THE n-BODY PROBLEM IN SPACES OF CONSTANT CURVATURE. PART II: SINGULARITIES

Florin Diacu
Pacific Institute for the Mathematical Sciences
and
Department of Mathematics and Statistics
University of Victoria
P.O. Box 3060 STN CSC
Victoria, BC, Canada, V8W 3R4
diacu@math.uvic.ca

Ernesto Pérez-Chavela
Departamento de Matemáticas
Universidad Autónoma Metropolitana-Iztapalapa
Apdo. 55534, México, D.F., México
epc@xanum.uam.mx

Manuele Santoprete
Department of Mathematics
Wilfrid Laurier University
75 University Avenue West,
Waterloo, ON, Canada, N2L 3C5.
msantopr@wlu.ca

November 15, 2011

Abstract. We analyze the singularities of the equations of motion and several types of singular solutions of the n-body problem in spaces of positive constant curvature. Apart from collisions, the equations encounter noncollision singularities, which occur when two or more bodies are antipodal. This conclusion leads, on the one hand, to hybrid solution singularities for as few as 3 bodies, whose orbits end up in a collision-antipodal configuration in finite time; on the other hand, it produces nonsingularity collisions, characterized by finite velocities and forces at the collision instant.

1. Introduction

Consider the curved n-body problem, the natural extension of the planar Newtonian n-body problem to surfaces of nonzero constant curvature: the unit sphere $\mathbf{S}^2$, for positive curvature, and the hyperbolic plane $\mathbf{H}^2$, for negative curvature. In [7], henceforth called Part I, we derived the equations of motion for $n \geq 2$, initiated the study of relative equilibria, and outlined the importance of the problem. We also showed that singularities occur at collisions, on both surfaces, and at antipodal configurations, in $\mathbf{S}^2$ alone. Here we provide a first study of the singularities and of the singular solutions of the equations. A continuation of this research appears in [3].

The set of singularities in $\mathbf{S}^2$ has a dynamical structure. When three bodies move along a geodesic, solutions close to binary collisions and far from antipodal singularities end up in collision, so binary collisions are attractive. But antipodal singularities are repulsive in the sense that no matter how close two bodies are to an antipodal singularity, they never reach it if the third body is far from a collision with any of them.

It is natural to ask whether the antipodal singularities of the equations of motion occur because of the coordinates we use. This answer is no. The equations in intrinsic coordinates, [11], preserve these singularities, which thus characterize the curved n-body problem in $\mathbf{S}^2$ and cannot be dismissed as artificial.

Another issue is that of solution singularities, which arise naturally when the analytic extension of the solution relative to time is impossible up to infinity. These singularities are due to collision or antipodal configurations. The main result of this paper proves the existence of hybrid singular solutions in the 3-body problem in $\mathbf{S}^2$ that end up in finite time in a collision-antipodal singularity. But, depending on the masses and the initial data, these configurations may be unreachable or be no singularities at all. If other types of noncollision singularities exist, such as pseudocollisions, [10], remains an open question. The reason why this problem is not easy to answer rests with the nonexistence of the linear-momentum and center-of-mass integrals (see Part I), as proved in [5].

2. Equations of motion

Consider the masses $m_1, \dots, m_n > 0$ in $\mathbb{R}^3$, whose positions are given by the vectors $\mathbf{q}_i = (x_i, y_i, z_i),\ i = \overline{1, n}$, and let $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_n)$ be the configuration of the system. The Hamiltonian function describing their motion on the unit sphere $\mathbf{S}^2$ is

$$H(\mathbf{q}, \mathbf{p}) = T(\mathbf{q}, \mathbf{p}) - U(\mathbf{q}),$$

where $T(\mathbf{q}, \mathbf{p}) = \frac{1}{2}\sum_{i=1}^n m_i^{-1}(\mathbf{p}_i \cdot \mathbf{p}_i)(\mathbf{q}_i \cdot \mathbf{q}_i)$ is the kinetic energy and

