\Lambda + v_i \colon i \in \mathbb{N}\}$ be a tessellation of $\mathbb{Z}^d$. For a given $N > 0$, set $S = \{i \colon (\Lambda + v_i) \cap \Lambda_N \neq \emptyset\}$. Then $\Lambda_N \subset \bigcup_{i \in S}(\Lambda + v_i) \subset \Lambda_{N+2\ell} - \ell(1, \ldots, 1)$.

We see that $|S|a(\Lambda) \geq N^d\alpha$. However, we also have $|S| \cdot |\Lambda| \leq |\Lambda_{N+2\ell}| = (N + 2\ell)^d$. Dividing, we see

$$\frac{a(\Lambda)}{|\Lambda|} \geq \alpha \left(\frac{N}{N+2\ell}\right)^d.$$

Letting N tend to infinity, the conclusion follows. □

For a convex set C in $\mathbb{R}^d$, we shall write $B_r(C)$ for the closed r-neighborhood of C (with respect to the Euclidean distance) and $I_r(C)$ for $\{x \in C \colon B_r(x) \subset C\}$, the r-interior of C.

Lemma 2.3. *Let $v \in \mathbb{R}^d$ be a vector satisfying $\|v\|_\infty \leq \frac{1}{2}$, and C be a convex set in $\mathbb{R}^d$. Then*

$$\left(C \cap \mathbb{Z}^d\right) \triangle \left((C + v) \cap \mathbb{Z}^d\right) \subset \left(B_{\sqrt{d}/2}(C) \setminus I_{\sqrt{d}/2}(C)\right) \cap \mathbb{Z}^d$$

Proof. We first note that for such a v, $\|v\|_2 \leq \sqrt{d}/2$. It is sufficient to show that

$$I_{\sqrt{d}/2}(C) \cap \mathbb{Z}^d \subset (C + v) \cap \mathbb{Z}^d \subset B_{\sqrt{d}/2}(C) \cap \mathbb{Z}^d.$$

This is clear since $I_{\sqrt{d}/2}(C) \subset C + v \subset B_{\sqrt{d}/2}(C)$. □

In stating the next lemma, we shall assume that C is a convex set in $\mathbb{R}^d$ containing an open ball of radius r_0 centered at x_0. We shall use non-standard notation, writing kC for the set obtained by scaling C about x_0 by the factor k (i.e. $kC = \{kx + (1-k)x_0 \colon x \in C\}$).

Lemma 2.4. *Suppose C is a convex set in $\mathbb{R}^d$ containing a ball of radius r_0 centered at x_0. Then*

$$\left(1 - \frac{r}{r_0}\right) C \subset I_r(C) \quad \textit{for } r \leq r_0$$

$$\textit{and} \quad B_r(C) \subset \left(1 + \frac{r}{r_0}\right) C.$$

Proof. To prove the first assertion, let $x \in (1-\frac{r}{r_0})C$ and let $y \in B_r(x)$. Then write $x = (1-\frac{r}{r_0})u + \frac{r}{r_0}x_0$ and $y - x = \frac{r}{r_0}v$ for some $u \in C$ and $v \in B_{r_0}(0)$. Now $y = (1-\frac{r}{r_0})u + \frac{r}{r_0}(x_0+v) \in C$ so we see $B_r(x) \subset C$. This establishes that $(1-\frac{r}{r_0})C \subset I_r(C)$ as required.

To demonstrate the second assertion, let $x \in B_r(C)$. Then $x = u + \frac{r}{r_0}v$, where $u \in C$ and $v \in B_{r_0}(0)$. Now $x = u + \frac{r}{r_0}(v+x_0) - \frac{r}{r_0}x_0$. Since $(u + \frac{r}{r_0}(v+x_0))/(1+\frac{r}{r_0}) \in C$, we see that $x \in (1+\frac{r}{r_0})C$ as required. □

Lemma 2.5. *If $C \subset \mathbb{R}^d$ is a convex set, then*

$$\mathrm{Vol}(I_{\sqrt{d}/2}(C)) \le |C \cap \mathbb{Z}^d| \le \mathrm{Vol}(B_{\sqrt{d}/2}(C))$$

Proof. Note first that $I_{\sqrt{d}/2}(C) \subset (C \cap \mathbb{Z}^d) + [-\frac{1}{2}, \frac{1}{2})^d \subset B_{\sqrt{d}/2}(C)$. Indeed if $x \in I_{\sqrt{d}/2}(C)$ then $x + (-\frac{1}{2}, \frac{1}{2}]^d \subset C$. Since the set on the left contains a unique point in $\mathbb{Z}^d$, the required containment follows. Since $(C \cap \mathbb{Z}^d) + [-\frac{1}{2}, \frac{1}{2})^d$ is a disjoint union of $|C \cap \mathbb{Z}^d|$ cubes of volume 1, the inequalities follow by taking the volume of the three sets. □

The following theorem appears in [3] and the reader is referred to that paper for a proof.

