Homographic solutions of the curved 3-body problem

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A R T I C L E I N F O

A B S T R A C T

In the 2-dimensional curved 3-body problem, we prove the existence of Lagrangian and Eulerian homographic orbits, and provide their complete classification in the case of equal masses. We also show that the only non-homothetic hyperbolic Eulerian solutions are the hyperbolic Eulerian relative equilibria, a result that proves their instability.

1. Introduction

We consider the 3-body problem in spaces of constant curvature ($\kappa \neq 0$), which we will call the curved 3-body problem, to distinguish it from its classical Euclidean ($\kappa = 0$) analogue. The potential (defined by the force function U_κ in (4), below) generalizes the Newtonian potential and preserves its basic properties: it is a harmonic function and generates a central field for which bounded orbits are closed.

This research direction started in the 1830s, when Janos Bolyai and Nikolai Lobachevsky proposed a curved 2-body problem, which was broadly studied by several mathematicians, including Ernest Schering [22,23], Wilhem Killing [12–14], and Heinrich Liebmann [17,18]. Many of the results obtained by these authors, as well as new ones, were recently proved with the help of modern methods by José Cariñena, Manuel Rañada, and Mariano Santander [1]. The study of the quantum analogue of the curved 2-body problem was proposed by Erwin Schrödinger [24], Leopold Infeld [10], and Alfred Schild [11]. Other attempts at extending the Newtonian case to spaces of constant curvature, such as the one of Rudolph Lipschitz [19], did not survive, mostly because the proposed potentials lacked the basic properties mentioned above.

The newest results occur in [3], a paper in which we obtained a unified framework that provides the equations of motion of the *curved n-body problem* for any $n \geqslant 2$ and $\kappa \neq 0$. We also proved there

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the existence of several classes of relative equilibria, including the Lagrangian orbits mentioned above. Relative equilibria are orbits for which the configuration of the system remains congruent with itself for all time, i.e. the distances between any two bodies are constant during the motion.

The study of the curved n-body problem, for $n \geqslant 3$, might help us understand the nature of the physical space. Gauss allegedly tried to determine the nature of space by measuring the angles of a triangle formed by the peaks of three mountains. Even if the goal of his topographic measurements was different from what anecdotical history attributes to him (see [20]), this method of deciding the nature of space remains valid for astronomical distances. But since we cannot measure the angles of cosmic triangles, we could alternatively check whether specific (potentially observable) motions of celestial bodies occur in spaces of negative, zero, or positive curvature, respectively.

We showed in [3] that while Lagrangian orbits (rotating equilateral triangles having the bodies at their vertices) of non-equal masses are known to occur for $\kappa = 0$, they must have equal masses for $\kappa \neq 0$. Since Lagrangian solutions of non-equal masses exist in our solar system (for example, the triangle formed by the Sun, Jupiter, and the Trojan asteroids), we can conclude that, if assumed to have constant curvature, the physical space is Euclidean for distances of the order 10^1 AU. The discovery of new orbits of the curved 3-body problem, as defined here in the spirit of an old tradition, might help us extend our understanding of space to larger scales.

So far, the only other existing paper on the curved n-body problem, treated in a unified context, deals with singularities [4], a subject we will not approach here. But relative equilibria can be put in a broader perspective. They are also the object of Saari's conjecture (see [21,5]), which we partially solved for the curved n-body problem [3]. Saari's conjecture has recently generated a lot of interest in classical celestial mechanics (see the references in [5,6]) and is still unsolved for $n > 3$. Moreover, it led to the formulation of Saari's homographic conjecture [21,6], a problem that is directly related to the purpose of this research.

We study here certain solutions that are more general than relative equilibria, namely orbits for which the configuration of the system remains similar with itself. In this class of solutions, the relative distances between particles may change proportionally during the motion, i.e. the size of the system may vary, though its shape remains the same. We will call these solutions *homographic*, in agreement with the classical terminology [25].

In the classical Newtonian case [25], as well as in more general classical contexts [2], the standard concept for understanding homographic solutions is that of central configuration. This notion, however, seems to have no meaningful analogue in spaces of constant curvature because it implies the existence of the first integrals of the centre of mass, which are absent in the curved case. The integrals of the centre of mass and linear momentum seem to be specific only to Euclidean space. Indeed, n-body problems derived by discretizing Einstein's field equations, as obtained by Tullio Levi-Civita [15,16], as well as Albert Einstein, Leopold Infeld, and Banesh Hoffmann [8], also lack such integrals.

We focus here on three types of homographic solutions. The first, which we call Lagrangian, form an equilateral triangle at every time instant. We ask that the plane of this triangle be always orthogonal to the rotation axis. This assumption seems to be natural because, as proved in [3], Lagrangian relative equilibria, which are particular homographic Lagrangian orbits, obey this property, and relative equilibria that lack this property fail to exist. We prove the existence of homographic Lagrangian orbits in Section 3, and provide their complete classification in the case of equal masses in Section 4, for $\kappa > 0$, and Section 5, for $\kappa < 0$. Moreover, we show in Section 6 that Lagrangian solutions with non-equal masses don't exist.

We then study another type of homographic solutions of the curved 3-body problem, which we call Eulerian, in analogy with the classical case that refers to bodies confined to a rotating straight line. At every time instant, the bodies of an Eulerian homographic orbit are on a (possibly) rotating geodesic. In Section 7 we prove the existence of these orbits. Moreover, for equal masses, we provide their complete classification in Section 8, for $\kappa > 0$, and Section 9, for $\kappa < 0$.

Finally, in Section 10, we discuss the existence of hyperbolic homographic solutions, which occur only for negative curvature. We prove that when the bodies are on the same hyperbolically rotating geodesic, a class of solutions we call hyperbolic Eulerian, every orbit is a hyperbolic Eulerian relative equilibrium. Therefore hyperbolic Eulerian relative equilibria are unstable, a fact which makes them unlikely observable candidates in a (hypothetical) hyperbolic physical universe.

2. Equations of motion

We consider the equations of motion on 2-dimensional manifolds of constant curvature, namely spheres embedded in $\mathbb{R}^3$, for $\kappa > 0$, and hyperboloids[1] embedded in the Minkovski space $\mathbb{M}^3$, for $\kappa < 0$.

Consider the masses $m_1, m_2, m_3 > 0$ in $\mathbb{R}^3$, for $\kappa > 0$, and in $\mathbb{M}^3$, for $\kappa < 0$, whose positions are given by the vectors $\mathbf{q}_i = (x_i, y_i, z_i)$, $i = 1, 2, 3$. Let $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$ be the configuration of the system, and $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$, with $\mathbf{p}_i = m_i \dot{\mathbf{q}}_i$, representing the momentum. We define the gradient operator with respect to the vector $\mathbf{q}_i$ as

$$\widetilde{\nabla}_{\mathbf{q}_i} = (\partial_{x_i}, \partial_{y_i}, \sigma \partial_{z_i}),$$

where σ is the signature function,

$$\sigma = \begin{cases} +1, & \text{for } \kappa > 0, \\ -1, & \text{for } \kappa < 0, \end{cases} \tag{1}$$

and let $\widetilde{\nabla}$ denote the operator $(\widetilde{\nabla}_{\mathbf{q}_1}, \widetilde{\nabla}_{\mathbf{q}_2}, \widetilde{\nabla}_{\mathbf{q}_3})$. For the 3-dimensional vectors $\mathbf{a} = (a_x, a_y, a_z)$ and $\mathbf{b} = (b_x, b_y, b_z)$, we define the inner product

$$\mathbf{a} \odot \mathbf{b} := (a_x b_x + a_y b_y + \sigma a_z b_z) \tag{2}$$

and the cross product

$$\mathbf{a} \otimes \mathbf{b} := \left(a_y b_z - a_z b_y, a_z b_x - a_x b_z, \sigma (a_x b_y - a_y b_x)\right). \tag{3}$$

The Hamiltonian function of the system describing the motion of the 3-body problem in spaces of constant curvature is

$$H_\kappa(\mathbf{q}, \mathbf{p}) = T_\kappa(\mathbf{q}, \mathbf{p}) - U_\kappa(\mathbf{q}),$$

where

$$T_\kappa(\mathbf{q}, \mathbf{p}) = \frac{1}{2} \sum_{i=1}^{3} m_i^{-1} (\mathbf{p}_i \odot \mathbf{p}_i)(\kappa \mathbf{q}_i \odot \mathbf{q}_i)$$

defines the kinetic energy and

$$U_\kappa(\mathbf{q}) = \sum_{1 \leqslant i < j \leqslant 3} \frac{m_i m_j |\kappa|^{1/2} \kappa \mathbf{q}_i \odot \mathbf{q}_j}{[\sigma (\kappa \mathbf{q}_i \odot \mathbf{q}_i)(\kappa \mathbf{q}_j \odot \mathbf{q}_j) - \sigma (\kappa \mathbf{q}_i \odot \mathbf{q}_j)^2]^{1/2}} \tag{4}$$

is the force function, $-U_\kappa$ representing the potential energy.[2] Then the Hamiltonian form of the equations of motion is given by the system

$$\begin{cases} \dot{\mathbf{q}}_i = m_i^{-1} \mathbf{p}_i, \\ \dot{\mathbf{p}}_i = \widetilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q}) - m_i^{-1} \kappa (\mathbf{p}_i \odot \mathbf{p}_i) \mathbf{q}_i, \quad i = 1, 2, 3, \ \kappa \neq 0, \end{cases} \tag{5}$$

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[1] The hyperboloid corresponds to Weierstrass's model of hyperbolic geometry (see the appendix in [3]).

[2] In [3] we showed how this expression of U_κ follows from the cotangent potential for $\kappa \neq 0$, and that U_0 is the Newtonian potential of the Euclidean problem, obtained as $\kappa \to 0$.

where the gradient of the force function has the expression

$$\widetilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q}) = \sum_{\substack{j=1 \\ j \neq i}}^{3} \frac{m_i m_j |\kappa|^{3/2} (\kappa \mathbf{q}_j \odot \mathbf{q}_j)[(\kappa \mathbf{q}_i \odot \mathbf{q}_i)\mathbf{q}_j - (\kappa \mathbf{q}_i \odot \mathbf{q}_j)\mathbf{q}_i]}{[\sigma(\kappa \mathbf{q}_i \odot \mathbf{q}_i)(\kappa \mathbf{q}_j \odot \mathbf{q}_j) - \sigma(\kappa \mathbf{q}_i \odot \mathbf{q}_j)^2]^{3/2}}. \tag{6}$$

The motion is confined to the surface of nonzero constant curvature κ, i.e. $(\mathbf{q}, \mathbf{p}) \in \mathbf{T}^*(\mathbf{M}_\kappa^2)^3$, where $\mathbf{T}^*(\mathbf{M}_\kappa^2)^3$ is the cotangent bundle of the configuration space $(\mathbf{M}_\kappa^2)^3$, and

$$\mathbf{M}_\kappa^2 = \left\{ (x, y, z) \in \mathbb{R}^3 \mid \kappa(x^2 + y^2 + \sigma z^2) = 1 \right\}.$$

In particular, $\mathbf{M}_1^2 = \mathbf{S}^2$ is the 2-dimensional sphere, and $\mathbf{M}_{-1}^2 = \mathbf{H}^2$ is the 2-dimensional hyperbolic plane, represented by the upper sheet of the hyperboloid of two sheets (see the appendix of [3] for more details). We will also denote $\mathbf{M}_\kappa^2$ by $\mathbf{S}_\kappa^2$ for $\kappa > 0$, and by $\mathbf{H}_\kappa^2$ for $\kappa < 0$.

Notice that the 3 constraints given by $\kappa \mathbf{q}_i \odot \mathbf{q}_i = 1$, $i = 1, 2, 3$, imply that $\mathbf{q}_i \odot \mathbf{p}_i = 0$, so the 18-dimensional system (5) has 6 constraints. The Hamiltonian function provides the integral of energy,

$$H_\kappa(\mathbf{q}, \mathbf{p}) = h,$$

where h is the energy constant. Eqs. (5) also have the integrals of the angular momentum,

$$\sum_{i=1}^{3} \mathbf{q}_i \otimes \mathbf{p}_i = \mathbf{c}, \tag{7}$$

where $\mathbf{c} = (\alpha, \beta, \gamma)$ is a constant vector. Unlike in the Euclidean case, there are no integrals of the center of mass and linear momentum. Their absence complicates the study of the problem since many of the standard methods don't apply anymore.