$$U(\mathbf{q}) = \frac{1}{2}\sum_{i=1}^n \sum_{j=1, j\neq i}^n \frac{m_i m_j \mathbf{q}_i \cdot \mathbf{q}_j}{[(\mathbf{q}_i \cdot \mathbf{q}_i)(\mathbf{q}_j \cdot \mathbf{q}_j) - (\mathbf{q}_i \cdot \mathbf{q}_j)^2]^{1/2}} \tag{1}$$

is the force function, $-U$ representing the potential energy (in Part I we showed how this expression of U follows from the cotangent potential). The Hamiltonian form of the equations of motion is given by the system

$$
\begin{cases} \dot{\mathbf{q}}_i = m_i^{-1}\mathbf{p}_i, \\ \dot{\mathbf{p}}_i = \nabla_{\mathbf{q}_i} U(\mathbf{q}) - m_i(\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i)\mathbf{q}_i, \quad i = \overline{1,n}, \end{cases} \tag{2}
$$

where the gradient of the force function has the expression

$$
\nabla_{\mathbf{q}_i} U(\mathbf{q}) = \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{m_i m_j (\mathbf{q}_j \cdot \mathbf{q}_j)[(\mathbf{q}_i \cdot \mathbf{q}_i)\mathbf{q}_j - (\mathbf{q}_i \cdot \mathbf{q}_j)\mathbf{q}_i]}{[(\mathbf{q}_i \cdot \mathbf{q}_i)(\mathbf{q}_j \cdot \mathbf{q}_j) - (\mathbf{q}_i \cdot \mathbf{q}_j)^2]^{3/2}}, \ i = \overline{1,n}. \tag{3}
$$

The motion is confined to the sphere, i.e. $(\mathbf{q}, \dot{\mathbf{q}}) \in \mathbf{T}^*(\mathbf{S}^2)^n$, where $\mathbf{T}^*(\mathbf{S}^2)^n$ is the cotangent bundle of the configuration space $(\mathbf{S}^2)^n$. The constraints $\mathbf{q}_i \cdot \mathbf{q}_i = 1$ imply that $\mathbf{q}_i \cdot \dot{\mathbf{q}}_i = 0$, so system (2) has dimension $4n$. The Hamiltonian provides the integral of energy,

$$
H(\mathbf{q}, \dot{\mathbf{q}}) = h,
$$

where h is the energy constant. Equations (2) also have the integrals of the angular momentum,

$$
\sum_{i=1}^{n} m_i \mathbf{q}_i \times \dot{\mathbf{q}}_i = \mathbf{c}, \tag{4}
$$

where $\mathbf{c}$ is a constant vector. Unlike in the Euclidean case, there are no integrals of the center of mass and linear momentum, [5], so the phase space is $(4n-4)$-dimensional.

3. Singularities of the equations

System (2) is undefined in the set $\mathbf{\Delta} := \cup_{1\le i<j\le n}\mathbf{\Delta}_{ij}$, with

$$
\mathbf{\Delta}_{ij} := \{\mathbf{q} \in (\mathbf{S}^2)^n \mid (\mathbf{q}_i \cdot \mathbf{q}_j)^2 = 1\}.
$$

The condition $(\mathbf{q}_i \cdot \mathbf{q}_j)^2 = 1$ suggests that we consider the set $\mathbf{\Delta}_{ij} = \mathbf{\Delta}_{ij}^+ \cup \mathbf{\Delta}_{ij}^-$, where

$$
\mathbf{\Delta}_{ij}^+ := \{\mathbf{q} \in (\mathbf{S}^2)^n \mid \mathbf{q}_i \cdot \mathbf{q}_j = 1\}, \quad \mathbf{\Delta}_{ij}^- := \{\mathbf{q} \in (\mathbf{S}^2)^n \mid \mathbf{q}_i \cdot \mathbf{q}_j = -1\}.
$$

Accordingly, we define $\mathbf{\Delta}^+ := \cup_{1\le i<j\le n}\mathbf{\Delta}_{ij}^+$ and $\mathbf{\Delta}^- := \cup_{1\le i<j\le n}\mathbf{\Delta}_{ij}^-$. Then $\mathbf{\Delta} = \mathbf{\Delta}^+ \cup \mathbf{\Delta}^-$. The elements of $\mathbf{\Delta}^+$ correspond to collisions, whereas the elements of $\mathbf{\Delta}^-$ correspond to antipodal configurations, when some bodies are at the opposite ends of a diameter. In both cases, the forces become infinite.