Theorem 2.6. *Let C be a bounded convex set in $\mathbb{R}^d$. Then there exists a parallelepiped $P \supset C$ such that $\mathrm{Vol}(P)/\mathrm{Vol}(C) \le d^d$.*

One can ensure that the parallelepiped P has rational vertices by making an arbitrarily small change to the constant d^d.

Proof of Theorem 2.1. We shall first prove a lower bound: given $\epsilon > 0$, if R is sufficiently large then any convex set in $\mathbb{R}^d$ containing a ball of radius R satisfies $a(C \cap \mathbb{Z}^d)/|C \cap \mathbb{Z}^d| \ge \alpha - \epsilon$

To prove this, we shall fix $\epsilon > 0$ and assume that C is a convex set in $\mathbb{R}^d$ containing a ball of radius r_0.

Combining Lemmas 2.3, 2.4 and 2.5, we see that for $\|v\|_\infty \le \frac{1}{2}$,

$$\begin{aligned}
|(C \cap \mathbb{Z}^d) \triangle ((C+v) \cap \mathbb{Z}^d)| &\le |B_{\sqrt{d}/2}(C) \cap \mathbb{Z}^d| - |I_{\sqrt{d}/2}(C) \cap \mathbb{Z}^d| \\
&\le \mathrm{Vol}(B_{\sqrt{d}}(C)) - \mathrm{Vol}(I_{\sqrt{d}}(C)) \\
&\le \left(\left(1 + \frac{\sqrt{d}}{r_0}\right)^d - \left(1 - \frac{\sqrt{d}}{r_0}\right)^d\right)\mathrm{Vol}(C) \qquad (2.2) \\
&= O\left(\frac{\mathrm{Vol}(C)}{r_0}\right).
\end{aligned}$$

It then follows (using the monotonicity and sub-additivity of a) that

$$\begin{aligned}\left|a\left(C \cap \mathbb{Z}^d\right) - a\left((C+v) \cap \mathbb{Z}^d\right)\right| &\leq \left|(C \cap \mathbb{Z}^d) \triangle ((C+v) \cap \mathbb{Z}^d)\right| \cdot a(\{0\}) \\ &= O\left(\frac{\operatorname{Vol}(C)}{r_0}\right)\end{aligned}$$

for all vectors v satisfying $\|v\|_\infty \leq \frac{1}{2}$, where the implicit constant in the error term is independent of v, C and r_0. By the translation invariance of $a(\cdot)$, we see that

$$\left|a\left(C \cap \mathbb{Z}^d\right) - a\left((C+v) \cap \mathbb{Z}^d\right)\right| = O\left(\frac{\operatorname{Vol}(C)}{r_0}\right) \tag{2.3}$$

for all $v \in \mathbb{R}^d$.

By Theorem 2.6, C is contained in a rational parallelepiped P with the property that $\operatorname{Vol}(C)/\operatorname{Vol}(P) \geq 1/(2d^d)$. The edges of P generate a lattice that gives a tessellation of $\mathbb{R}^d$ by translates of P. Let the translates of P be given by $P+v_1, P+v_2, \ldots$ and define

$$K = \bigcup_i (C+v_i) \cap \mathbb{Z}^d.$$

This is the intersection of an approximate tessellation of $\mathbb{R}^d$ by copies of C with $\mathbb{Z}^d$. We then want to study the intersection with a square lattice. For $r < r_0$, we consider translates of the set Λ_r through vectors in $r\mathbb{Z}^d$. We then define two further subsets G (for good) and B (for bad) of $\mathbb{Z}^d$:

$$\begin{aligned}G &= \bigcup \{\Lambda_r + rv \colon v \in \mathbb{Z}^d,\ \Lambda_r + rv \subset \mathbb{R}^d \setminus K\}, \\ B &= \mathbb{Z}^d \setminus (G \cup K).\end{aligned}$$

Thus we have divided $\mathbb{Z}^d$ into 3 parts: K, G and B.

Set $B_i = (B_{r\sqrt{d}}(C+v_i) \setminus (C+v_i)) \cap \mathbb{Z}^d$. If $x \in B$, then since $x \notin K$, we see that x does not belong to one of the translates of C; and since $x \notin G$, we see that there exists a y in the same Λ_r-translate as x such that $y \in K$ so $x \in B_{r\sqrt{d}}(C+v_i)$ for some i. In particular, we see that $x \in \bigcup_i B_i$.