Using the fact that $\kappa \mathbf{q}_i \odot \mathbf{q}_i = 1$ for $i = 1, 2, 3$, we can write system (5) as

$$\ddot{\mathbf{q}}_i = \sum_{\substack{j=1 \\ j \neq i}}^{3} \frac{m_j |\kappa|^{3/2} [\mathbf{q}_j - (\kappa \mathbf{q}_i \odot \mathbf{q}_j)\mathbf{q}_i]}{[\sigma - \sigma(\kappa \mathbf{q}_i \odot \mathbf{q}_j)^2]^{3/2}} - (\kappa \dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i)\mathbf{q}_i, \quad i = 1, 2, 3, \tag{8}$$

which is the form of the equations of motion we will use in this paper. The sums in the right-hand side of the above equations represent the gradient of the potential. When $\kappa \to 0$, both the sphere (for $\kappa \to 0$ through positive values) and the hyperboloid (for $\kappa \to 0$ through negative values) become planes at infinity, relative to the centre of the frame. The segments through the origin of the frame whose angle measures the distance between two points on the curved surface become parallel and infinite, so the distance in the limit plane is the Euclidean distance. Consequently the potential tends to the Newtonian potential as $\kappa \to 0$ (see [3] for more details). The terms involving the velocities occur because of the constraints imposed by the curvature. When $\kappa \to 0$, these terms obviously vanish.

3. Local existence and uniqueness of Lagrangian solutions

In this section we define the Lagrangian solutions of the curved 3-body problem, which form a particular class of homographic orbits. Then, for equal masses and suitable initial conditions, we prove their local existence and uniqueness.

Definition 1. A solution of Eqs. (8) is called Lagrangian if, at every time t, the masses form an equilateral triangle that is orthogonal to the z axis.

Though we explained in the Introduction why the orthogonality condition seems to be natural, we were, so far, unable to prove that, if they don't obey it, homographic orbits that change size do not exist. We conjecture, however, that this property is true.

According to Definition 1, the size of a Lagrangian solution can vary, but its shape is always the same. Moreover, all masses have the same coordinate $z(t)$, which may also vary in time, though the triangle is always perpendicular to the z axis.

We can represent a Lagrangian solution of the curved 3-body problem in the form

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3), \quad \text{with } \mathbf{q}_i = (x_i, y_i, z_i), \ i = 1, 2, 3, \tag{9}$$

$$
\begin{aligned}
x_1 &= r\cos\omega, & y_1 &= r\sin\omega, & z_1 &= z, \\
x_2 &= r\cos(\omega + 2\pi/3), & y_2 &= r\sin(\omega + 2\pi/3), & z_2 &= z, \\
x_3 &= r\cos(\omega + 4\pi/3), & y_3 &= r\sin(\omega + 4\pi/3), & z_3 &= z,
\end{aligned}
$$

where $z = z(t)$ satisfies $z^2 = \sigma\kappa^{-1} - \sigma r^2$; σ is the signature function defined in (1); $r := r(t)$ is the *size function*; and $\omega := \omega(t)$ is the *angular function*.

Indeed, for every time t, we have that $x_i^2(t) + y_i^2(t) + \sigma z_i^2(t) = \kappa^{-1}$, $i = 1, 2, 3$, which means that the bodies stay on the surface $\mathbf{M}_\kappa^2$, each body has the same z coordinate, i.e. the plane of the triangle is orthogonal to the z axis, and the angles between any two bodies, seen from the geometric center of the triangle, are always the same, so the triangle remains equilateral. Therefore representation (9) of the Lagrangian orbits agrees with Definition 1.

Definition 2. A Lagrangian solution of Eqs. (8) is called Lagrangian homothetic if the equilateral triangle expands or contracts, but does not rotate around the z axis.

In terms of representation (9), a Lagrangian solution is Lagrangian homothetic if $\omega(t)$ is constant, but $r(t)$ is not constant. Such orbits occur, for instance, when three bodies of equal masses lying initially in the same open hemisphere are released with zero velocities from an equilateral configuration, to end up in a triple collision.

Definition 3. A Lagrangian solution of Eqs. (8) is called a Lagrangian relative equilibrium if the triangle rotates around the z axis without expanding or contracting.

In terms of representation (9), a Lagrangian relative equilibrium occurs when $r(t)$ is constant, but $\omega(t)$ is not constant. Of course, Lagrangian homothetic solutions and Lagrangian relative equilibria, whose existence we proved in [3], are particular Lagrangian orbits, but we expect that the Lagrangian orbits are not reduced to them. We now show this by proving the local existence and uniqueness of Lagrangian solutions that are neither Lagrangian homothetic, nor Lagrangian relative equilibria.

Theorem 1. *In the curved 3-body problem of equal masses, $m_1 = m_2 = m_3 := m > 0$, for every set of initial conditions belonging to a certain class, the local existence and uniqueness of a Lagrangian solution, which both rotates and changes size, is assured.*

Proof. We will check to see if Eqs. (8) admit solutions of the form (9) that start in the region $z > 0$ and for which both $r(t)$ and $\omega(t)$ are not constant. We compute then that

$$\kappa\mathbf{q}_i \odot \mathbf{q}_j = 1 - 3\kappa r^2/2 \quad \text{for } i, j = 1, 2, 3, \text{ with } i \neq j,$$

$$\dot{x}_1 = \dot{r}\cos\omega - r\dot{\omega}\sin\omega, \qquad \dot{y}_1 = \dot{r}\sin\omega + r\dot{\omega}\cos\omega,$$

$$\dot{x}_2 = \dot{r}\cos\left(\omega + \frac{2\pi}{3}\right) - r\dot{\omega}\sin\left(\omega + \frac{2\pi}{3}\right),$$

$$\dot{y}_2 = \dot{r}\sin\left(\omega + \frac{2\pi}{3}\right) + r\dot{\omega}\cos\left(\omega + \frac{2\pi}{3}\right),$$

$$\dot{x}_3 = \dot{r}\cos\left(\omega + \frac{4\pi}{3}\right) - r\dot{\omega}\sin\left(\omega + \frac{4\pi}{3}\right),$$

$$\dot{y}_3 = \dot{r}\sin\left(\omega + \frac{4\pi}{3}\right) + r\dot{\omega}\cos\left(\omega + \frac{4\pi}{3}\right),$$

$$\dot{z}_1 = \dot{z}_2 = \dot{z}_3 = -\sigma r\dot{r}\left(\sigma\kappa^{-1} - \sigma r^2\right)^{-1/2}, \tag{10}$$

$$\kappa\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i = \kappa r^2\dot{\omega}^2 + \frac{\kappa\dot{r}^2}{1 - \kappa r^2} \quad \text{for } i = 1, 2, 3,$$

$$\ddot{x}_1 = \left(\ddot{r} - r\dot{\omega}^2\right)\cos\omega - (r\ddot{\omega} + 2\dot{r}\dot{\omega})\sin\omega,$$

$$\ddot{y}_1 = \left(\ddot{r} - r\dot{\omega}^2\right)\sin\omega + (r\ddot{\omega} + 2\dot{r}\dot{\omega})\cos\omega,$$

$$\ddot{x}_2 = \left(\ddot{r} - r\dot{\omega}^2\right)\cos\left(\omega + \frac{2\pi}{3}\right) - (r\ddot{\omega} + 2\dot{r}\dot{\omega})\sin\left(\omega + \frac{2\pi}{3}\right),$$

$$\ddot{y}_2 = \left(\ddot{r} - r\dot{\omega}^2\right)\sin\left(\omega + \frac{2\pi}{3}\right) + (r\ddot{\omega} + 2\dot{r}\dot{\omega})\cos\left(\omega + \frac{2\pi}{3}\right),$$

$$\ddot{x}_3 = \left(\ddot{r} - r\dot{\omega}^2\right)\cos\left(\omega + \frac{4\pi}{3}\right) - (r\ddot{\omega} + 2\dot{r}\dot{\omega})\sin\left(\omega + \frac{4\pi}{3}\right),$$

$$\ddot{y}_3 = \left(\ddot{r} - r\dot{\omega}^2\right)\sin\left(\omega + \frac{4\pi}{3}\right) + (r\ddot{\omega} + 2\dot{r}\dot{\omega})\cos\left(\omega + \frac{4\pi}{3}\right),$$

$$\ddot{z}_1 = \ddot{z}_2 = \ddot{z}_3 = -\sigma r\ddot{r}\left(\sigma\kappa^{-1} - \sigma r^2\right)^{-1/2} - \kappa^{-1}\dot{r}^2\left(\sigma\kappa^{-1} - \sigma r^2\right)^{-3/2}.$$

Substituting these expressions into system (8), we are led to the system below, where the double-dot terms on the left indicate to which differential equation each algebraic equation corresponds:

$$\ddot{x}_1: \quad A\cos\omega - B\sin\omega = 0,$$

$$\ddot{x}_2: \quad A\cos\left(\omega + \frac{2\pi}{3}\right) - B\sin\left(\omega + \frac{2\pi}{3}\right) = 0,$$

$$\ddot{x}_3: \quad A\cos\left(\omega + \frac{4\pi}{3}\right) - B\sin\left(\omega + \frac{4\pi}{3}\right) = 0,$$

$$\ddot{y}_1: \quad A\sin\omega + B\cos\omega = 0,$$

$$\ddot{y}_2: \quad A\sin\left(\omega + \frac{2\pi}{3}\right) + B\cos\left(\omega + \frac{2\pi}{3}\right) = 0,$$

$$\ddot{y}_3: \quad A\sin\left(\omega + \frac{4\pi}{3}\right) + B\cos\left(\omega + \frac{4\pi}{3}\right) = 0,$$

$$\ddot{z}_1, \ddot{z}_2, \ddot{z}_3: \quad A = 0,$$

where

$$A := A(t) = \ddot{r} - r\left(1 - \kappa r^2\right)\dot{\omega}^2 + \frac{\kappa r\dot{r}^2}{1 - \kappa r^2} + \frac{24m(1 - \kappa r^2)}{r^2(12 - 9\kappa r^2)^{3/2}},$$

$$B := B(t) = r\ddot{\omega} + 2\dot{r}\dot{\omega}.$$

Obviously, the above system has solutions if and only if $A = B = 0$, which means that the local existence and uniqueness of Lagrangian orbits with equal masses is equivalent to the existence of solutions of the system of differential equations

$$\begin{cases} \dot{r} = v, \\ \dot{w} = -\dfrac{2vw}{r}, \\ \dot{v} = r(1 - \kappa r^2)w^2 - \dfrac{\kappa r v^2}{1 - \kappa r^2} - \dfrac{24m(1 - \kappa r^2)}{r^2(12 - 9\kappa r^2)^{3/2}} \end{cases} \tag{11}$$

with initial conditions $r(0) = r_0$, $w(0) = w_0$, $v(0) = v_0$, where $w = \dot{\omega}$. The functions r, ω, and w are analytic, and as long as the initial conditions satisfy the conditions $r_0 > 0$ for all κ, as well as $r_0 < \kappa^{-1/2}$ for $\kappa > 0$, standard results of the theory of differential equations guarantee the local existence and uniqueness of a solution (r, w, v) of Eqs. (11), and therefore the local existence and uniqueness of a Lagrangian orbit with $r(t)$ and $\omega(t)$ not constant. The proof is now complete. $\quad\square$

4. Classification of Lagrangian solutions for $\kappa > 0$

We can now state and prove the following result:

Theorem 2. *In the curved 3-body problem with equal masses, $m_1 = m_2 = m_3 := m$, and $\kappa > 0$ there are five classes of Lagrangian solutions:*

  (i) *Lagrangian homothetic orbits that begin or end in total collision in finite time;*
 (ii) *Lagrangian relative equilibria that move on a circle;*
(iii) *Lagrangian periodic orbits that both rotate and change size;*
(iv) *Lagrangian non-periodic, non-collision orbits that eject at time $-\infty$, with zero velocity, from the equator, reach a maximum distance from the equator, which depends on the initial conditions, and return to the equator, with zero velocity, at time $+\infty$.*

None of the above orbits can cross the equator, defined as the great circle of the sphere orthogonal to the z axis.

 (v) *Lagrangian equilibrium points, when the three equal masses are fixed on the equator at the vertices of an equilateral triangle.*

The rest of this section is dedicated to the proof of this theorem.

Let us start by noticing that the first two equations of system (11) imply that $\dot{w} = -\frac{2\dot{r}w}{r}$, which leads to

$$w = \frac{c}{r^2},$$

where c is a constant. The case $c = 0$ can occur only when $w = 0$, which means $\dot{\omega} = 0$. Under these circumstances the angular velocity is zero, so the motion is homothetic. These are the orbits whose existence is stated in Theorem 2(i). They occur only when the angular momentum is zero, and lead to a triple collision in the future or in the past, depending on the sense of the velocity vectors.

For the rest of this section, we assume that $c \neq 0$. Then system (11) takes the form

$$\begin{cases} \dot{r} = v, \\ \dot{v} = \dfrac{c^2(1 - \kappa r^2)}{r^3} - \dfrac{\kappa r v^2}{1 - \kappa r^2} - \dfrac{24m(1 - \kappa r^2)}{r^2(12 - 9\kappa r^2)^{3/2}}. \end{cases} \tag{12}$$

Notice that the term $\frac{\kappa r v^2}{1-\kappa r^2}$ of the last equation arises from the derivatives $\dot{z}_1$, $\dot{z}_2$, $\dot{z}_3$ in (10). But these derivatives would be zero if the equilateral triangle rotates along the equator, because r is constant in this case, so the term $\frac{\kappa r v^2}{1-\kappa r^2}$ vanishes. Therefore the existence of equilateral relative equilibria on the equator (included in statement (ii) above), and the existence of equilibrium points (stated in (v))—results proved in [3]—are in agreement with the above equations. Nevertheless, the term $\frac{\kappa r v^2}{1-\kappa r^2}$ stops any orbit from crossing the equator, a fact mentioned before statement (v) of Theorem 2.