In the 2-body problem, $\mathbf{\Delta}^+$ and $\mathbf{\Delta}^-$ are disjoint. Indeed, $\mathbf{q}_1 \cdot \mathbf{q}_2$ is 1 or -1, but not both. But $\mathbf{\Delta}^+ \cap \mathbf{\Delta}^-$ is not empty for $n \ge 3$. In the 3-body problem, for instance, the configuration in which two bodies are at collision and the third lies at the opposite end of the corresponding diameter will be called collision-antipodal.

The theory of differential equations regards singularities as points where the equations break down. But singularities often exhibit a dynamical role. In the rectilinear 3-body problem, for instance, the set of binary collisions is attractive in the sense that for any given initial velocities, there are initial positions such that if two bodies come close enough to each other, but far enough from other collisions, then the collision occurs. (Close to triple collisions, things get more complicated: two of the bodies may form a binary, while the third gets expelled at high speed, [8].)

Something similar happens for binary collisions of the 3-body problem in $\mathbf{S}^1$. Given initial velocities, we can choose initial positions that put m_1 and m_2 close enough to a binary collision, and m_3 far enough from an antipodal singularity with either m_1 or m_2, such that the binary collision takes place. This is indeed the case because the attraction between m_1 and m_2 can be made as large as desired by placing the bodies close enough to each other. Since m_3 is far enough from an antipodal position, and no comparable force can oppose the attraction between m_1 and m_2, these bodies collide.

Antipodal singularities lead to a new phenomenon. Given initial velocities, no matter how close we choose the initial positions near an antipodal singularity, the corresponding solution gets repelled from this singularity as long as no collision force competes. So while binary collisions can be regarded as attractive if far away from antipodal singularities, binary antipodal singularities can be seen as repulsive if far away from collisions. But what happens when collision and antipodal singularities are close to each other? As we will see in the next section, the behaviour of orbits in that region of the phase space is sensitive to the choice of masses and initial data. In particular, we will prove the existence of hybrid singular solutions in the 3-body problem, i.e. those that end in finite time in a collision-antipodal singularity, as well as of solutions that reach a collision-antipodal configuration but remain analytic at this point.

4. Solution singularities

$\mathbf{\Delta}$ is related to singularities arising from the analytic continuation of solutions. For $(\mathbf{q}, \dot{\mathbf{q}})(0) \in \mathbf{T}^*(\mathbf{S}^2)^n$ with $\mathbf{q}(0) \notin \mathbf{\Delta}$, standard results ensure the local existence and uniqueness of an analytic solution $(\mathbf{q}, \dot{\mathbf{q}})$ defined on some interval $[0, t^+)$. Since $\mathbf{S}^2$ is a connected set, this solution can be analytically extended to an interval $[0, t^*)$, with $0 < t^+ \leq t^* \leq \infty$. If $t^* = \infty$, the solution is globally defined. But if $t^* < \infty$, the solution is called singular, and we say that it has a singularity at time t^*.

There is a close connection between singular solutions and singularities of the equations of motion. At the end of the 19th century, Painlevé pointed out this connection in the Euclidean case. In his Stockholm lectures, [10], he showed that every singular solution $(\mathbf{q}, \dot{\mathbf{q}})$ is such that $\mathbf{q}(t) \to \mathbf{\Delta}$ when $t \to t^*$, for otherwise it would be globally defined. In flat space, $\mathbf{\Delta}$ is formed by all collision configurations, so when $\mathbf{q}$ tends to an element of $\mathbf{\Delta}$, the solution ends in a collision singularity. But it is also possible that $\mathbf{q}$ reaches $\mathbf{\Delta}$ by oscillating among various elements without ever settling for any of them. Painlevé conjectured that such pseudocollisions exist. In 1908, von Zeipel showed that a necessary condition for a pseudocollision is that the motion becomes unbounded in finite time, [12], [9]. Xia produced the first example of this kind in 1992, [13]. Historical accounts of this development appear in [1] and [6].