Using Lemmas 2.5 and 2.4, we see that

$$\begin{aligned}|B_i| &\leq \operatorname{Vol}(B_{(r+1)\sqrt{d}}(C)) - \operatorname{Vol}(I_{\sqrt{d}}(C)) \\ &\leq \left(\left(1 + \frac{(r+1)\sqrt{d}}{r_0}\right)^d - \left(1 - \frac{\sqrt{d}}{r_0}\right)^d\right) \operatorname{Vol}(C) \\ &= O\left(\frac{\operatorname{Vol}(C) r}{r_0}\right).\end{aligned}$$

Since the fundamental parallelepiped for the lattice is rational, there exists a multiple of it, $\tilde{P}$, for which the edge displacements are integer vectors. The partition of the points inside this part of the lattice into points of K, B and

G is then repeated periodically throughout $\mathbb{Z}^d$. Letting T be the intersection of $\mathbb{Z}^d$ with $\tilde{P}$, we have $a(T) \geq |T|\alpha$ by Lemma 2.2.

Now let $C_1, C_2, \ldots, C_s$ be the intersection of $\mathbb{Z}^d$ with the translates of C inside the large parallelepiped $\tilde{P}$, where $C_1 = C \cap \mathbb{Z}^d$. Similarly, let $G_1, G_2, \ldots, G_t$ be the translates of Λ_r that are contained in $T \setminus (C_1 \cup \ldots \cup C_s)$. The remainder $B_T = T \setminus (C_1 \cup \ldots \cup C_s \cup G_1 \cup \ldots \cup G_t)$ consists of points that belong to the B_i corresponding to $C_1, \ldots C_s$ and other points within an ℓ_∞ distance r of the boundary of T. Hence

$$\begin{aligned}|B_T| &\leq s \cdot O\left(\frac{\mathrm{Vol}(C)r}{r_0}\right) + \mathrm{Vol}(B_{\sqrt{d}}(\tilde{P}) \setminus I_{(r+1)\sqrt{d}}(\tilde{P}))\\ &\leq s \cdot O\left(\frac{\mathrm{Vol}(C)r}{r_0}\right) + \left(\left(1+\frac{\sqrt{d}}{r_0}\right)^d - \left(1 - \frac{(r+1)\sqrt{d}}{r_0}\right)^d\right)\mathrm{Vol}(\tilde{P})\\ &= O\left(\frac{\mathrm{Vol}(\tilde{P})r}{r_0}\right).\end{aligned}$$

We now have by the subadditivity of $a(\cdot)$ and Lemma 2.2,

$$a(C_1) + \ldots + a(C_s) + a(G_1) + \ldots + a(G_t) + |B_T|a(\{0\}) \geq |T|\alpha.$$

By (2.3), $a(C_i) = a(C_1) + O\left(\mathrm{Vol}(C)/r_0\right)$. So we see that

$$\begin{aligned}s \cdot a(C_1) &\geq |T|\alpha - t \cdot a(\Lambda_r) + O\left(\frac{\mathrm{Vol}(\tilde{P})r}{r_0}\right)\\ &= (|T| - tr^d)\alpha - tr^d\left(\frac{a(\Lambda_r)}{r^d} - \alpha\right) + O\left(\frac{\mathrm{Vol}(\tilde{P})r}{r_0}\right).\end{aligned}$$

Since by (2.2),

$$|T| - tr^d \geq |C_1| + \ldots + |C_s| = s|C_1| + O\left(\frac{s\,\mathrm{Vol}(C)}{r_0}\right) = s|C_1| + O\left(\frac{\mathrm{Vol}(\tilde{P})}{r_0}\right),$$

we have

$$s \cdot a(C_1) \geq s|C_1|\alpha - tr^d\left(\frac{a(\Lambda_r)}{|\Lambda_r|} - \alpha\right) + O\left(\frac{\mathrm{Vol}(\tilde{P})r}{r_0}\right).$$

Dividing through by $s|C_1|$, and making use of the facts $\operatorname{Vol}(P)/\operatorname{Vol}(C) \leq 2d^d$, $\operatorname{Vol}(C)/|C_1| \leq (1+\sqrt{d}/r_0)^d$ and $|T|/\operatorname{Vol}(\tilde{P}) \leq (1-\sqrt{d}/r_0)^{-d}$, we see that

$$\begin{aligned}\frac{a(C_1)}{|C_1|} &\geq \alpha - \frac{|T|}{s|C_1|}\left(\frac{a(\Lambda_r)}{|\Lambda_r|} - \alpha\right) + O\left(\frac{r}{r_0}\right)\\ &\geq \alpha - \frac{|T|}{\operatorname{Vol}(\tilde{P})}\frac{\operatorname{Vol}(\tilde{P})}{s\operatorname{Vol}(C)}\frac{\operatorname{Vol}(C)}{|C_1|}\left(\frac{a(\Lambda_r)}{|\Lambda_r|} - \alpha\right) + O\left(\frac{r}{r_0}\right)\\ &\geq \alpha - 2d^d\left(\frac{a(\Lambda_r)}{|\Lambda_r|} - \alpha\right) + O\left(\frac{r}{r_0}\right).\end{aligned}$$