Understanding system (12) is the key to proving Theorem 2. We start with the following facts:

Lemma 1. *Assume* $\kappa, m > 0$ *and* $c \neq 0$. *Then for* $\kappa^{1/2}c^2 - (8/\sqrt{3})m < 0$, *system* (12) *has two fixed points, while for* $\kappa^{1/2}c^2 - (8/\sqrt{3})m \geqslant 0$ *it has one fixed point.*

Proof. The fixed points of system (12) are given by $\dot{r} = 0 = \dot{v}$. Substituting $v = 0$ in the second equation of (12), we obtain

$$\frac{1-\kappa r^2}{r^2}\left[\frac{c^2}{r} - \frac{24m}{(12-9\kappa r^2)^{3/2}}\right] = 0.$$

The above remarks show that, for $\kappa > 0$, $r = \kappa^{-1/2}$ is a fixed point, which physically represents an equilateral relative equilibrium moving along the equator. Other potential fixed points of system (12) are given by the equation

$$c^2(12 - 9\kappa r^2)^{3/2} = 24mr,$$

whose solutions are the roots of the polynomial

$$729c^4\kappa^3 r^6 - 2916c^4\kappa^2 r^4 + 144(27c^4\kappa + 4m^2)r^2 - 1728. \tag{13}$$

Writing $x = r^2$ and assuming $\kappa > 0$, this polynomial takes the form

$$p(x) = 729\kappa^3 x^3 - 2916c^4\kappa^2 x^2 + 144(27c^4\kappa + 4m^2)x - 1728, \tag{14}$$

and its derivative is given by

$$p'(x) = 2187c^4\kappa^3 x^2 - 5832c^4\kappa^2 x + 144(27c^4\kappa + 4m^2). \tag{15}$$

The discriminant of p' is $-5038848c^4\kappa^3 m^2 < 0$.

By Descartes's rule of signs, p can have one or three positive roots. If p has three positive roots, then p' must have two positive roots, but this is not possible because its discriminant is negative. Consequently p has exactly one positive root.

For the point $(r, v) = (r_0, 0)$ to be a fixed point of Eqs. (12), r_0 must satisfy the inequalities $0 < r_0 \leqslant \kappa^{-1/2}$. If we denote

$$g(r) = \frac{c^2}{r} - \frac{24m}{(12-9\kappa r^2)^{3/2}}, \tag{16}$$

we see that, for $\kappa > 0$, g is a decreasing function since

$$\frac{d}{dr}g(r) = -\frac{c^2}{r^2} - \frac{648m\kappa r}{(12-9\kappa r^2)^{5/2}} < 0. \tag{17}$$

When $r \to 0$, we obviously have that $g(r) > 0$ since we assumed $c \neq 0$. When $r \to \kappa^{-1/2}$, we have $g(r) \to \kappa^{1/2}c^2 - (8/\sqrt{3})m$. If $\kappa^{1/2}c^2 - (8/\sqrt{3})m > 0$, then $r_0 > \kappa^{-1/2}$, so $(r_0, 0)$ is not a fixed point. Therefore, assuming $c \neq 0$, a necessary condition that $(r_0, 0)$ is a fixed point of system (12) with $0 < r_0 < \kappa^{-1/2}$ is that

$$\kappa^{1/2}c^2 - (8/\sqrt{3})m < 0.$$

For $\kappa^{1/2}c^2 - (8/\sqrt{3})m \geqslant 0$, the only fixed point of system (12) is $(r, v) = (\kappa^{-1/2}, 0)$. This conclusion completes the proof of the lemma. $\square$

4.1. The flow in the (r, v) plane for $\kappa > 0$

We will now study the flow of system (12) in the (r, v) plane for $\kappa > 0$. At every point with $v \neq 0$, the slope of the vector field is given by $\frac{dv}{dr}$, i.e. by the ratio $\frac{\dot{v}}{\dot{r}} = h(r, v)$, where

$$h(r, v) = \frac{c^2(1 - \kappa r^2)}{vr^3} - \frac{\kappa r v}{1 - \kappa r^2} - \frac{24m(1 - \kappa r^2)}{vr^2(12 - 9\kappa r^2)^{3/2}}.$$

Since $h(r, -v) = -h(r, v)$, the flow of system (12) is symmetric with respect to the r axis for $r \in (0, \kappa^{-1/2}]$. Also notice that, except for the fixed point $(\kappa^{-1/2}, 0)$, system (12) is undefined on the lines $r = 0$ and $r = \kappa^{-1/2}$. Therefore the flow of system (12) exists only for points (r, v) in the band $(0, \kappa^{-1/2}) \times \mathbb{R}$ and for the point $(\kappa^{-1/2}, 0)$.

Since $\dot{r} = v$, no interval on the r axis can be an invariant set for system (12). Then the symmetry of the flow relative to the r axis implies that orbits cross the r axis perpendicularly. But since $g(r) \neq 0$ at every non-fixed point, the flow crosses the r axis perpendicularly everywhere, except at the fixed points.

Let us further treat the case of one fixed point and the case of two fixed points separately.

4.1.1. The case of one fixed point

A single fixed point, namely $(\kappa^{-1/2}, 0)$, appears when $\kappa^{1/2}c^2 - (8/\sqrt{3})m \geqslant 0$. Then the function g, which is decreasing, has no zeroes for $r \in (0, \kappa^{-1/2})$, therefore $g(r) > 0$ in this interval, so the flow always crosses the r axis upwards.

For $v \neq 0$, the right-hand side of the second equation of (12) can be written as

$$G(r, v) = g_1(r)g(r) + g_2(r, v), \tag{18}$$

where

$$g_1(r) = \frac{1 - \kappa r^2}{r^2} \quad \text{and} \quad g_2(r, v) = -\frac{\kappa r v^2}{1 - \kappa r^2}. \tag{19}$$

But $\frac{d}{dr}g_1(r) = -2/r^3 < 0$ and $\frac{\partial}{\partial r}g_2(r, v) = -\frac{\kappa v^2(1 + \kappa r^2)}{(1 - \kappa r^2)^2} < 0$. So, like g, the functions g_1 and g_2 are decreasing in $(0, \kappa^{-1/2})$, with $g_1, g > 0$, therefore G is a decreasing function as well. Consequently, for $v = \text{constant} > 0$, the slope of the vector field decreases from $+\infty$ at $r = 0$ to $-\infty$ at $r = \kappa^{-1/2}$. For $v = \text{constant} < 0$, the slope of the vector field increases from $-\infty$ at $r = 0$ to $+\infty$ at $r = \kappa^{-1/2}$.

This behavior of the vector field forces every orbit to eject downwards from the fixed point, at time $t = -\infty$ and with zero velocity, on a trajectory tangent to the line $r = \kappa^{-1/2}$, reach slope zero at some moment in time, then cross the r axis perpendicularly upwards and symmetrically return with final zero velocity, at time $t = +\infty$, to the fixed point (see Fig. 1(a)). So the flow of system (12) consists in this case solely of homoclinic orbits to the fixed point $(\kappa^{-1/2}, 0)$, orbits whose existence is claimed in Theorem 2(iv). Some of these trajectories may come very close to a total collapse, which they will never reach because only solutions with zero angular momentum (like the homothetic orbits) encounter total collisions, as proved in [4].
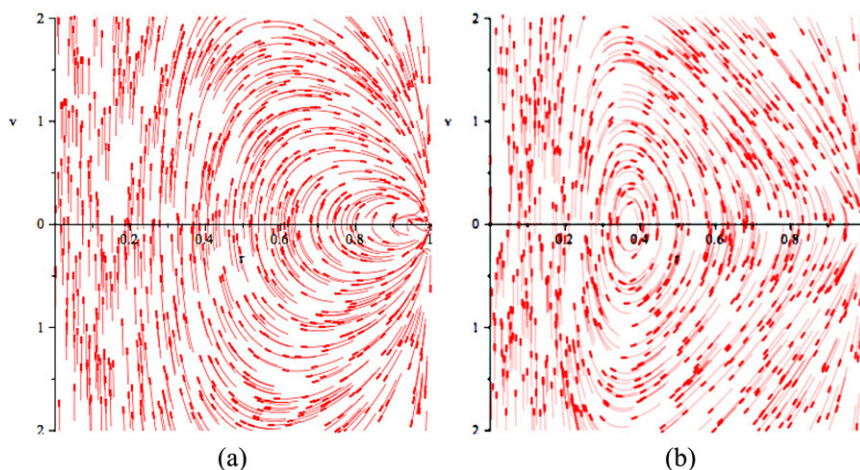
Fig. 1. A sketch of the flow of system (12) for (a) $\kappa = c = 1$, $m = 0.24$, typical for one fixed point, and (b) $\kappa = c = 1$, $m = 4$, typical for two fixed points.

So the orbits cannot reach any singularity of the line $r = 0$, and neither can they begin or end in a singularity of the line $r = \kappa^{-1/2}$. The reason for the latter is that such points are of the form $(\kappa^{-1/2}, \nu)$ with $\nu \neq 0$, therefore $\dot{r} \neq 0$ at such points. But the vector field tends to infinity when approaching the line $r = \kappa^{-1/2}$, so the flow must be tangent to it, consequently $\dot{r}$ must tend to zero, which is a contradiction. Therefore only homoclinic orbits exist in this case.

4.1.2. The case of two fixed points

Two fixed points, $(\kappa^{-1/2}, 0)$ and $(r_0, 0)$, with $0 < r_0 < \kappa^{-1/2}$, occur when $\kappa^{1/2}c^2 - (8/\sqrt{3})m < 0$. Since g is decreasing in the interval $(0, \kappa^{-1/2})$, we can conclude that $g(t) > 0$ for $t \in (0, r_0)$ and $g(t) < 0$ for $t \in (r_0, \kappa^{-1/2})$. Therefore the flow of system (12) crosses the r axis upwards when $r < r_0$, but downwards for $r > r_0$ (see Fig. 1(b)).

The function $G(r, \nu)$, defined in (18), fails to be decreasing in the interval $(0, \kappa^{-1/2})$ along lines of constant ν, but it has no singularities in this interval and still maintains the properties

$$\lim_{r \to 0^+} G(r, \nu) = +\infty \quad \text{and} \quad \lim_{r \to (\kappa^{-1/2})^-} G(r, \nu) = -\infty.$$

Therefore G must vanish at some point, so due to the symmetry of the vector field with respect to the r axis, the fixed point $(r_0, 0)$ is surrounded by periodic orbits. The points where G vanishes are given by the nullcline $\dot{\nu} = 0$, which has the expression

$$\nu^2 = \frac{(1 - \kappa r^2)^2}{\kappa r^3} \left[\frac{c^2}{r} - \frac{24m}{(12 - 9\kappa r^2)^{3/2}} \right].$$

This nullcline is a disconnected set, formed by the fixed point $(\kappa^{-1/2}, 0)$ and a continuous curve, symmetric with respect to the r axis. Indeed, since the equation of the nullcline can be written as $\nu^2 = \frac{(1 - \kappa r^2)^2}{\kappa r^3} g(r)$, and $\lim_{r \to (\kappa^{-1/2})^-} g(r) = \kappa^{1/2}c^2 - (8/\sqrt{3})m < 0$ in the case of two fixed points (as shown in the proof of Lemma 1), only the point $(\kappa^{-1/2}, 0)$ satisfies the nullcline equation away from the fixed point $(r_0, 0)$.

The asymptotic behavior of G near $r = \kappa^{-1/2}$ also forces the flow to produce homoclinic orbits for the fixed point $(\kappa^{-1/2}, 0)$, as in the case discussed in Section 4.1.1. The existence of these two kinds of solutions is stated in Theorem 2(iii) and (iv), respectively. The fact that orbits cannot begin

or end at any of the singularities of the lines $r = 0$ or $r = \kappa^{-1/2}$ follows as in Section 4.1.1. This remark completes the proof of Theorem 2.