The results of Painlevé don't remain intact in our problem, [4], [3], so whether pseudocollisions exist is not clear. Nevertheless, we can show that there are solutions ending in, or repelled from, collision-antipodal singularities, as well as solutions that are not singular at such configurations. To prove these facts, we need the result stated below, which provides a criterion for determining the direction of motion along a great circle in the framework of an isosceles problem defined in some invariant great circle $\mathbf{S}^1$.

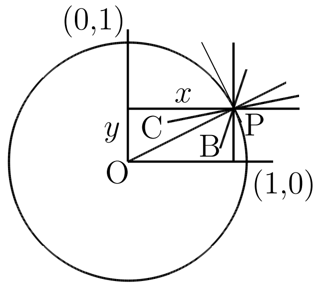

FIGURE 1. The relative positions of the force acting on m, while the body is on the geodesic $z = 0$.

Lemma 1. *Consider the n-body problem in $\mathbf{S}^2$, and assume that a body of mass m is at rest at time t_0 on the geodesic $z = 0$ within its first quadrant, $x, y > 0$. Then, if*

(a) $\ddot{x}(t_0) > 0$ and $\ddot{y}(t_0) < 0$, the force pulls the body towards $(1, 0)$.

(b) $\ddot{x}(t_0) < 0$ and $\ddot{y}(t_0) > 0$, the force pulls the body towards $(0, 1)$.

(c) $\ddot{x}(t_0) \leq 0$ and $\ddot{y}(t_0) \leq 0$, the force pulls the body towards $(1, 0)$ if $\ddot{y}(t_0)/\ddot{x}(t_0) > y(t_0)/x(t_0)$, towards $(0, 1)$ if $\ddot{y}(t_0)/\ddot{x}(t_0) < y(t_0)/x(t_0)$, but no force acts on the body if neither inequality holds.

(d) $\ddot{x}(t_0) > 0$ and $\ddot{y}(t_0) > 0$, the motion is impossible.

Proof. Differentiating the constraints, we obtain $x\ddot{x} + y\ddot{y} = -(\dot{x}^2 + \dot{y}^2) \leq 0$, which means that the force acting on m is directed along the tangent at m to the circle $z = 0$ or inside the half-plane containing this circle. Assuming that an xy-coordinate system is fixed at the origin of the acceleration vector, this vector lies in the half-plane below the line of slope $-x(t_0)/y(t_0)$. It is now a simple exercise to prove each point of the theorem separately. We only detail (c), since it's a bit more involved.

If $\ddot{x}(t_0) \leq 0$ and $\ddot{y}(t_0) \leq 0$, the force acting on m is a vector in the third quadrant. Its direction depends on whether the acceleration vector lies: (i) below the line of slope $y(t_0)/x(t_0)$ (PB is below OP in Figure 1); (ii) above it (PC is above OP); or (iii) on it (on the line OP). Case (iii) includes the possibility of zero acceleration.

In case (i), the acceleration vector lies on a line whose slope is larger than $y(t_0)/x(t_0)$, i.e. $\ddot{y}(t_0)/\ddot{x}(t_0) > y(t_0)/x(t_0)$, so the force pulls m toward $(1, 0)$. In case (ii), the acceleration vector lies on a line of slope that is smaller than $y(t_0)/x(t_0)$, i.e. $\ddot{y}(t_0)/\ddot{x}(t_0) < y(t_0)/x(t_0)$, so the force pulls m toward $(0, 1)$. In case (iii), the acceleration vector is either zero or lies on the line of slope $y(t_0)/x(t_0)$, i.e. $\ddot{y}(t_0)/\ddot{x}(t_0) = y(t_0)/x(t_0)$. But the latter alternative never happens. This fact follows from the equations of motion (2), which show that the acceleration is the difference between the gradient of the force function and a multiple of the position vector. But according to Euler's formula for homogeneous functions (see (3) in Part I) and the fact that the velocities are zero, these vectors are orthogonal, so their difference can have the same direction as one of them only if it is zero. This vectorial argument agrees with the kinematic facts, which show that if $\dot{x}(t_0) = \dot{y}(t_0) = 0$ and the acceleration has the same direction as the position vector, then m doesn't move, so $\dot{x}(t) = \dot{y}(t) = 0$, and therefore $\ddot{x}(t) = \ddot{y}(t) = 0$ for all t. In particular, this means that when $\ddot{y}(t_0) = \ddot{x}(t_0) = 0$, no force acts on m, so the body remains fixed. □