Choose r such that

$$\frac{a(\Lambda_r)}{r^d} - \alpha \leq \frac{\epsilon}{4d^d}.$$

Next, for any $r_0 > r$ such that the error term is less than $\epsilon/2$, one sees that $a(C \cap \mathbb{Z}^d)/|C \cap \mathbb{Z}^d| \geq \alpha - \epsilon$. This establishes the existence of a suitable R.

To establish an upper bound, we consider the translates $A_1, A_2, \ldots, A_n$ of Λ_r through vectors in $r\mathbb{Z}^d$ that intersect $C \cap \mathbb{Z}^d$. We have $a(C \cap \mathbb{Z}^d) \leq n \cdot a(\Lambda_r)$. Letting H be the union of the translates $A_1, \ldots, A_n$, we observe that $H \subset B_{r\sqrt{d}}(C) \cap \mathbb{Z}^d$. Dividing the above inequality by $|C \cap \mathbb{Z}^d|$, we see

$$\frac{a(C \cap \mathbb{Z}^d)}{|C \cap \mathbb{Z}^d|} \leq \frac{|H|}{|C \cap \mathbb{Z}^d|}\frac{a(\Lambda_r)}{|\Lambda_r|} \leq \frac{|B_{r\sqrt{d}}(C) \cap \mathbb{Z}^d|}{|C \cap \mathbb{Z}^d|}\frac{a(\Lambda_r)}{|\Lambda_r|}.$$

Since $|B_{r\sqrt{d}}(C) \cap \mathbb{Z}^d| - |C \cap \mathbb{Z}^d| = O\left(\operatorname{Vol}(C)\frac{r}{r_0}\right)$, we see

$$\frac{a(C \cap \mathbb{Z}^d)}{|C \cap \mathbb{Z}^d|} \leq \frac{a(\Lambda_r)}{|\Lambda_r|} + O\left(\frac{r}{r_0}\right),$$

where the implicit constant is independent of C, r and r_0. Choosing r large enough so that $a(\Lambda_r)/|\Lambda_r| \leq \alpha + \epsilon/2$, we then see that for all sufficiently large r_0,

$$\frac{a(C \cap \mathbb{Z}^d)}{|C \cap \mathbb{Z}^d|} \leq \alpha + \epsilon.$$

This completes the proof of Theorem 2.1. □

3. High entropy Bernoulli measures

We begin with the following basic lemma.

Lemma 3.1. *Let X be a strongly irreducible shift of finite type with filling length r. Suppose that $A_1, A_2, \ldots$ are non-empty subsets of $\mathbb{Z}^d$ such that $d(A_i, A_j) > r$ for all $i \neq j$. If $\xi_1, \xi_2, \ldots$ are points of X, then there exists $\xi \in X$ such that $\xi|_{A_i} = \xi_i|_{A_i}$ for all i.*

Proof. We start by proving inductively that for each n, there exists a point $\eta_n \in X$ such that $\eta_n|_{A_i} = \xi_i|_{A_i}$ for all $i \leq n$. Clearly for $n = 1$, we may take $\eta_1 = \xi_1$. Supposing that such an η_n exists, the existence of η_{n+1} exists from the definition of strong irreducibility applied to $\eta_n|_{A_1\cup\ldots\cup A_n}$ and $\xi_{n+1}|_{A_{n+1}}$. Since the space X is compact (hence sequentially compact), we may take a subsequence of (η_n) convergent to a point $\xi \in X$. Since the alphabet was topologized with the discrete topology, we see that the restriction of any convergent sequence to a finite part of $\mathbb{Z}^d$ is eventually constant and therefore $\xi|_{A_i} = \xi_i|_{A_i}$ for all i. □

We now apply the results on entropy above to give an explicit construction of a factor mapping from an independent identically distributed process to a measure on our strongly irreducible shift of finite type X.