5. Classification of Lagrangian solutions for $\kappa < 0$

We can now state and prove the following result:

Theorem 3. *In the curved* 3-*body problem with equal masses, $m_1 = m_2 = m_3 := m$, and $\kappa < 0$ there are eight classes of Lagrangian solutions:*

   (i) *Lagrangian homothetic orbits that begin or end in total collision in finite time;*
  (ii) *Lagrangian relative equilibria, for which the bodies move on a circle parallel with the xy plane;*
 (iii) *Lagrangian periodic orbits that change size;*
 (iv) *Lagrangian orbits that eject at time $-\infty$ from a certain relative equilibrium solution $\mathbf{s}$ (whose existence and position depend on the values of the parameters) and return to it at time $+\infty$;*
  (v) *Lagrangian orbits that come from infinity at time $-\infty$ and reach the relative equilibrium $\mathbf{s}$ at time $+\infty$;*
 (vi) *Lagrangian orbits that eject from the relative equilibrium $\mathbf{s}$ at time $-\infty$ and reach infinity at time $+\infty$;*
(vii) *Lagrangian orbits that come from infinity at time $-\infty$ and symmetrically return to infinity at time $+\infty$, never able to reach the Lagrangian relative equilibrium $\mathbf{s}$;*
(viii) *Lagrangian orbits that come from infinity at time $-\infty$, reach a position close to a total collision, and symmetrically return to infinity at time $+\infty$. The minimum size of this orbit is, for given κ, m, and c, smaller than the corresponding orbit described in* (ii).

The rest of this section is dedicated to the proof of this theorem. Notice first that the orbits described in Theorem 3(i) occur for zero angular momentum, when $c = 0$, as for instance when the three equal masses are released with zero velocities from the Lagrangian configuration, a case in which a total collapse takes place at the point $(0, 0, |\kappa|^{-1/2})$. Depending on the initial conditions, the motion can be bounded or unbounded. The existence of the orbits described in Theorem 3(ii) was proved in [3]. To address the other points of Theorem 3, and show that no other orbits than the ones stated there exist, we need to study the flow of system (12) for $\kappa < 0$. Let us first prove the following fact:

Lemma 2. *Assume $\kappa < 0$, $m > 0$, and $c \neq 0$. Then system* (12) *has no fixed points when $27c^4\kappa + 4m^2 \leqslant 0$, and can have two, one, or no fixed points when $27c^4\kappa + 4m^2 > 0$.*

Proof. The number of fixed points of system (12) is the same as the number of positive zeroes of the polynomial p defined in (14). If $27c^4\kappa + 4m^2 \leqslant 0$, all coefficients of p are negative, so by Descartes's rule of signs, p has no positive roots.

Now assume that $27c^4\kappa + 4m^2 > 0$. Then the zeroes of p are the same as the zeroes of the monic polynomial (i.e. with leading coefficient 1):

$$\bar{p}(x) = x^3 - 4\kappa^{-1}x^2 + \left[48\kappa^{-2} + (64/81)c^{-4}\kappa^{-3}m^2\right]x - (64/27)\kappa^{-3},$$

obtained when dividing p by the leading coefficient. But a monic cubic polynomial can be written as

$$x^3 - (a_1 + a_2 + a_3)x^2 + (a_1a_2 + a_2a_3 + a_3a_1)x - a_1a_2a_3,$$

where a_1, a_2, and a_3 are its roots. One of these roots is always real and has the opposite sign of $-a_1a_2a_3$. Since the free term of $\bar{p}$ is positive, one of its roots is always negative, independently of the allowed values of the coefficients κ, m, c. Consequently p can have two positive roots (including the possibility of a double positive root) or no positive root at all. Therefore system (12) can have two, one, or no fixed points. As we will see later, all three cases occur. $\square$

We further state and prove a property, which we will use to establish Lemma 4:

Lemma 3. *Assume $\kappa < 0, m > 0, c \neq 0$, let $(r_*, 0)$ be a fixed point of system (12), and consider the function g defined in (16). Then $\frac{d}{dr} g(r_*) = 0$ if and only if $r_* = (-\frac{2}{3\kappa})^{1/2}$. Moreover, $\frac{d^2}{dr^2} g(r_*) > 0$.*

Proof. Since $(r_*, 0)$ is a fixed point of system (12), it follows that $g(r_*) = 0$. Then it follows from relation (16) that $(12 - 9\kappa r_*^2)^{3/2} = 24mr_*/c^2$. Substituting this value of $(12 - 9\kappa r_*^2)^{3/2}$ into the equation $\frac{d}{dr} g(r_*) = 0$, which is equivalent to

$$\frac{648 m\kappa r_*}{(12 - 9\kappa r_*^2)^{5/2}} = -\frac{c^2}{r_*^2},$$

it follows that $27\kappa/(12 - 9\kappa r_*^2) = -1/r_*^2$. Therefore $r_* = (-\frac{2}{3\kappa})^{1/2}$. Obviously, for this value of r_*, $g(r_*) = 0$, so the first part of Lemma 3 is proved. To prove the second part, substitute $r_* = (-\frac{2}{3\kappa})^{1/2}$ into the equation $g(r_*) = 0$, which is then equivalent with the relation

$$9\sqrt{3}c^2(-\kappa)^{1/2} - 4m = 0. \tag{20}$$

Notice that

$$\frac{d^2}{dr^2} g(r) = \frac{2c^2}{r^3} - \frac{648 m\kappa}{(12 - 9\kappa r^2)^{5/2}} - \frac{29160 m\kappa^2 r^2}{(12 - 9\kappa r^2)^{7/2}}.$$

Substituting for $r_* = (-\frac{2}{3\kappa})^{1/2}$ in the above equation, and using (20), we are led to the conclusion that $\frac{d^2}{dr^2} g(r_*) = -(2\sqrt{3} + 6\sqrt{2})m\kappa/9\sqrt{6}$, which is positive for $\kappa < 0$. This completes the proof.   □

The following result is important for understanding a qualitative aspect of the flow of system (12), which we will discuss later in this section.

Lemma 4. *Assume $\kappa < 0, m > 0, c \neq 0$, and let $(r_*, 0)$ be a fixed point of system (12). If $\frac{\partial}{\partial r} G(r_*, 0) = 0$, then $\frac{\partial^2}{\partial r^2} G(r_*, 0) > 0$, where G is defined in (18).*

Proof. Since $(r_*, 0)$ is a fixed point of (12), $G(r_*, 0) = 0$. But for $\kappa < 0$, we have $g_1(r_*) > 0$, so necessarily $g(r_*) = 0$. Moreover, $\frac{d}{dr} g_1(r_*) \neq 0$, and since $\frac{\partial}{\partial r} g_2(r, \nu) = -\frac{\kappa\nu^2(1+\kappa r^2)}{(1-\kappa r^2)^2}$, it follows that $\frac{\partial}{\partial r} g_2(r_*, 0) = 0$. But

$$\frac{\partial G}{\partial r}(r, \nu) = \frac{d}{dr} g_1(r) \cdot g(r) + g_1(r)\frac{d}{dr} g(r) + \frac{\partial}{\partial r} g_2(r, \nu),$$

so the condition $\frac{\partial}{\partial r} G(r_*, 0) = 0$ implies that $\frac{d}{dr} g(r_*) = 0$. By Lemma 3, $r_* = (-\frac{2}{3\kappa})^{1/2}$ and $\frac{d^2}{dr^2} g(r_*) > 0$. Using now the fact that

$$\frac{\partial^2 G}{\partial r^2}(r, \nu) = \frac{d^2}{dr^2} g_1(r)g(r) + 2\frac{d}{dr} g_1(r)\frac{d}{dr} g(r) + g_1(r)\frac{d^2}{dr^2} g(r) + \frac{\partial^2}{\partial r^2} g_2(r, \nu),$$

it follows that $\frac{\partial^2}{\partial r^2} G(r_*, 0) = g_1(r_*)\frac{d^2}{dr^2} g(r_*)$. Since Lemma 3 implies that $\frac{d^2}{dr^2} g(r_*) > 0$, and we know that $g_1(r_*) > 0$, it follows that $\frac{\partial^2}{\partial r^2} G(r_*, 0) > 0$, a conclusion that completes the proof.   □
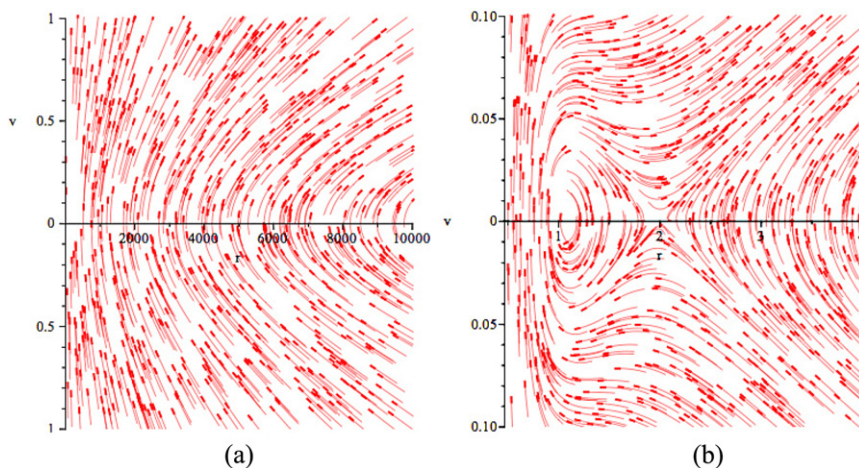
Fig. 2. A sketch of the flow of system (12) for (a) $\kappa = -2$, $c = 1/3$, and $m = 1/2$, typical for no fixed points; (b) $\kappa = -0.3$, $c = 0.23$, and $m = 0.12$, typical for two fixed points, which are in this case on the line $\nu = 0$ at approximately $r_1 = 1.0882233$ and $r_2 = 2.0007055$.

5.1. The flow in the (r, ν) plane for $\kappa < 0$

We will now study the flow of system (12) in the (r, ν) plane for $\kappa < 0$. As in the case $\kappa > 0$, and for the same reason, the flow is symmetric with respect to the r axis, which it crosses perpendicularly at every non-fixed point with $r > 0$. Since we can have two, one, or no fixed points, we will treat each case separately.

5.1.1. The case of no fixed points

No fixed points occur when $g(r)$ has no zeroes. Since $g(r) \to \infty$ as $r \to 0$ with $r > 0$, it follows that $g(r) > 0$. Since $g_1(r)$ and $g_2(r, \nu)$ are also positive, it follows that $G(r, \nu) > 0$ for $r > 0$. But $h(r, \nu) = G(r, \nu)/\nu$. Then $h(r, \nu) > 0$ for $\nu > 0$ and $h(r, \nu) < 0$ for $\nu < 0$, so the flow comes from infinity at time $-\infty$, crosses the r axis perpendicularly upwards, and symmetrically reaches infinity at time $+\infty$ (see Fig. 2(a)). These are orbits as in the statements of Theorem 3(vii) and (viii) but without any reference to the Lagrangian relative equilibrium **s**.

5.1.2. The case of two fixed points

In this case, the function g defined at (16) has two distinct zeroes, one for $r = r_1$ and the other for $r = r_2$, with $0 < r_1 < r_2$. In Theorem 3, we denoted the fixed point $(r_2, 0)$ by **s**. Moreover, $g(r) > 0$ for $r \in (0, r_1) \cup (r_2, \infty)$, and $g(r) < 0$ for $r \in (r_1, r_2)$. Therefore the vector field crosses the r axis downwards between r_1 and r_2, but upwards for $r < r_1$ as well as for $r > r_2$.

To determine the behavior of the flow near the fixed point $(r_1, 0)$, we linearize system (12). For this let $F(r, \nu) = \nu$ to be the right-hand side of the first equation in (12), and notice that $\frac{\partial F}{\partial r}(r_1, 0) = 1$, $\frac{\partial F}{\partial \nu}(r_1, 0) = 0$, and $\frac{\partial G}{\partial \nu}(r_1, 0) = 0$. Since, along the r axis, $G(r, 0)$ is positive for $r < r_1$, but negative for $r_1 < r < r_2$, it follows that either $\frac{\partial G}{\partial r}(r_1, 0) < 0$ or $\frac{\partial G}{\partial r}(r_1, 0) = 0$. But according to Lemma 4, if $\frac{\partial G}{\partial r}(r_1, 0) = 0$, then $\frac{\partial^2}{\partial r^2} G(r_*, 0) > 0$, so $G(r, 0)$ is convex up at $(r_1, 0)$. Then $G(r, 0)$ cannot not change sign when r passes through r_1 along the line $\nu = 0$, so the only existing possibility is $\frac{\partial G}{\partial r}(r_1, 0) < 0$.

The eigenvalues of the linearized system corresponding to the fixed point $(r_1, 0)$ are then given by the equation

$$\det \begin{bmatrix} -\lambda & 1 \\ \frac{\partial G}{\partial r}(r_1, 0) & -\lambda \end{bmatrix} = 0. \tag{21}$$

Since $\frac{\partial G}{\partial r}(r_1, 0)$ is negative, the eigenvalues are purely imaginary, so $(r_1, 0)$ is not a hyperbolic fixed point for Eqs. (12). Therefore this fixed point could be a spiral sink, a spiral source, or a center for the nonlinear system. But the symmetry of the flow of system (12) with respect to the r axis, and the fact that, near r_1, the flow crosses the r axis upwards to the left of r_1, and downwards to the right of r_1, eliminates the possibility of spiral behavior, so $(r_1, 0)$ is a center (see Fig. 2(b)).