5. Main result

We next prove the existence of solutions with collision-antipodal singularities, solutions repelled from collision-antipodal singularities in positive time, as well as solutions that remain analytic at a collision-antipodal configuration. So the dynamics of $\mathbf{\Delta}^+ \cap \mathbf{\Delta}^-$ is more complicated than the dynamics of $\mathbf{\Delta}^+$ and $\mathbf{\Delta}^-$ away from the intersection, since orbits can go towards or away from this set for $t > 0$, and can even avoid singularities. This result represents a first example of a noncollision singularity reached by only three bodies as well as a first example of a nonsingular collision in celestial mechanics.

Theorem 1. *Consider the* 3*-body problem in* $\mathbf{S}^2$ *with the bodies* m_1 *and* m_2 *having mass* $M > 0$ *and the body* m_3 *having mass* $m > 0$*. Then*

(i) there are values of m *and* M*, as well as initial data, for which the solutions end in finite time in a collision-antipodal singularity;*

(ii) other choices of masses and initial data lead to solutions that are repelled from a collision-antipodal singularity;

(iii) and yet other choices of masses and initial data correspond to solutions that reach a collision-antipodal configuration but remain analytic at this point.

Proof. We start with some initial data we will refine on the way. During the refinement process, we will also choose suitable masses. Consider

$$
\begin{aligned}
x_1(0) &= -x(0), & y_1(0) &= y(0), & z_1(0) &= 0,\\
x_2(0) &= x(0), & y_2(0) &= y(0), & z_2(0) &= 0,\\
x_3(0) &= 0, & y_3(0) &= -1, & z_3(0) &= 0,
\end{aligned}
$$

and zero initial velocities, where all $0 < x(t), y(t) < 1$ are functions with $x^2 + y^2 = 1$. Since all z coordinates are zero, only the equations of coordinates x and y play a role in the motion. The symmetries imply that m_3 stays fixed, the angular momentum is zero, and it is enough to study m_2. Substituting the above initial conditions into the equations of motion, we obtain

$$
\ddot{x}(0) = -\frac{y(0)}{x^2(0)}\left(\frac{M}{4y^2(0)} - m\right) \quad \text{and} \quad \ddot{y}(0) = \frac{1}{x(0)}\left(\frac{M}{4y^2(0)} - m\right). \tag{5}
$$

Several situations occur, depending on the choice of masses and initial data. Here are two significant possibilities.

1. For $M \geq 4m$, it follows that $\ddot{x}(0) < 0$ and $\ddot{y}(0) > 0$ for any choices of initial positions with $0 < x(0), y(0) < 1$.
2. For $M < 4m$, there are initial positions for which:
   (a) $\ddot{x}(0) < 0$ and $\ddot{y}(0) > 0$,
   (b) $\ddot{x}(0) > 0$ and $\ddot{y}(0) < 0$,
   (c) $\ddot{x}(0) = \ddot{y}(0) = 0$.

In 2(c), the orbits are fixed points, such as when $M = 2m$ and $x(0) = y(0) = \sqrt{2}/2$. Of interest for us are 1 and 2(b). In 1, m_2 moves from rest towards a collision with m_1 at $(0, 1)$, but whether this collision takes place also depends on velocities. In 2(b), m_2 moves away from the same collision, and we need to see again how the velocities alter

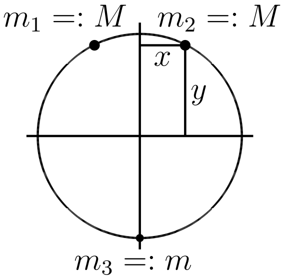

FIGURE 2. The initial positions of m_1, m_2, and m_3 on the geodesic $z = 0$.