Proof of Theorem B. We may assume that X does not consist of a single fixed point, as if so, the conclusion is trivial. Let l be the filling length of X and $\epsilon > 0$. Given $\Lambda \in \mathcal{F}$, define $H(\Lambda)$ to be the logarithm of the number of configurations on Λ that extend to points of X. We see that $H(\Lambda)$ is a monotonic, subadditive, translation-invariant functional. We shall write h for $h_{\rm top}(X)$. Since X is not the trivial shift, we have $h > 0$. By Theorem 2.1, there exists an R_1 such that if C is a convex set in $\mathbb{R}^d$ containing a ball of radius R_1, then $H(C\cap\mathbb{Z}^d)/|C\cap\mathbb{Z}^d| \geq h - \epsilon/2$. From the lemmas above, we further see that there exists an $R_2 \geq R_1 + 2l\sqrt{d}$ such that if C is a convex set containing a ball of radius R_2, then

$$\frac{|I_{l\sqrt{d}}(C)\cap\mathbb{Z}^d|}{|C\cap\mathbb{Z}^d|}(h - \epsilon/2) \geq h - \epsilon. \tag{3.1}$$

Our strategy is then to construct explicitly a product measure on a space $\bar{\Omega}$ and define a measurable factor map from $\bar{\Omega}$ into X. We shall then estimate the entropy of the resulting push-forward measure.

Write S for the space $\{0,1\}^{\mathbb{Z}^d}$, Ω for $[0,1)^{\mathbb{Z}^d}$ and take $\bar{\Omega}$ to be the product $\Omega\times\prod_{i=1}^{\infty} S^{(i)}$, where each $S^{(i)}$ is a copy of S. We then define a measure $\bar{\mu}$ on $\bar{\Omega}$ by taking a product of measures (actually a product of product measures) on the factors. The measure on Ω is the product of copies of Lebesgue measure on the interval $[0,1)$ indexed by $\mathbb{Z}^d$. The measure on the factor $S^{(i)}$ is a product measure $\mu^{(i)}$ where the probability of a 1 at any site is given by $\varrho^{(i)}$ and the probability of a 0 is $1-\varrho^{(i)}$, where the $\varrho^{(i)}$ satisfy $0 < \varrho^{(i)} < 1$ and are to be specified below. Clearly $\bar{\Omega}$ could equally be regarded as the space $\left([0,1)\times\prod_{n=1}^{\infty}\{0,1\}\right)^{\mathbb{Z}^d}$ and the measure defined above is just a product of identical measures on each of the factors indexed by $\mathbb{Z}^d$. It is now clear that the group $\mathbb{Z}^d$ acts naturally on $\bar{\Omega}$ by translating all the data through the same displacement. The image of the measure $\bar{\mu}$ by any shift-commuting measurable mapping will necessarily be a Bernoulli measure.

Define $F\colon S^{(1)} \to S$ by

$$F(\xi)_v = \begin{cases} 1 & \text{if } \xi_v = 1 \text{ and } \xi_u = 0 \text{ for all } 0 < \|u-v\|_\infty \le 2R_2 \\ 0 & \text{otherwise} \end{cases}$$

Thus after applying the measurable shift-commuting mapping F, the point $F(\xi)$ has no two 1s separated by an ℓ_∞ distance of less than $2R_2$. Clearly, for $\mu^{(1)}$-almost every ξ, $F(\xi)$ contains infinitely many 1s.

We then use $F(\xi)$ to define a set of Voronoi tiles: Let $v_1^{(1)}, v_2^{(1)}, \ldots$ be the points at which $F(\xi)$ takes the value 1. Fix a vector δ whose components together with 1 form a set that is linearly independent over $\mathbb{Q}$ that satisfies in addition $\|\delta\|_\infty \le 1/(2\sqrt{d})$ and set

$$T_i^{(1)} = \{x \in \mathbb{R}^d \colon d(x, v_i^{(1)}) \le d(x, v_j^{(1)}), \text{ for all } j.\} + \delta.$$

The δ ensures that no point of $\mathbb{Z}^d$ is contained in more than one tile. By definition of F, $T_i^{(1)}$ necessarily contains a ball of radius R_2 about $v_i + \delta$. Now letting C_i be the region $I_{l\sqrt{d}}(T_i^{(1)}) \cap \mathbb{Z}^d$, we see that $I_{l\sqrt{d}}(T_i^{(1)})$ is convex and contains a ball of radius R_1. By choice of R_1, we have $H(C_i) \ge (h - \epsilon/2)|C_i|$. By choice of R_2, we have $H(C_i) \ge (h-\epsilon)|T_i^{(1)} \cap \mathbb{Z}^d|$.

We now describe a way of filling in each C_i with a random uniformly chosen one of the $\exp H(C_i)$ choices and then extending it to be a configuration on the whole of $\mathbb{Z}^d$. Note that by Lemma 3.1, the existence of configurations on $\mathbb{Z}^d$ extending the configurations on the C_i is assured. We do the extension in stages: having filled in the C_i, we repeatedly amalgamate finite sets of tiles to form larger tiles. The limit of this process is the required configuration on $\mathbb{Z}^d$.