We can understand the generic behavior of the flow near the isolated fixed point $(r_2, 0)$ through linearization as well. For this purpose, notice that $\frac{\partial F}{\partial r}(r_2, 0) = 1$, $\frac{\partial F}{\partial v}(r_2, 0) = 0$, and $\frac{\partial G}{\partial v}(r_2, 0) = 0$. Since, along the r axis, $G(r)$ is negative for $r_1 < r < r_2$, but positive for $r > r_2$, it follows that $\frac{\partial G}{\partial r}(r_2, 0) > 0$ or $\frac{\partial G}{\partial r}(r_2, 0) = 0$. But using Lemma 4 the same way we did above for the fixed point $(r_1, 0)$, we can conclude that the only possibility is $\frac{\partial G}{\partial r}(r_2, 0) > 0$.

The eigenvalues corresponding to the fixed point $(r_2, 0)$ are given by the equation

$$\det \begin{bmatrix} -\lambda & 1 \\ \frac{\partial G}{\partial r}(r_2, 0) & -\lambda \end{bmatrix} = 0. \tag{22}$$

Consequently the fixed point $(r_2, 0)$ is hyperbolic, its two eigenvalues are $\lambda_1 > 0$ and $\lambda_2 < 0$, so $(r_2, 0)$ is a saddle.

Indeed, for small $v > 0$, the slope of the vector field decreases to $-\infty$ on lines $r = $ constant, with $r_1 < r < r_2$, when v tends to 0. On the same lines, with $r > r_2$, the slope decreases from $+\infty$ as v increases. This behavior gives us an approximate idea of how the eigenvectors corresponding to the eigenvalues λ_1 and λ_2 are positioned in the rv plane.

On lines of the form $v = \eta r$, with $\eta > 0$, the slope $h(r, v)$ of the vector field becomes

$$h(r, \eta r) = \frac{1 - \kappa r^2}{\eta r^3} \left[\frac{c^2}{r} - \frac{24m(1 - \kappa r^2)}{(12 - 9\kappa r^2)^{3/2}} \right] - \frac{\kappa \eta r^2}{1 - \kappa r^2}.$$

So, as r tends to ∞, the slope $h(r, \eta r)$ tends to η. Consequently the vector field doesn't bound the flow with negative slopes, and thus allows it to go to infinity.

With the fixed point $(r_1, 0)$ as a center, the fixed point $(r_2, 0)$ as a saddle, and a vector field that doesn't bound the orbits as $r \to \infty$, the flow must behave qualitatively as in Fig. 2(b).

This behavior of the flow proves the existence of the following types of solutions:

(a) periodic orbits around the fixed point $(r_1, 0)$, corresponding to Theorem 3(iii);

(b) a homoclinic orbit to the fixed point $(r_1, 0)$, corresponding to Theorem 3(iv);

(c) an orbit that tends to the fixed point $(r_2, 0)$, corresponding to Theorem 3(v);

(d) an orbit that is ejected from the fixed point $(r_2, 0)$, corresponding to Theorem 3(vi);

(e) orbits that come from infinity in the direction of the stable manifold of $(r_2, 0)$ and inside it, hit the r axis to the right of r_2, and return symmetrically to infinity in the direction of the unstable manifold of $(r_2, 0)$; these orbits correspond to Theorem 3(vii);

(f) orbits that come from infinity in the direction of the stable manifold of $(r_2, 0)$ and outside it, turn around the homoclinic loop, and return symmetrically to infinity in the direction of the unstable manifold of $(r_2, 0)$; since such an orbit hits the r axis at a point between 0 and r_1, its minimum size is, for given κ, m, and c, smaller than the corresponding orbit described in Theorem 3(ii); these solutions correspond to Theorem 3(viii).

Since no other orbits show up, the proof of this case is complete.

5.1.3. The case of one fixed point

We left the case of one fixed point at the end because it is non-generic. It occurs when the two fixed points of the previous case overlap. Let us denote this fixed point by $(r_0, 0)$. Then the function $g(r)$ is positive everywhere except at the fixed point, where it is zero. So near r_0, g is decreasing for $r < r_0$ and increasing for $r > r_0$, and the r axis is tangent to the graph of g. Consequently, $\frac{\partial G}{\partial r}(r_0, 0) = 0$, and the eigenvalues obtained from Eq. (21) are $\lambda_1 = \lambda_2 = 0$. In this degenerate case, the orbits near the fixed point influence the asymptotic behavior of the flow at $(r_0, 0)$. Since the flow away from the fixed point looks very much like in the case of no fixed points, the only difference

between the flow sketched in Fig. 2(a) and the current case is that at least an orbit ends at $(r_0, 0)$, and at least another orbit one ejects from it. These orbits are described in Theorem 3(iv) and (v).

The proof of Theorem 3 is now complete.

6. Mass equality of Lagrangian solutions

In this section we show that all Lagrangian solutions that satisfy Definition 1 must have equal masses. In other words, we will prove the following result:

Theorem 4. *In the curved 3-body problem, the bodies of masses m_1, m_2, m_3 can lead to a Lagrangian solution if and only if $m_1 = m_2 = m_3$.*

Proof. The fact that three bodies of equal masses can lead to Lagrangian solutions for suitable initial conditions was proved in Theorem 1. So we will further prove that Lagrangian solutions can occur only if the masses are equal. Since the case of relative equilibria was settled in [3], we need to consider only the Lagrangian orbits that are not relative equilibria. This means we can treat only the case when $r(t)$ is not constant. Recall also that the Lagrangian orbits were defined to form a triangle orthogonal to the z axis.

Assume now that the masses are m_1, m_2, m_3, and substitute a solution of the form

$$x_1 = r\cos\omega, \qquad y_1 = r\sin\omega, \qquad z_1 = (\sigma\kappa^{-1} - \sigma r^2)^{1/2},$$
$$x_2 = r\cos(\omega + 2\pi/3), \qquad y_2 = r\sin(\omega + 2\pi/3), \qquad z_2 = (\sigma\kappa^{-1} - \sigma r^2)^{1/2},$$
$$x_3 = r\cos(\omega + 4\pi/3), \qquad y_3 = r\sin(\omega + 4\pi/3), \qquad z_3 = (\sigma\kappa^{-1} - \sigma r^2)^{1/2}$$

into the equations of motion. Computations and a reasoning similar to the ones performed in the proof of Theorem 1 lead us to the system:

$$\ddot{r} - r(1 - \kappa r^2)\dot{\omega}^2 + \frac{\kappa r\dot{r}^2}{1 - \kappa r^2} + \frac{12(m_1 + m_2)(1 - \kappa r^2)}{r^2(12 - 9\kappa r^2)^{3/2}} = 0,$$

$$\ddot{r} - r(1 - \kappa r^2)\dot{\omega}^2 + \frac{\kappa r\dot{r}^2}{1 - \kappa r^2} + \frac{12(m_2 + m_3)(1 - \kappa r^2)}{r^2(12 - 9\kappa r^2)^{3/2}} = 0,$$

$$\ddot{r} - r(1 - \kappa r^2)\dot{\omega}^2 + \frac{\kappa r\dot{r}^2}{1 - \kappa r^2} + \frac{12(m_3 + m_1)(1 - \kappa r^2)}{r^2(12 - 9\kappa r^2)^{3/2}} = 0,$$

$$r\ddot{\omega} + 2\dot{r}\dot{\omega} - \frac{4\sqrt{3}(m_1 - m_2)}{r^2(12 - 9\kappa r^2)^{3/2}} = 0,$$

$$r\ddot{\omega} + 2\dot{r}\dot{\omega} - \frac{4\sqrt{3}(m_2 - m_3)}{r^2(12 - 9\kappa r^2)^{3/2}} = 0,$$

$$r\ddot{\omega} + 2\dot{r}\dot{\omega} - \frac{4\sqrt{3}(m_3 - m_1)}{r^2(12 - 9\kappa r^2)^{3/2}} = 0,$$

which, obviously, can have solutions only if $m_1 = m_2 = m_3$. This conclusion completes the proof. □

7. Local existence and uniqueness of Eulerian solutions

In this section we define the Eulerian solutions of the curved 3-body problem and prove their local existence for suitable initial conditions in the case of equal masses.

Definition 4. A solution of Eqs. (8) is called Eulerian if, at every time instant, the bodies are on a geodesic that contains the point $(0, 0, |\kappa|^{-1/2})$.

Unlike the orthogonality condition imposed in Definition 1, the requirement of Definition 4 that the geodesic contains a certain point is not restrictive, but rather a normalization, which provides a more convenient setting.

According to Definition 4, the size of an Eulerian solution may change, but the particles are always on a (possibly rotating) geodesic. If the masses are equal, it is natural to assume that one body lies at the point $(0, 0, |\kappa|^{-1/2})$, while the other two bodies find themselves at diametrically opposed points of a circle. Thus, in the case of equal masses, which we further consider, we ask that the moving bodies have the same coordinate z, which may vary in time.

We can thus represent such an Eulerian solution of the curved 3-body problem in the form

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3), \quad \text{with } \mathbf{q}_i = (x_i, y_i, z_i), \ i = 1, 2, 3, \tag{23}$$

$$
\begin{aligned}
x_1 &= 0, & y_1 &= 0, & z_1 &= (\sigma\kappa)^{-1/2}, \\
x_2 &= r\cos\omega, & y_2 &= r\sin\omega, & z_2 &= z, \\
x_3 &= -r\cos\omega, & y_3 &= -r\sin\omega, & z_3 &= z,
\end{aligned}
$$

where $z = z(t)$ satisfies $z^2 = \sigma\kappa^{-1} - \sigma r^2 = (\sigma\kappa)^{-1}(1 - \kappa r^2)$; σ is the signature function defined in (1); $r := r(t)$ is the *size function*; and $\omega := \omega(t)$ is the *angular function*.

Notice that, for every time t, we have $x_i^2(t) + y_i^2(t) + \sigma z_i^2(t) = \kappa^{-1}$, $i = 1, 2, 3$, which means that the bodies stay on the surface $\mathbf{M}_\kappa^2$. Eqs. (23) also express the fact that the bodies are on the same (possibly rotating) geodesic. Therefore representation (23) of the Eulerian orbits agrees with Definition 4 in the case of equal masses.

Definition 5. An Eulerian solution of Eqs. (8) is called Eulerian homothetic if the configuration expands or contracts, but does not rotate.

In terms of representation (23), an Eulerian homothetic orbit for equal masses occurs when $\omega(t)$ is constant, but $r(t)$ is not constant. If, for instance, all three bodies are initially in the same open hemisphere, while the two moving bodies have the same mass and the same z coordinate, and are released with zero initial velocities, then we are led to an Eulerian homothetic orbit that ends in a triple collision.

Definition 6. An Eulerian solution of Eqs. (8) is called an Eulerian relative equilibrium if the configuration of the system rotates without expanding or contracting.

In terms of representation (23), an Eulerian relative equilibrium orbit occurs when $r(t)$ is constant, but $\omega(t)$ is not constant. Of course, Eulerian homothetic solutions and elliptic Eulerian relative equilibria, whose existence we proved in [3], are particular Eulerian orbits, but we expect that the Eulerian orbits are not reduced to them. We now show this fact by proving the local existence and uniqueness of Eulerian solutions that are neither Eulerian homothetic, nor Eulerian relative equilibria.