this tendency. For arbitrary M and m, the equations of motion are

$$\begin{cases} \ddot{x} = -\frac{M}{4x^2y} + \frac{my}{x^2} - (\dot{x}^2 + \dot{y}^2)x \\ \ddot{y} = \frac{M}{4xy^2} - \frac{m}{x} - (\dot{x}^2 + \dot{y}^2)y \end{cases} \tag{6}$$

and the energy integral is

$$\dot{x}^2 + \dot{y}^2 = \frac{h}{M} - \frac{2my}{x} + \frac{M(2y^2 - 1)}{2xy}.$$

Substituting this expression of $\dot{x}^2 + \dot{y}^2$ into system (6), we obtain

$$\begin{cases} \ddot{x} = \frac{4(M-2m)x^4 - 2(M-2m)x^2 - M + 4m}{4x^2y} - \frac{h}{M}x \\ \ddot{y} = \frac{M + 2(M-2m)y^2 - 4(M-2m)y^4}{4xy^2} - \frac{h}{M}y. \end{cases} \tag{7}$$

We further focus on the first class of orbits announced in this theorem.

(i) Take $M = 8m$, which brings system (7) to the form

$$\begin{cases} \ddot{x} = \frac{6mx^2}{y} - \frac{3m}{y} - \frac{m}{x^2y} - \frac{h}{8m}x \\ \ddot{y} = \frac{2m}{xy^2} + \frac{3m}{x} - \frac{6my^2}{x} - \frac{h}{8m}y, \end{cases} \tag{8}$$

with the energy integral

$$\dot{x}^2 + \dot{y}^2 + \frac{4mx}{y} - \frac{2my}{x} = \frac{h}{8m}. \tag{9}$$

Then, as $x \to 0$ and $y \to 1$, both $\ddot{x}$ and $\ddot{y}$ tend to $-\infty$, so they are ultimately negative, a fact corresponding to Lemma 1(c). But a simple computation shows that $\ddot{y}/\ddot{x}$ tends to zero as $x \to 0$ and $y \to 1$. Since $y/x > 0$, it follows that if $(x(0), y(0))$ is chosen close enough to $(0, 1)$, then $\ddot{y}(0)/\ddot{x}(0) < y(0)/x(0)$, so according to Lemma 1(c) the collision-antipodal configuration is reached. As the forces and the potential are infinite at this point, using the energy relation (9) it follows that the velocities are also infinite. Consequently the motion cannot be analytically extended beyond the collision-antipodal configuration, which thus proves to be a singularity.

(ii) Take $M = 2m$. Then equations (7) have the form

$$\begin{cases} \ddot{x} = \frac{m}{2x^2y} - \frac{h}{2m}x \\ \ddot{y} = \frac{m}{2xy^2} - \frac{h}{2m}y, \end{cases} \tag{10}$$

with the energy integral

$$\dot{x}^2 + \dot{y}^2 + \frac{m}{xy} = \frac{h}{2m}, \tag{11}$$

which implies that $h > 0$. Obviously, as $x \to 0$ and $y \to 1$, the forces and the kinetic energy become infinite, so the collision-antipodal configuration is a singularity, if reached. But this cannot happen. Indeed, from 2(c), the initial position $x(0) = y(0) = \sqrt{2}/2$ corresponds to a fixed point of the equations of motion for zero initial velocities. So we must seek the desired solution for initial conditions with $0 < x(0) < \sqrt{2}/2$ and the corresponding choice of $y(0) > 0$. Let's pick initial positions as close to the collision-antipodal singularity as we want and choose zero initial velocities. For $x \to 0$, however, equations (10) show that both $\ddot{x}$ and $\ddot{y}$ grow positive. But by Lemma 1(d), this outcome is impossible, so the motion cannot come infinitesimally close to the corresponding collision-antipodal singularity, which repels any such solution.

(iii) Take $M = 4m$, which brings system (7) to the form

$$\begin{cases} \ddot{x} = \frac{m(2x^2-1)}{y} - \frac{h}{4m}x \\ \ddot{y} = \frac{mx(2y^2+1)}{y^2} - \frac{h}{4m}y \end{cases} \tag{12}$$

with energy integral

$$\dot{x}^2 + \dot{y}^2 + \frac{2mx}{y} = \frac{h}{4m}. \tag{13}$$

We can compute h from the initial data. Thus, for initial positions $x(0), y(0)$ and initial velocities $\dot{x}(0) = \dot{y}(0) = 0$, the energy constant is $h = 8m^2x(0)/y(0) > 0$.