For the first stage, we start by listing all the shapes (up to translation) $\Lambda^{(r)} \in \mathcal{F}$ that can arise as C_i. On each $\Lambda^{(r)}$, we list all the configurations $\zeta_j^{(r)}$, $1 \le j \le n^{(r)} = \exp H(\Lambda^{(r)})$, that arise as the restriction of points in X.

If $\mathcal{A}$ is the alphabet of X, we write $\mathcal{A}^*$ for the alphabet with a 'gap' symbol adjoined. We then write X^* for the set of points of $\mathcal{A}^{*\mathbb{Z}^d}$ that are obtained by taking a point $\xi \in X$ and replacing the symbols on an arbitrary subset $A \subset \mathbb{Z}^d$ by the gap symbol.

We shall write a point $\bar\omega \in \bar\Omega$ as $(\beta, \xi^{(1)}, \xi^{(2)}, \ldots)$, where $\beta \in \Omega$ and $\xi^{(i)} \in S^{(i)}$. We then define a map $\Phi_1 \colon \bar\Omega \to X^*$. To specify $\Phi_1(\bar\omega)$, one finds the Voronoi tiles $T_1^{(1)}, T_2^{(1)}, \ldots$ as above from $F(\xi^{(1)})$. The $v_i^{(1)}$ are referred to as the *centers* of the tiles $T_i^{(1)}$. For each i, one finds the r_i such that C_i is a translate of $\Lambda^{(r_i)}$. The number β_{v_i} is then used to fill in C_i as follows: there exists a unique j such that $j/n^{(r_i)} \le \beta_{v_i} < (j+1)/n^{(r_i)}$. The region C_i is then filled in with a translated copy of $\zeta_j^{(r_i)}$. The remainder of $\mathbb{Z}^d$ is filled with the gap symbol. This mapping is clearly measurable and by Lemma 3.1, the image lies in X^*. It also commutes with the shift:

$$\Phi_1(\sigma_u(\bar{\omega})) = \sigma_u(\Phi_1(\bar{\omega}))$$

for all $u \in \mathbb{Z}^d$. This completes the first stage of the tiling.

Suppose that the $n-1$st stage tiling is complete, giving a sequence of mappings $\Phi_1 \dots \Phi_{n-1}$ from $\bar{\Omega}$ to X^*, that is a sequence of extensions in the sense that for $m < n$, $\Phi_m(\bar{\omega})$ is obtained from $\Phi_{m-1}(\bar{\omega})$ by replacing some of the gap symbols by symbols from $\mathcal{A}$.

We now show how to define the nth stage tiling by specifying the map Φ_n. Let the tiles in $\Phi_{n-1}(\bar{\omega})$ be $T_1^{(n-1)}, T_2^{(n-1)}, \dots$, with centers $v_1^{(n-1)}, v_2^{(n-1)}, \dots$. We now use the point $\xi^{(n)}$ of $S^{(n)}$ as follows: Let $v_i^{(n)}$ be the points at which $\xi^{(n)}$ takes the value 1. Then define nth stage tiles to be the union of the $T_j^{(n-1)}$ whose central points $v_j^{(n-1)}$ are closer to $v_i^{(n)} + \delta$ than to any other such point. These tiles are denoted $T_i^{(n)}$. We shall assume that all of these tiles are non-empty by simply removing them from the list otherwise. We complete this stage by filling in the gap symbols of $\Phi_{n-1}(\bar{\omega})$ in the $I_{l\sqrt{d}}(T_i^{(n)}) \cap \mathbb{Z}^d$. To ensure that this is done in a shift-commuting way, we introduce an ordering on $\mathcal{A}$ and do the filling in the (unique) smallest way in the lexicographic ordering that extends to a configuration in X. We note that by Lemma 3.1, the point after this filling in of gap symbols is completed belongs to X^*.

Let p_n denote the probability that the origin (or any arbitrary point of $\mathbb{Z}^d$) is contained in the $l\sqrt{d}$-interior of one of the nth stage tiles defined above. We claim that if the density $\varrho^{(n)}$ is chosen sufficiently small, then p_n can be made as close to 1 as desired. Let $\tau > 0$ and let r_n be chosen so that with probability at least $1 - \tau/2$, the origin is contained in the $l\sqrt{d}$-interior of the union of the $T_i^{(n-1)}$ whose central points (defined in the $n-1$st stage) are within r_n of the origin.

