N-Body Problem

Encyclopedia of Nonlinear Science

by Florin Diacu

The Newtonian N-body problem of celestial mechanics is a mathematical generalization of the solar system. In the Euclidean space $\mathbb{R}^3$ consider N points of masses $m_i > 0$, $i = 1, \dots, N$, which attract each other by a force directly proportional with the product of the masses and inversely proportional with the square of the distance. The equations of motion are given by the $6N$-dimensional system of differential equations

$$\begin{cases} \dot{\mathbf{q}} = M^{-1}\mathbf{p} \\ \dot{\mathbf{p}} = G\nabla U(q), \end{cases} \tag{1}$$

where the upper dot represents differentiation with respect to the time variable, $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N)$ is the *configuration* of the particle system, with $\mathbf{q}_i = (q_i^1, q_i^2, q_i^3)$ giving the coordinates of the point of mass m_i, $\mathbf{p} = M\dot{\mathbf{q}}$ is the *momentum*, where M is a $3N$-dimensional square matrix having on the diagonal the elements $m_1, m_1, m_1, \dots, m_N, m_N, m_N$ and zeros in rest, G is the gravitational constant, and $U(\mathbf{q}) = \sum_{1\le i<j\le N} \frac{m_i m_j}{|\mathbf{q}_i - \mathbf{q}_j|}$ is the *potential function*, $-U(\mathbf{q})$ representing the *potential energy*. Standard results of the theory of differential equations ensure the existence and uniqueness of an analytic solution for any initial value problem as long as the initial data do not belong to the *collision set* $\Delta = \bigcup_{1\le i<j\le N} \Delta_{ij}$, where $\Delta_{ij} = \{\mathbf{q} \in \mathbb{R}^{3N} \mid \mathbf{q}_i = \mathbf{q}_j\}$.

Newton formulated this problem in his master work *Principia*, but Euler was the first who wrote the equations as we know them now. The case $N = 2$, also called the *Kepler problem*, is completely solved (see [A] for a recent discussion of some early solutions). The relative motion of one body with respect to the other is planar and, depending on the initial conditions, it can be a circle, an ellipse, a parabola, a branch of a hyperbola, or a line, case in which a collision takes place in the future or in the past. Kepler's laws (see the "celestial mechanics" entry) can be recovered from system (1).

For $N \geq 3$ very little is known about the N-body problem in spite of thousands of research papers written over more than three centuries. The case $N = 3$ was the most studied since many of its results obtained can be generalized to any larger N. The first attempts to understand the 3-body problem were quantitative, aiming at finding explicit solutions. In 1767 Euler found the collinear periodic orbits, in which three bodies of any masses move such that they oscillate along a rotating line [E] (Figure 1) and in 1772 Lagrange discovered some periodic solutions which lie at the vertices of a rotating equilateral triangle that shrinks and expands periodically [L] (Figure 2). Those solutions led to the study of *central configurations*, for which $\mathbf{q}'' = k\mathbf{q}$ for some constant $k > 0$. Each central configuration provides a class of periodic orbits.