Assuming that $x \to 0$ and $y \to 1$, equations (12) imply that $\ddot{x}(t) \to -m < 0$ and $\ddot{y}(t) \to -h/4m < 0$, which means that the forces are finite at the collision-antipodal configuration. We are thus in the case of Lemma 1(c), so to determine the direction of motion for m_2 when it comes close to $(0,1)$, we need to take into account the ratio $\ddot{y}/\ddot{x}$, which tends to $h/4m^2$ as $x \to 0$. Since $h = 8m^2x(0)/y(0)$, $\lim_{x\to 0}(\ddot{y}/\ddot{x}) = 2x(0)/y(0)$. Then $2x(0)/y(0) < y(0)/x(0)$ for any $x(0)$ and $y(0)$ with $0 < x(0) < 1/\sqrt{3}$ and the corresponding choice of $y(0) > 0$ given by the constraint $x^2(0) + y^2(0) = 1$. But the inequality $2x(0)/y(0) < y(0)/x(0)$ is equivalent to the condition $\ddot{y}(t_0)/\ddot{x}(t_0) < y(t_0)/x(t_0)$ in Lemma 1(c), according to which the force pulls m_2 toward $(0,1)$. Therefore the velocity and the force acting on m_2 keep this body on the same path until the collision-antipodal configuration occurs.

It is also clear from equation (13) that the velocity is positive and finite at collision. Since the distance between the initial position and $(0,1)$ is bounded, m_2 collides with m_1 in finite time. Therefore the choice of masses with $M = 4m$, initial positions $x(0), y(0)$ with $0 < x(0) < 1/\sqrt{3}$ and the corresponding value of $y(0)$, and initial velocities $\dot{x}(0) = \dot{y}(0) = 0$, leads to a solution that remains analytic at the collision-antipodal configuration, so the motion naturally extends beyond this point. □

Acknowledgments. This research has been supported in part by grants from NSERC (Canada) and CONACYT (Mexico).

References

[1] F. Diacu, Painlevé's conjecture, *Math. Intelligencer* **15**, 2 (1993), 6-12.

[2] F. Diacu, Near-collision dynamics for particle systems with quasihomogeneous potential laws, *J. Differential Equations* **128** (1996), 58-77.

[3] F. Diacu, On the singularities of the curved n-body problem, *Trans. Amer. Math. Soc.* **363**, 4 (2011), 2249-2264.

[4] F. Diacu, Singularities of the N-body problem, in *Classical and Celestial Mechanics: The Recife Lectures*, H. Cabral and F. Diacu, eds., Princeton University Press, Princeton, N.J., 2002, pp. 35-62.

[5] F. Diacu, The nonexistence of linear-momentum and centre-of-mass integrals in the curved N-body problem (in preparation).

[6] F. Diacu and P. Holmes, Celestial Encounters: The Origins of Chaos and Stability, Princeton Univ. Press, Princeton, N.J., 1995.

[7] F. Diacu, E. Pérez-Chavela, and M. Santoprete, The n-body problem in spaces of constant curvature. Part I: Relative equilibria (simultaneous submission to this journal).

[8] R. McGehee, Triple collision in the collinear three-body problem, *Invent. Math.* **27** (1974), 191-227.

[9] R. McGehee, Von Zeipel's theorem on singularities in celestial mechanics, *Expo. Math.* **4** (1986), 335-345.

[10] P. Painlevé, *Leçons sur la théorie analytique des équations différentielles*, Hermann, Paris, 1897.

[11] E. Pérez-Chavela and J.G. Reyes Victoria, An intrinsic approach in the curved N-body problem. The positive curvature case, *Trans. Amer. Math. Soc.* (to appear).

[12] H. von Zeipel, Sur les singularités du problème des n corps, *Ark. Mat. Astr. Fys.* **4**, 32 (1908), 1-4.

[13] Z. Xia, The existence of noncollision singularities in the N-body problem, *Ann. of Math.* **135** (1992), 411-468.