To calculate p_n, we estimate the probability q_n that the nearest $v_i^{(n)}$ and second nearest $v_j^{(n)}$ to the origin have the property that $d(v_j^{(n)}, 0) \le d(v_i^{(n)}, 0) + 2r_n$ and show that this probability converges to 0 as $\varrho^{(n)} \to 0$. We see that we may bound q_n by $\sum_k q_{n,k}$ where $q_{n,k}$ is the probability that the ball of radius $2kr_n$ about the origin does not contain any 1s in $\xi^{(n)}$ while the ball of radius $2(k+2)r_n$ contains at least two. Let N_k be the number of points of $\mathbb{Z}^d$ contained in a ball of radius $2kr_n$ about the origin. We then see that

$$q_{n,k} = (1 - \varrho^{(n)})^{N_k} \sum_{i=2}^{N_{k+2}-N_k} \binom{N_{k+2} - N_k}{i} {\varrho^{(n)}}^i (1 - \varrho^{(n)})^{N_{k+2}-N_k-i}.$$

For convenience, we set $\varrho^{(n)} = M^{-d+\frac{1}{4}}$, where M is to be determined. We shall make use of the following estimates: $N_k \ge Ck^d$, $N_{k+1} - N_k \ge C'k^{d-1}$ and $N_{k+2} - N_k \le C''k^{d-1}$.

For $k \geq M$, we estimate

$$q_{n,k} \leq \exp(-\varrho^{(n)} N_k) \leq \exp(-M^{-d+\frac{1}{4}}(CM^d + (k-M)C'M^{d-1})).$$

We then see that

$$\sum_{k \geq M} q_{n,k} \leq \frac{\exp(-CM^{\frac{1}{4}})}{1 - \exp(C'M^{-\frac{3}{4}})} = O(M^{\frac{3}{4}} \exp(-CM^{\frac{1}{4}})) = o(1).$$

For $k \leq M$ and M sufficiently large, we have $(N_{k+2} - N_k)\varrho^{(n)}/(1-\varrho^{(n)}) < \frac{1}{2}$ and hence comparing the binomial sum to a geometric series, we estimate $q_{n,k} \leq (\varrho^{(n)})^2 (N_{k+2} - N_k)^2$. It follows that

$$\sum_{k \leq M} q_{n,k} \leq M \cdot M^{-2d+\frac{1}{2}} C''^2 M^{2d-2} = C''^2 M^{-\frac{1}{2}} = o(1).$$

By choosing M large, we see that q_n can be made arbitrarily small.

Supposing that the origin is contained in the $l\sqrt{d}$-interior of the union of the level $n-1$ tiles with centers distant at most r_n from the origin (a probability $1 - \frac{\tau}{2}$ event), if the difference between the first and second radii of neighbors of the origin in $\xi^{(n)}$ is more than $2r_n$ (a probability $1 - q_n$ event), then it follows that all of the above tiles are amalgamated into a single tile containing the origin in its $l\sqrt{d}$-interior. Hence choosing $q_n < \tau/2$, we see that $p_n > 1 - \tau$. This establishes that the probability that the origin is contained in the $l\sqrt{d}$-interior of a level n tile may be made arbitrarily close to 1. This completes the inductive step.

Since the process is stationary, the above estimates apply to all points of $\mathbb{Z}^d$, so by ensuring that the p_n converge to 1, it will follow that almost surely, each point of $\mathbb{Z}^d$ is contained in the $l\sqrt{d}$-interior of a tile at level n for some n and hence a symbol is assigned to $\Phi_n(\omega)$. Since these values are never subsequently altered, we may take the limit of the $\Phi_n(\omega)$ to get a point of X. Since ω is chosen according to a product measure and the mapping $\Phi(\omega) = \lim_{n\to\infty} \Phi_n(\omega)$ is shift-commuting, it follows that the image measure is Bernoulli. Measurability of Φ is clear since it is the limit of measurable mappings.

To check the entropy of this measure, consider the first stage tile interiors C_i that are completely contained in Λ_N. Let $\eta > 0$. We claim that for sufficiently large N, with probability at least $1 - \eta$, the proportion of Λ_N occupied by these tiles is at least $1 - \eta$. To see this, consider the probability that a given point of $\mathbb{Z}^d$ has no stage 1 center within a Euclidean distance $\sqrt{N}$. Without loss of generality, we take the point to be the origin. We now demonstrate that this probability is $O\left(\exp(-CN^{d/2})\right)$. A point v is a stage 1 center if $\xi_v^{(1)} = 1$ and all points within a Euclidean distance $2R_2$ are 0s of $\xi^{(1)}$. Clearly for the lattice $4R_2\mathbb{Z}^d$, the events that the lattice points are stage 1 centers are independent and equiprobable. The probability that the origin has no stage 1 center within a Euclidean distance $\sqrt{N}$ is dominated by the probability that the origin has no such stage 1 center belonging to the lattice, but this latter probability can clearly be shown to be $O\left(\exp(-CN^{d/2})\right)$. It follows that the probability that there is a point of Λ_N with no stage 1 center within a Euclidean distance $\sqrt{N}$ is $O(N^d \exp(-CN^{d/2}))$. Choose N such that this probability is less that η and such that $(N-4\sqrt{N})^d/N^d > 1-\eta$. We then see that with probability at least $1-\eta$, points inside the box B of side $N-4\sqrt{N}$ centered at $(\frac{N}{2},\ldots,\frac{N}{2})$ belong to tiles with centers inside the box B' of side $N-2\sqrt{N}$ centered at the same point, whereas points outside Λ_N belong to tiles with centers outside B'. Accordingly, with probability at least $1-\eta$, B is entirely covered with stage 1 tiles that lie entirely within Λ_N as required.