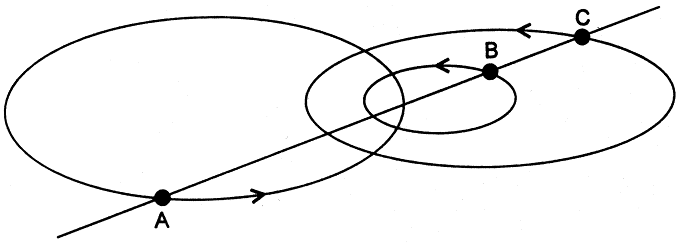

Figure 1. The Eulerian solutions of the 3-body problem

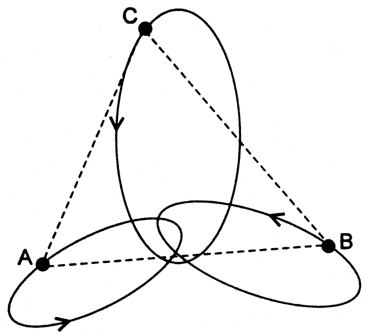

Figure 2. The Lagrangean solutions of the 3-body problem

Another idea was to reduce the order of the system with the help of first integrals. Ten linearly independent integrals are known: three for the centre of mass, three for the momentum, three for the angular momentum, and one for the energy (see [W]). Together with a certain symmetry, these integrals allow the reduction of the 3-body problem from dimension 18 to 7. But unfortunately the dimension of the problem cannot be further reduced. In 1887 Bruns proved that there exist no more linearly independent integrals, algebraic with respect to $\mathbf{q}$, $\mathbf{p}$ and t [Br], thus showing the limitations of the quantitative methods. This led Poincaré to attempt a qualitative approach. His first prolific ideas appeared in a memoir published in 1890, [P], for which he was awarded the King Oscar Prize (see [B] and [DH]). There he laid the foundations of several branches of mathematics, including dynamical systems, chaos, KAM theory, and algebraic topology.

Poincaré aimed to understand the geometry of the phase space and the relative behaviour of orbits and thus to answer questions regarding stability, asymptotic motion at infinity, existence of periodic orbits, etc. An important problem in this direction was that of singular solutions, i.e., those that tend to the *collision set* Δ. It took mathematicians almost a century to prove that for $N \geq 5$ there exist singular solutions that do not end in collisions but in *pseudocollisions*, which are orbits that become unbounded in finite time (see [DH]). For $N = 4$ the problem is still open.

Recently a lot of interest has been shown to finding *choreographies*, i.e., periodic orbits for which all the bodies move on the same closed curve. For more than two centuries the only known example was the class of Lagrangean solutions in the particular case that all the masses are equal and the ellipses are circles. With the help of variational methods, in 2000 a spectacular new class was proved to exist: three bodies of equal mass chase each other along a curve resembling the figure eight (Figure 3) [CM], [M]. There is numerical evidence that this periodic orbit is KAM stable, i.e., the solutions through most initial conditions in some sufficiently small neighbourhood of the orbit stay close to it for all time, while the other solutions leave the neighbourhood very slowly. Unfortunately the stability region seems to be very small. Numerical experiments suggest that the probability of finding an eight in the universe is somewhere between one per galaxy and one per universe. Hundreds of other choreographies have been numerically put into the evidence.

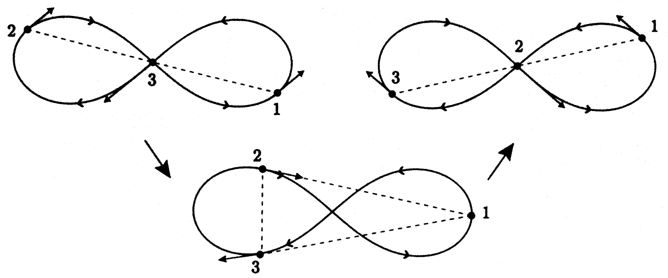

Figure 3. The figure eight solution of the 3-body problem

Still far from fully understood are the questions regarding various restricted 3-body problems. In the elliptic one, for example, it is asked to determine the motion of one body, assumed to have zero mass, while the other two move on ellipses like in the Kepler problem with negative energy.

Numerical methods are also of help for getting insight into the problem. But due to the apparently chaotic character of the motion, they must be implemented with care. Recently a lot of progress has been made in successfully applying scientific computation to various aspects of the general and restricted 3-body problem.

Many of the ideas of the classical N-body problem can be adapted to related problems for understanding the motion of particle systems given by other potentials, like those of Manev and Schwarzschild (also used in celestial mechanics), Coulomb (atomic and molecular theories), Lennard-Jones (crystal formation), etc. Based on the Coulomb potential, Bohr's model of the hydrogen atom led to the development of quantum mechanics. Several other branches of science have profited from the study of the N-body problem.

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