Theorem 5. *In the curved 3-body problem of equal masses, $m_1 = m_2 = m_3 := m > 0$, for every set of initial conditions belonging to a certain class, the local existence and uniqueness of an Eulerian solution, which both rotates and changes size, is assured.*

Proof. To check whether Eqs. (8) admit solutions of the form (23) that start in the region $z > 0$ and for which both $r(t)$ and $\omega(t)$ are not constant, we first compute that

$$\kappa \mathbf{q}_1 \odot \mathbf{q}_2 = \kappa \mathbf{q}_1 \odot \mathbf{q}_3 = \left(1 - \kappa r^2\right)^{1/2},$$

$$\kappa \mathbf{q}_2 \odot \mathbf{q}_3 = 1 - 2\kappa r^2,$$

$$\dot{x}_1 = 0, \qquad\qquad\qquad \dot{y}_1 = 0,$$

$$\dot{x}_2 = \dot{r}\cos\omega - r\dot{\omega}\sin\omega, \qquad \dot{y}_2 = \dot{r}\sin\omega + r\dot{\omega}\cos\omega,$$

$$\dot{x}_3 = -\dot{r}\cos\omega + r\dot{\omega}\sin\omega, \qquad \dot{y}_2 = -\dot{r}\sin\omega - r\dot{\omega}\cos\omega,$$

$$\dot{z}_1 = 0, \qquad\qquad\qquad \dot{z}_2 = \dot{z}_3 = -\frac{\sigma r\dot{r}}{(\sigma\kappa)^{1/2}(1 - \kappa r^2)^{1/2}},$$

$$\kappa \dot{\mathbf{q}}_1 \odot \dot{\mathbf{q}}_1 = 0,$$

$$\kappa \dot{\mathbf{q}}_2 \odot \dot{\mathbf{q}}_2 = \kappa \dot{\mathbf{q}}_3 \odot \dot{\mathbf{q}}_3 = \kappa r^2\dot{\omega}^2 + \frac{\kappa \dot{r}^2}{1 - \kappa r^2},$$

$$\ddot{x}_1 = \ddot{y}_1 = \ddot{z}_1 = 0,$$

$$\ddot{x}_2 = \left(\ddot{r} - r\dot{\omega}^2\right)\cos\omega - (r\ddot{\omega} + 2\dot{r}\dot{\omega})\sin\omega,$$

$$\ddot{y}_2 = \left(\ddot{r} - r\dot{\omega}^2\right)\sin\omega + (r\ddot{\omega} + 2\dot{r}\dot{\omega})\cos\omega,$$

$$\ddot{x}_3 = -\left(\ddot{r} - r\dot{\omega}^2\right)\cos\omega + (r\ddot{\omega} + 2\dot{r}\dot{\omega})\sin\omega,$$

$$\ddot{y}_3 = -\left(\ddot{r} - r\dot{\omega}^2\right)\sin\omega - (r\ddot{\omega} + 2\dot{r}\dot{\omega})\cos\omega,$$

$$\ddot{z}_2 = \ddot{z}_3 = -\sigma r\ddot{r}\left(\sigma\kappa^{-1} - \sigma r^2\right)^{-1/2} - \kappa^{-1}\dot{r}^2\left(\sigma\kappa^{-1} - \sigma r^2\right)^{-3/2}.$$

Substituting these expressions into Eqs. (8), we are led to the system below, where the double-dot terms on the left indicate to which differential equation each algebraic equation corresponds:

$$\ddot{x}_2, \ddot{x}_3: \quad C\cos\omega - D\sin\omega = 0,$$

$$\ddot{y}_2, \ddot{y}_3: \quad C\sin\omega + D\cos\omega = 0,$$

$$\ddot{z}_2, \ddot{z}_3: \quad C = 0,$$

where

$$C := C(t) = \ddot{r} - r\left(1 - \kappa r^2\right)\dot{\omega}^2 + \frac{\kappa r\dot{r}^2}{1 - \kappa r^2} + \frac{m(5 - 4\kappa r^2)}{4r^2(1 - \kappa r^2)^{1/2}},$$

$$D := D(t) = r\ddot{\omega} + 2\dot{r}\dot{\omega}.$$

(The equations corresponding to $\ddot{x}_1$, $\ddot{y}_1$, and $\ddot{z}_1$ are identities, so they don't show up). The above system has solutions if and only if $C = D = 0$, which means that the existence of Eulerian homographic orbits of the curved 3-body problem with equal masses is equivalent to the existence of solutions of the system of differential equations:

$$\begin{cases} \dot{r} = v, \\[2mm] \dot{w} = -\dfrac{2vw}{r}, \\[3mm] \dot{v} = r\left(1 - \kappa r^2\right)w^2 - \dfrac{\kappa r v^2}{1 - \kappa r^2} - \dfrac{m(5 - 4\kappa r^2)}{4r^2(1 - \kappa r^2)^{1/2}}, \end{cases} \qquad (24)$$

with initial conditions $r(0) = r_0$, $w(0) = w_0$, $v(0) = v_0$, where $w = \dot{\omega}$. The functions r, ω, and w are analytic, and as long as the initial conditions satisfy the conditions $r_0 > 0$ for all κ, as well as $r_0 < \kappa^{-1/2}$ for $\kappa > 0$, standard results of the theory of differential equations guarantee the local existence and uniqueness of a solution (r, w, v) of Eqs. (24), and therefore the local existence and uniqueness of an Eulerian orbit with $r(t)$ and $\omega(t)$ not constant. This conclusion completes the proof. $\square$

8. Classification of Eulerian solutions for $\kappa > 0$

We can now state and prove the following result:

Theorem 6. *In the curved 3-body problem with equal masses, $m_1 = m_2 = m_3 := m > 0$, and $\kappa > 0$ there are three classes of Eulerian solutions*:

 (i) *homothetic orbits that begin or end in total collision in finite time*;
 (ii) *relative equilibria, for which one mass is fixed at one pole of the sphere while the other two move on a circle parallel with the xy plane*;
(iii) *periodic homographic orbits that change size*.

None of the above orbits can cross the equator, defined as the great circle orthogonal to the z axis.

The rest of this section is dedicated to the proof of this theorem.

Let us start by noticing that the first two equations of system (24) imply that $\dot{w} = -\frac{2\dot{r}w}{r}$, which leads to

$$ w = \frac{c}{r^2}, $$

where c is a constant. The case $c = 0$ can occur only when $w = 0$, which means $\dot{\omega} = 0$. Under these circumstances the angular velocity is zero, so the motion is homothetic. The existence of these orbits is stated in Theorem 6(i). They occur only when the angular momentum is zero, and lead to a triple collision in the future or in the past, depending on the direction of the velocity vectors. The existence of the orbits described in Theorem 6(ii) was proved in [3].

For the rest of this section, we assume that $c \neq 0$. System (24) is thus reduced to

$$ \begin{cases} \dot{r} = v, \\ \dot{v} = \dfrac{c^2(1 - \kappa r^2)}{r^3} - \dfrac{\kappa r v^2}{1 - \kappa r^2} - \dfrac{m(5 - 4\kappa r^2)}{4r^2(1 - \kappa r^2)^{1/2}}. \end{cases} \tag{25} $$

To address the existence of the orbits described in Theorem 6(iii), and show that no other Eulerian orbits than those of Theorem 6 exist for $\kappa > 0$, we need to study the flow of system (25) for $\kappa > 0$. Let us first prove the following fact:

Lemma 5. *Regardless of the values of the parameters $m, \kappa > 0$, and $c \neq 0$, system (25) has one fixed point $(r_0, 0)$ with $0 < r_0 < \kappa^{-1/2}$.*

Proof. The fixed points of system (25) are of the form $(r, 0)$ for all values of r that are zeroes of $u(r)$, where

$$ u(r) = \frac{c^2(1 - \kappa r^2)}{r} - \frac{m(5 - 4\kappa r^2)}{4(1 - \kappa r^2)^{1/2}}. \tag{26} $$

But finding the zeroes of $u(r)$ is equivalent to obtaining the roots of the polynomial

$$ 16\kappa^2 \left(c^4\kappa + m^2 \right) r^6 - 8\kappa \left(6c^4\kappa + 5m^2 \right) r^4 + \left(48c^4\kappa + 25m^2 \right) r^2 - 16c^4. $$

Denoting $x = r^2$, this polynomial becomes

$$q(x) = 16\kappa^2\left(c^4\kappa + m^2\right)x^3 - 8\kappa\left(6c^4\kappa + 5m^2\right)x^2 + \left(48c^4\kappa + 25m^2\right)x - 16c^4.$$

Since $\kappa > 0$, Descarte's rule of signs implies that q can have one or three positive roots. The derivative of q is the polynomial

$$q'(x) = 48\kappa^2\left(c^4\kappa + m^2\right)x^2 - 16\kappa\left(6c^4\kappa + 5m^2\right)x + 48c^4\kappa + 25m^2, \tag{27}$$

whose discriminant is $64\kappa^2 m^2(21c^4\kappa + 25m^2)$. But, as $\kappa > 0$, this discriminant is always positive, so it offers no additional information on the total number of positive roots.

To determine the exact number of positive roots, we will use the resultant of two polynomials. Denoting by a_i, $i = 1, 2, \ldots, \zeta$, the roots of a polynomial P, and by b_j, $j = 1, 2, \ldots, \xi$, those of a polynomial Q, the resultant of P and Q is defined by the expression

$$\text{Res}(P, Q) = \prod_{i=1}^{\zeta} \prod_{j=1}^{\xi}(a_i - b_j).$$

Then P and Q have a common root if and only if $\text{Res}[P, Q] = 0$. Consequently the resultant of q and q' is a polynomial in κ, c, and m whose zeroes are the double roots of q. But

$$\text{Res}\left(q, q'\right) = 1024c^4\kappa^5 m^4\left(c^4\kappa + m^2\right)\left(108c^4\kappa + 125m^2\right).$$

Then, for $m, \kappa > 0$ and $c \neq 0$, $\text{Res}[q, q']$ never cancels, therefore q has exactly one positive root. Indeed, should q have three positive roots, a continuous variation of κ, m, and c, would lead to some values of the parameters that correspond to a double root. Since double roots are impossible, the existence of a unique equilibrium $(r_0, 0)$ with $r_0 > 0$ is proved. To conclude that $r_0 < \kappa^{-1/2}$ for all $m, \kappa > 0$ and $c \neq 0$, it is enough to notice that $\lim_{r \to 0} u(r) = +\infty$ and $\lim_{r \to \kappa^{-1/2}} u(r) = -\infty$. This conclusion completes the proof. $\quad \square$

8.1. The flow in the (r, v) plane for $\kappa > 0$

We can now study the flow of system (25) in the (r, v) plane for $\kappa > 0$. The vector field is not defined along the lines $r = 0$ and $r = \kappa^{-1/2}$, so it lies in the band $(0, \kappa^{-1/2}) \times \mathbb{R}$. Consider now the slope $\frac{dv}{dr}$ of the vector field. This slope is given by the ratio $\frac{\dot{v}}{\dot{r}} = v(r, v)$, where

$$v(r, v) = \frac{c^2(1 - \kappa r^2)}{vr^3} - \frac{\kappa r v}{1 - \kappa r^2} - \frac{m(5 - 4\kappa r^2)}{4vr^2(1 - \kappa r^2)^{1/2}}. \tag{28}$$

But v is odd with respect to v, i.e. $v(r, -v) = -v(r, v)$, so the vector field is symmetric with respect to the r axis.

Since $\lim_{r \to 0} v(r) = +\infty$ and $\lim_{r \to \kappa^{-1/2}} v(r) = -\infty$, the flow crosses the r axis perpendicularly upwards to the left of r_0 and downwards to its right, where $(r_0, 0)$ is the fixed point of the system (25) whose existence and uniqueness we proved in Lemma 5. But the right-hand side of the second equation in (25) is of the form

$$W(r, v) = u(r)/r^2 + g_2(r, v), \tag{29}$$

where g_2 was defined earlier as $g_2(r, v) = -\frac{\kappa r v^2}{1 - \kappa r^2}$, while $u(r)$ was defined in (26). Notice that

$$\lim_{r \to 0} W(r, v) = +\infty \quad \text{and} \quad \lim_{r \to \kappa^{-1/2}} W(r, v) = -\infty.$$
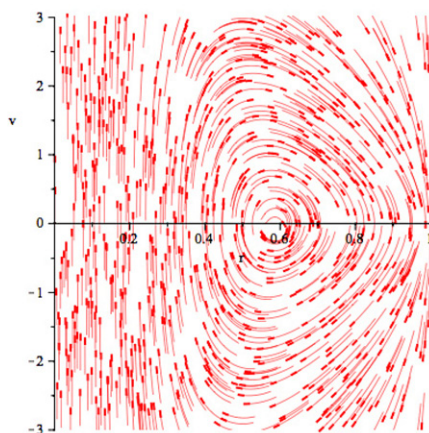
Fig. 3. A sketch of the flow of system (25) for $\kappa = 1$, $c = 2$, and $m = 2$, typical for Eulerian solutions with $\kappa > 0$.

Moreover, $W(r_0, 0) = 0$, and W has no singularities for $r \in (0, \kappa^{-1/2})$. Therefore the flow that enters the region $v > 0$ to the left of r_0 must exit it to the right of the fixed point. The symmetry with respect to the r axis forces all orbits to be periodic around $(r_0, 0)$ (see Fig. 3). This proves the existence of the solutions described in Theorem 6(iii), and shows that no orbits other than those in Theorem 6 occur for $\kappa > 0$. The proof of Theorem 6 is now complete.

9. Classification of Eulerian solutions for $\kappa < 0$

We can now state and prove the following result:

Theorem 7. *In the curved 3-body problem with equal masses, $m_1 = m_2 = m_3 := m > 0$, and $\kappa > 0$ there are four classes of Eulerian solutions*:

 (i) *Eulerian homothetic orbits that begin or end in total collision in finite time*;
 (ii) *Eulerian relative equilibria, for which one mass is fixed at the vertex of the hyperboloid while the other two move on a circle parallel with the xy plane*;
 (iii) *Eulerian periodic orbits that change size; the line connecting the two moving bodies is always parallel with the xy plane, but their z coordinate changes in time*;
 (iv) *Eulerian orbits that come from infinity at time $-\infty$, reach a position when the size of the configuration is minimal, and then return to infinity at time $+\infty$.*

The rest of this section is dedicated to the proof of this theorem.

The homothetic orbits of the type stated in Theorem 7(i) occur only when $c = 0$. Then the two moving bodies collide simultaneously with the fixed one in the future or in the past. Depending on the initial conditions, the motion can be bounded or unbounded.