We then estimate the entropy of the partition as follows: for each arrangement of the tiles intersecting Λ_N, we get a measure, ν_i, on $\mathcal{A}^{\Lambda_N}$. Each of these measures has an associated probability p_i that is given by the probability that the corresponding arrangement of tiles occurs. The measure ν on the configurations on Λ_N is then given by $\nu = \sum_i p_i \nu_i$. By a convexity calculation, we have $H_\nu(\mathcal{P}^{\Lambda_N}) \geq \sum_i p_i H_{\nu_i}(\mathcal{P}^{\Lambda_N})$. Notice that $H_{\nu_i}(\mathcal{P}^{\Lambda_N}) \geq H_{\nu_i}(\mathcal{P}^A)$ where A is the subset of Λ_N consisting of the C_i completely contained in Λ_N. If the tiles completely contained in Λ_N occupy at least a proportion $1-\eta$ of Λ_N, we see using (3.1) that

$$H_{\nu_i}(\mathcal{P}^A) = \sum_{C_i \subset \Lambda_N} H(C_i) \geq (h - \frac{\epsilon}{2})|A| \geq (h-\epsilon)(1-\eta)|\Lambda_N|.$$

Considering the measures ν_i whose corresponding sets A occupy at least a proportion $1-\eta$ of Λ_N, since their probabilities p_i sum to at least $1-\eta$, it follows that $H_\mu(\mathcal{P}^{\Lambda_N}) = H_\nu(\mathcal{P}^{\Lambda_N}) \geq (1-\eta)^2(h-\epsilon)|\Lambda_N|$. Since η is arbitrary, we see that $h_\mu(X) \geq h-\epsilon$ completing the proof.

□

The above theorem provides a constructive proof of the Variational Principle for strongly irreducible shifts of finite type. In [8], an example of a mixing shift of finite type is presented in which this theorem fails badly. A shift of finite type is constructed that is mixing, but has the property that even the weak-mixing measures (a large class containing the Bernoulli measures) have measure-theoretic entropies that are bounded away from the topological entropy.

References

[1] R. Burton and J. Steif. Nonuniqueness of measures of maximal entropy for subshifts of finite type. *Ergodic Theory Dynamical Systems*, 14:213–235, 1994.

[2] R. Burton and J. Steif. Some 2-d symbolic dynamical systems: Entropy and mixing. In M. Pollicott and K. Schmidt, editors, *Ergodic Theory of* $\mathbb{Z}^d$ *actions*, pages 297–305, Cambridge, U.K., 1996. Cambridge Univ. Press.

[3] G. D. Chakerian and S. K. Stein. Some intersection properties of convex bodies. *Proc. Amer. Math. Soc.*, 18:109–112, 1967.

[4] G. Keller. *Equilibrium states in ergodic theory.* Cambridge Univ. Press, Cambridge, U.K., 1998.

[5] S. Lightwood. Morphisms from non-periodic $\mathbb{Z}^2$ Subshifts I: Constructing Embeddings from Homomorphisms. *Ergodic Theory Dynamical Systems* (to appear).

[6] M. Misiurewicz. A short proof of the variational principle for a $\mathbb{Z}_+^N$ action on a compact space. *Astérisque*, 40:147–157, 1976.

[7] D. Ornstein and B. Weiss. The Shannon–McMillan–Breiman theorem for a class of amenable groups. *Israel J. Math.*, 44(1):53–60, 1983.

[8] A. Quas and A. A. Şahin. Local entropy maxima and measure connectedness in subshifts. 2000. preprint.

[9] E. A. Robinson and A. A. Şahin. Mixing properties of nearly maximal entropy measures for $\mathbf{Z}^d$ shifts of finite type, *Colloq. Math.*, 84/85:43–50, 2000.

[10] E. A. Robinson and A. A. Şahin. Modeling ergodic measure-preserving transformations on $\mathbf{Z}^d$ shifts of finite type, *Monatsh. Math.* (to appear).

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