The existence of the orbits stated in Theorem 7(ii) was proved in [3]. To prove the existence of the solutions stated in Theorem 7(iii) and (iv), and show that there are no other kinds of orbits, we start with the following result:

Lemma 6. *In the curved three body problem with $\kappa < 0$, the polynomial q defined in the proof of Lemma 5 has no positive roots for $c^4\kappa + m^2 \leqslant 0$, but has exactly one positive root for $c^4\kappa + m^2 > 0$.*

Proof. We split our analysis in three different cases depending on the sign of $c^4\kappa + m^2$:

(1) $c^4\kappa + m^2 = 0$. In this case q has form $8\kappa m^2 x^2 + 23c^4\kappa x - 16c^4$, a polynomial that does not have any positive root.

(2) $c^4\kappa + m^2 < 0$. Writing $6c^4\kappa + 5m^2 = 6(c^4\kappa + m^2) - m^2$, we see that the term of q corresponding to x^2 is always negative, so by Descartes's rule of signs the number of positive roots depends on the sign of the coefficient corresponding to x, i.e. $48c^4\kappa + 25m^2 = 48(c^4\kappa + m^2) - 23m^2$, which is also negative, and therefore q has no positive root.

(3) $c^4\kappa + m^2 > 0$. This case leads to three subcases:

– if $6c^4\kappa + 5m^2 < 0$, then necessarily $48c^4\kappa + 25m^2 < 0$ and, so q has exactly one positive root;

– if $6c^4\kappa + 5m^2 > 0$ and $48c^4\kappa + 25m^2 < 0$, then q has one change of sign and therefore exactly one positive root;

– if $48c^4\kappa + 25m^2 > 0$, then all coefficients, except for the free term, are positive, therefore q has exactly one positive root.

These conclusions complete the proof. $\square$

The following result will be used towards understanding the case when system (25) has one fixed point.

Lemma 7. *Regardless of the values of the parameters $\kappa < 0$, $m > 0$, and $c \neq 0$, there is no fixed point, $(r_*, 0)$, of system* (25) *for which $\frac{\partial}{\partial r}W(r_*, 0) = 0$, where W is defined in* (29).

Proof. Since $u(r_*) = 0$, $\frac{\partial}{\partial r}g_2(r_*, 0) = 0$, and

$$\frac{\partial W}{\partial r}(r, v) = -\left(2/r^3\right)u(r) + \left(1/r^2\right)\frac{d}{dr}u(r) + \frac{\partial}{\partial r}g_2(r, v),$$

it means that $W(r_*, 0) = 0$ if and only if $\frac{d}{dr}u(r_*) = 0$. Consequently our result would follow if we can prove that there is no fixed point $(r_*, 0)$ for which $\frac{d}{dr}u(r_*) = 0$. To show this fact, notice first that

$$\frac{d}{dr}u(r) = -\frac{c^2(1 + \kappa r^2)}{r^2} - \frac{\kappa mr(4\kappa r^2 - 3)}{4(1 - \kappa r^2)^{3/2}}. \tag{30}$$

From the definition of $u(r)$ in (26), the identity $u(r_*) = 0$ is equivalent to

$$\left(1 - \kappa r_*^2\right)^{1/2} = \frac{mr_*(5 - 4\kappa r_*^2)}{4c^2(1 - \kappa r_*^2)}.$$

Regarding $(1 - \kappa r^2)^{3/2}$ as $(1 - \kappa r^2)^{1/2}(1 - \kappa r^2)$, and substituting the above expression of $(1 - \kappa r_*^2)^{1/2}$ into (30) for $r = r_*$, we obtain that

$$\frac{\kappa(4\kappa r_*^2 - 3)}{5 - 4\kappa r_*^2} + \frac{1 + \kappa r_*^2}{r_*^2} = 0,$$

which leads to the conclusion that $r_*^2 = 5/2\kappa < 0$. Therefore there is no fixed point $(r_*, 0)$ such that $\frac{d}{dr}u(r_*) = 0$. This conclusion completes the proof. $\square$

9.1. The flow in the (r, v) plane for $\kappa < 0$

To study the flow of system (25) for $\kappa < 0$, we will consider the two cases given by Lemma 6, namely when system (25) has no fixed points and when it has exactly one fixed point.
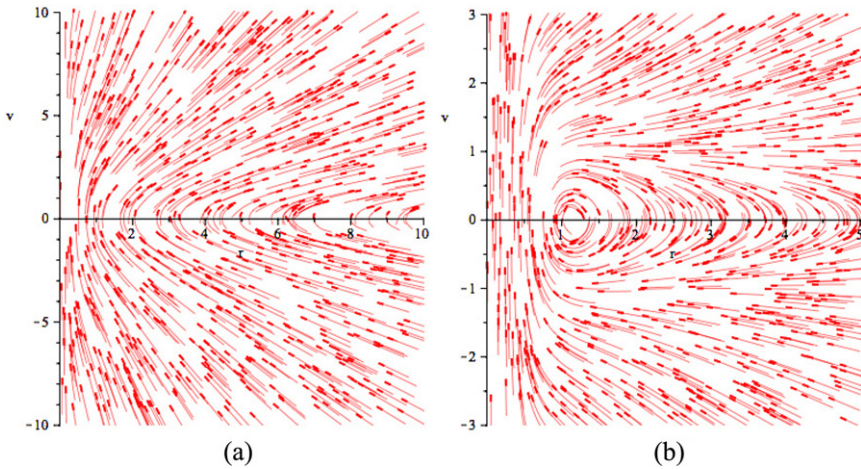
Fig. 4. A sketch of the flow of system (25) for (a) $\kappa = -2$, $c = 2$, and $m = 4$, typical for no fixed points; (b) $\kappa = -2$, $c = 2$, and $m = 6.2$, typical for one fixed point.

9.1.1. The case of no fixed points

Since $\kappa < 0$, and system (25) has no fixed points, the function $u(r)$, defined in (26), has no zeroes. But $\lim_{r \to 0} u(r) = +\infty$, so $u(r) > 0$ for all $r > 0$. Then $u(r)/r^2 > 0$ for all $r > 0$. Since $\lim_{r \to 0} g_2(r, v) = 0$, it follows that $\lim_{r \to 0} W(r, v) = +\infty$, where $W(r, v)$ (defined in (29)) forms the right-hand side of the second equation in system (25). Since system (25) has no fixed points, W doesn't vanish. Therefore $W(r, v) > 0$ for all $r > 0$ and v.

Notice that the slope of the vector field, $v(r, v)$, defined in (28), is of the form $v(r, v) = W(r, v)/v$, which implies that the flow crosses the r axis perpendicularly at every point with $r > 0$. Also, for r fixed, $\lim_{v \to \pm\infty} W(r, v) = +\infty$. Moreover, for v fixed, $\lim_{r \to \infty} W(r, v) = 0$. This means that the flow has a simple behavior as in Fig. 4(a). These orbits correspond to those stated in Theorem 7(iv).

9.1.2. The case of one fixed point

We start with analyzing the behavior of the flow near the unique fixed point $(r_0, 0)$. Let $F(r, v) = v$ denote the right-hand side in the first equation of system (25). Then $\frac{\partial}{\partial r} F(r_0, 0) = 0$, $\frac{\partial}{\partial v} F(r_0, 0) = 1$, and $\frac{\partial}{\partial v} W(r_0, 0) = 0$. To determine the sign of $\frac{\partial}{\partial r} W(r_0, 0)$, notice first that $\lim_{r \to 0} W(r, v) = +\infty$. Since the equation $W(r, v) = 0$ has a single root of the form $(r_0, 0)$, with $r_0 > 0$, it follows that $W(r, 0) > 0$ for $0 < r < r_0$.

To show that $W(r, 0) < 0$ for $r > r_0$, assume the contrary, which (given the fact that r_0 is the only zero of $W(r, 0)$) means that $W(r, 0) > 0$ for $r > r_0$. So $W(r, 0) \geqslant 0$, with equality only for $r = r_0$. But recall that we are in the case when the parameters satisfy the inequality $c^4 \kappa + m^2 > 0$. Then a slight variation of the parameters $\kappa < 0$, $m > 0$, and $c \neq 0$, within the region defined by the above inequality, leads to two zeroes for $W(r, 0)$, a fact which contradicts Lemma 6. Therefore, necessarily, $W(r, 0) < 0$ for $r > r_0$.

Consequently $W(r, 0)$ is decreasing in a small neighborhood of r_0, so $\frac{\partial}{\partial r} W(r_0, 0) \leqslant 0$. But by Lemma 7, $\frac{\partial}{\partial r} W(r_0, 0) \neq 0$, so necessarily $\frac{\partial}{\partial r} W(r_0, 0) < 0$. The eigenvalues corresponding to the system obtained by linearizing Eqs. (25) around the fixed point $(r_0, 0)$ are given by the equation

$$\det \begin{bmatrix} -\lambda & 1 \\ \frac{\partial W}{\partial r}(r_0, 0) & -\lambda \end{bmatrix} = 0, \tag{31}$$

so these eigenvalues are purely imaginary. In terms of system (25), this means that the fixed point $(r_0, 0)$ could be a spiral sink, a spiral source, or a center. The symmetry of the flow with respect to the r axis excludes the first two possibilities, consequently $(r_0, 0)$ is a center (see Fig. 4(b)). We

thus proved that, in a neighborhood of this fixed point, there exist infinitely many periodic Eulerian solutions whose existence was stated in Theorem 7(iii).

To complete the analysis of the flow of system (25), we will use the nullcline $\dot{v} = 0$, which is given by the equation

$$v^2 = \frac{1 - \kappa r^2}{\kappa r} \left[\frac{c^2(1 - \kappa r^2)}{r^3} - \frac{m(5 - 4\kappa^2)}{4r^2(1 - \kappa r^2)^{1/2}} \right]. \tag{32}$$

Along this curve, which passes through the fixed point $(r_0, 0)$, and is symmetric with respect to the r axis, the vector field has slope zero. To understand the qualitative behavior of this curve, notice that

$$\lim_{r \to \infty} \frac{1 - \kappa r^2}{\kappa r} \left[\frac{c^2(1 - \kappa r^2)}{r^3} - \frac{m(5 - 4\kappa^2)}{4r^2(1 - \kappa r^2)^{1/2}} \right] = m(-\kappa)^{1/2} + \kappa c^2.$$

But we are restricted to the parameter region given by the inequality $m^2 + \kappa c^4 > 0$, which is equivalent to

$$\left[m(-\kappa)^{1/2} - (-\kappa)c^2 \right]\left[m(-\kappa)^{1/2} + (-\kappa)c^2 \right] > 0.$$

Since the second factor of this product is positive, it follows that the first factor must be positive, therefore the above limit is positive. Consequently the curve given in (32) is bounded by the horizontal lines

$$v = \left[m(-\kappa)^{1/2} + \kappa c^2 \right]^{1/2} \quad \text{and} \quad v = -\left[m(-\kappa)^{1/2} + \kappa c^2 \right]^{1/2}.$$

Inside the curve, the vector field has negative slope for $v > 0$ and positive slope for $v < 0$. Outside the curve, the vector field has positive slope for $v > 0$, but negative slope for $v < 0$. So the orbits of the flow that stay outside the nullcline curve are unbounded. They correspond to solutions whose existence was stated in Theorem 7(iv). This conclusion completes the proof of Theorem 7.

10. Hyperbolic homographic solutions

In this last section we consider a certain class of homographic orbits, which occur only in spaces of negative curvature. They are related to hyperbolic rotations. More precisely, the Principal Axis Theorem for hyperbolic space (see [7,9], and the appendix of [3]) states that, in a convenient basis, a hyperbolic rotation can be given by a matrix of the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh s & \sinh s \\ 0 & \sinh s & \cosh s \end{bmatrix},$$

where s is a fixed real value. In the case $\kappa = -1$, we proved in [3] the existence of hyperbolic Eulerian relative equilibria of the curved 3-body problem with equal masses. These orbits behave as follows: three bodies of equal masses move along three fixed hyperbolas, each body on one of them; the middle hyperbola, which is a geodesic passing through the vertex of the hyperboloid, lies in a plane of $\mathbb{R}^3$ that is parallel and equidistant from the planes containing the other two hyperbolas, none of which is a geodesic. At every moment in time, the bodies lie equidistantly from each other on a geodesic hyperbola that rotates hyperbolically. These solutions are the hyperbolic counterpart of Eulerian solutions, in the sense that the bodies stay on the same geodesic, which rotates hyperbolically, instead of circularly. The existence proof we gave in [3] works for any $\kappa < 0$. We therefore provide the following definitions.

Definition 7. A solution of the curved 3-body problem is called hyperbolic homographic if the bodies maintain a configuration similar to itself while rotating hyperbolically. When the bodies remain on the same hyperbolically rotating geodesic, the solution is called hyperbolic Eulerian.

While there is, so far, no evidence of hyperbolic non-Eulerian homographic solutions, we showed in [3] that hyperbolic Eulerian orbits exist in the case of equal masses. In the particular case of equal masses, it is natural to assume that the middle body moves on a geodesic passing through the point $(0, 0, |\kappa|^{-1/2})$ (the vertex of the hyperboloid's upper sheet), while the other two bodies are on the same (hyperbolically rotating) geodesic, and equidistant from it.

Consequently we can seek hyperbolic Eulerian solutions of equal masses of the form:

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3), \quad \text{with } \mathbf{q}_i = (x_i, y_i, z_i), \ i = 1, 2, 3, \quad \text{and} \tag{33}$$

$$
\begin{aligned}
&x_1 = 0, &&y_1 = |\kappa|^{-1/2} \sinh\omega, &&z_1 = |\kappa|^{-1/2} \cosh\omega, \\
&x_2 = (\rho^2 + \kappa^{-1})^{1/2}, &&y_2 = \rho \sinh\omega, &&z_2 = \rho \cosh\omega, \\
&x_3 = -(\rho^2 + \kappa^{-1})^{1/2}, &&y_3 = \rho \sinh\omega, &&z_3 = \rho \cosh\omega,
\end{aligned}
$$

where $\rho := \rho(t)$ is the *size function* and $\omega := \omega(t)$ is the *angular function*.

Indeed, for every time t, we have that $x_i^2(t) + y_i^2(t) - z_i^2(t) = \kappa^{-1}$, $i = 1, 2, 3$, which means that the bodies stay on the surface $\mathbf{H}_\kappa^2$, while lying on the same, possibly (hyperbolically) rotating, geodesic. Therefore representation (33) of the hyperbolic Eulerian homographic orbits agrees with Definition 7.

With the help of this analytic representation, we can define Eulerian homothetic orbits and hyperbolic Eulerian relative equilibria.

Definition 8. A hyperbolic Eulerian homographic solution is called Eulerian homothetic if the configuration of the system expands or contracts, but does not rotate hyperbolically.

In terms of representation (33), an Eulerian homothetic solution occurs when $\omega(t)$ is constant, but $\rho(t)$ is not constant. A straightforward computation shows that if $\omega(t)$ is constant, the bodies lie initially on a geodesic, and the initial velocities are such that the bodies move along the geodesic towards or away from a triple collision at the point occupied by the fixed body.

Notice that Definition 8 leads to the same orbits produced by Definition 5. While the configuration of the former solution does not rotate hyperbolically, and the configuration of the latter solution does not rotate elliptically, both fail to rotate while expanding or contracting. This is the reason why Definitions 5 and 8 use the same name (Eulerian homothetic) for these types of orbits.

Definition 9. A hyperbolic Eulerian homographic solution is called a hyperbolic Eulerian relative equilibrium if the configuration rotates hyperbolically while its size remains constant.

In terms of representation (33), hyperbolic Eulerian relative equilibria occur when $\rho(t)$ is constant, while $\omega(t)$ is not constant.

Unlike for Lagrangian and Eulerian solutions, hyperbolic Eulerian homographic orbits exist only in the form of homothetic solutions or relative equilibria. As we will further prove, any composition between a homothetic orbit and a relative equilibrium fails to be a solution of system (8).

Theorem 8. *In the curved 3-body problem of equal masses, $m_1 = m_2 = m_3 := m$, and $\kappa < 0$, the only hyperbolic Eulerian homographic solutions are either Eulerian homothetic orbits or hyperbolic Eulerian relative equilibria.*

Proof. Consider for system (8) a solution of the form (33) that is not homothetic. Then

$$\kappa \mathbf{q}_1 \odot \mathbf{q}_2 = \kappa \mathbf{q}_1 \odot \mathbf{q}_3 = |\kappa|^{1/2} \rho,$$

$$\kappa \mathbf{q}_2 \odot \mathbf{q}_3 = -1 - 2\kappa\rho^2,$$

$$\dot{x}_1 = \ddot{x}_1 = 0, \qquad \dot{y}_1 = |\kappa|^{-1/2}\dot{\omega}\cosh\omega, \qquad \dot{z}_1 = |\kappa|^{-1/2}\sinh\omega,$$

$$\dot{x}_2 = -\dot{x}_3 = \frac{\rho\dot{\rho}}{(\rho^2 + \kappa^{-1})^{1/2}},$$

$$\dot{y}_2 = \dot{y}_3 = \dot{\rho}\sinh\omega + \rho\dot{\omega}\cosh\omega,$$

$$\dot{z}_2 = \dot{z}_3 = \dot{\rho}\cosh\omega + \rho\dot{\omega}\sinh\omega,$$

$$\kappa \dot{\mathbf{q}}_2 \odot \dot{\mathbf{q}}_2 = -\dot{\omega}^2, \qquad \kappa \dot{\mathbf{q}}_2 \odot \dot{\mathbf{q}}_2 = \kappa \dot{\mathbf{q}}_3 \odot \dot{\mathbf{q}}_3 = \kappa\rho^2\dot{\omega}^2 - \frac{\kappa\dot{\rho}^2}{1 + \kappa\rho^2},$$

$$\ddot{x}_2 = -\ddot{x}_3 = \frac{\rho\ddot{\rho}}{(\rho^2 + \kappa^{-1})^{1/2}} + \frac{\kappa^{-1}\dot{\rho}^2}{(\rho^2 + \kappa^{-1})^{3/2}},$$

$$\ddot{y}_2 = \ddot{y}_3 = (\ddot{\rho} + \rho\dot{\omega}^2)\sinh\omega + (\rho\ddot{\omega} + 2\dot{\rho}\dot{\omega})\cosh\omega,$$

$$\ddot{z}_2 = \ddot{z}_3 = (\ddot{\rho} + \rho\dot{\omega}^2)\cosh\omega + (\rho\ddot{\omega} + 2\dot{\rho}\dot{\omega})\sinh\omega.$$

Substituting these expressions in system (8), we are led to an identity corresponding to $\ddot{x}_1$. The other equations lead to the system

$$\ddot{x}_2, \ddot{x}_3: \quad E = 0,$$

$$\ddot{y}_1: \quad |\kappa|^{-1/2}\ddot{\omega}\cosh\omega = 0,$$

$$\ddot{z}_1: \quad |\kappa|^{-1/2}\ddot{\omega}\sinh\omega = 0,$$

$$\ddot{y}_2, \ddot{y}_3: \quad E\sinh\omega + F\cosh\omega = 0,$$

$$\ddot{z}_2, \ddot{z}_3: \quad E\cosh\omega + F\sinh\omega = 0,$$

where

$$E := E(t) = \ddot{\rho} + \rho(1 + \kappa\rho^2)\dot{\omega}^2 - \frac{\kappa\rho\dot{\rho}^2}{1 + \kappa\rho^2} + \frac{m(1 - 4\kappa\rho^2)}{4\rho^2|1 + \kappa\rho^2|^{1/2}},$$

$$F := F(t) = \rho\ddot{\omega} + 2\dot{\rho}\dot{\omega}.$$

This system can be obviously satisfied only if

$$\begin{cases} \ddot{\omega} = 0, \\ \ddot{\omega} = -\dfrac{2\dot{\rho}\dot{\omega}}{\rho}, \\ \ddot{\rho} = -\rho(1 + \kappa\rho^2)\dot{\omega}^2 + \dfrac{\kappa\rho\dot{\rho}^2}{1 + \kappa\rho^2} - \dfrac{m(1 + 4\kappa\rho^2)}{4\rho^2|1 + \kappa\rho^2|^{1/2}}. \end{cases} \tag{34}$$

The first equation implies that $\omega(t) = at + b$, where a and b are constants, which means that $\dot{\omega}(t) = a$. Since we assumed that the solution is not homothetic, we necessarily have $a \neq 0$. But from the second equation, we can conclude that

$$\dot{\omega}(t) = \frac{c}{\rho^2(t)},$$

where c is a constant. Since $a \neq 0$, it follows that $\rho(t)$ is constant, which means that the homographic solution is a relative equilibrium. This conclusion is also satisfied by the third equation, which reduces to

$$a^2 = \dot{\omega}^2 = \frac{m(1 - 4\kappa\rho)}{4\rho^3 |1 + \kappa\rho^2|^{1/2}},$$

being verified by two values of a (equal in absolute value, but of opposite signs) for every κ and ρ fixed. Therefore every hyperbolic Eulerian homographic solution that is not Eulerian homothetic is a hyperbolic Eulerian relative equilibrium. This conclusion completes the proof. $\quad\square$

Since a slight perturbation of hyperbolic Eulerian relative equilibria, within the set of hyperbolic Eulerian homographic solutions, produces no orbits with variable size, it means that hyperbolic Eulerian relative equilibria of equal masses are unstable. Therefore though they exist in a mathematical sense, as proved above (as well as in [3], using a direct method), such equal-mass orbits are unlikely to be found in a (hypothetical) hyperbolic physical universe.

Acknowledgments

References

[1] J.F. Cariñena, M.F. Rañada, M. Santander, Central potentials on spaces of constant curvature: The Kepler problem on the two-dimensional sphere $\mathbf{S}^2$ and the hyperbolic plane $\mathbf{H}^2$, J. Math. Phys. 46 (2005) 052702.
[2] F. Diacu, Near-collision dynamics for particle systems with quasihomogeneous potentials, J. Differential Equations 128 (1996) 58–77.
[3] F. Diacu, E. Pérez-Chavela, M. Santoprete, The n-body problem in spaces of constant curvature, arXiv:0807.1747, 2008, 54 pp.
[4] F. Diacu, On the singularities of the curved n-body problem, Trans. Amer. Math. Soc., in press..
[5] F. Diacu, E. Pérez-Chavela, M. Santoprete, Saari's conjecture for the collinear n-body problem, Trans. Amer. Math. Soc. 357 (10) (2005) 4215–4223.
[6] F. Diacu, T. Fujiwara, E. Pérez-Chavela, M. Santoprete, Saari's homographic conjecture of the 3-body problem, Trans. Amer. Math. Soc. 360 (12) (2008) 6447–6473.
[7] F. Dillen, W. Kühnel, Ruled Weingarten surfaces in Minkowski 3-space, Manuscripta Math. 98 (1999) 307–320.
[8] A. Einstein, L. Infeld, B. Hoffmann, The gravitational equations and the problem of motion, Ann. of Math. 39 (1) (1938) 65–100.
[9] J. Hano, K. Nomizu, On isometric immersions of the hyperbolic plane into the Lorentz–Minkovski space and the Monge–Ampère equation of certain type, Math. Ann. 262 (1983) 245–253.
[10] L. Infeld, On a new treatment of some eigenvalue problems, Phys. Rev. 59 (1941) 737–747.
[11] L. Infeld, A. Schild, A note on the Kepler problem in a space of constant negative curvature, Phys. Rev. 67 (1945) 121–122.
[12] W. Killing, Die Rechnung in den nichteuklidischen Raumformen, J. Reine Angew. Math. 89 (1880) 265–287.
[13] W. Killing, Die Mechanik in den nichteuklidischen Raumformen, J. Reine Angew. Math. 98 (1885) 1–48.
[14] W. Killing, Die Nicht-Eukildischen Raumformen in Analytischer Behandlung, Teubner, Leipzig, 1885.
[15] T. Levi-Civita, The relativistic problem of several bodies, Amer. J. Math. 59 (1) (1937) 9–22.
[16] T. Levi-Civita, Le problème des n corps en relativité générale, Gauthier–Villars, Paris, 1950; or the English translation: The n-Body Problem in General Relativity, D. Reidel, Dordrecht, 1964.
[17] H. Liebmann, Über die Zentralbewegung in der nichteuklidische Geometrie, Berichte Königl. Sächsischen Gesell. Wiss. Math. Phys. Kl. 55 (1903) 146–153.
[18] H. Liebmann, Nichteuklidische Geometrie, G.J. Göschen, Leipzig, 1905; second ed., G.J. Göschen, 1912; third ed., Walter de Gruyter, Berlin, Leipzig, 1923.

[19] R. Lipschitz, Extension of the planet-problem to a space of *n* dimensions and constant integral curvature, Quart. J. Pure Appl. Math. 12 (1873) 349–370.

[20] A.I. Miller, The myth of Gauss's experiment on the Euclidean nature of physical space, Isis 63 (3) (1972) 345–348.

[21] D. Saari, Collisions, Rings, and Other Newtonian *N*-Body Problems, CBMS Reg. Conf. Ser. Math., No. 104, American Mathematical Society, Providence, RI, 2005.

[22] E. Schering, Die Schwerkraft im Gaussischen Raume, Nachr. Königl. Gesell. Wiss. Göttingen 15 (1870) 311–321.

[23] E. Schering, Die Schwerkraft in mehrfach ausgedehnten Gaussischen und Riemmanschen Räumen, Nachr. Königl. Gesell. Wiss. Göttingen 6 (1873) 149–159.

[24] E. Schrödinger, A method for determining quantum-mechanical eigenvalues and eigenfunctions, Proc. Roy. Irish Acad. Sect. A 46 (1940) 9–16.

[25] A. Wintner, The Analytical Foundations of Celestial Mechanics, Princeton University Press, Princeton, NJ, 1941.