COLOURINGS OF m-EDGE-COLOURED GRAPHS AND SWITCHING

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Abstract. Graphs with m disjoint edge sets are considered, both in the presence of a switching operation and without one. The operation of switching at a vertex x with respect to a finite permutation group Γ involves using some group element to change the sets to which the edges incident with the vertex x belong. We show that some of the colouring theory developed for signed graphs, and for oriented graphs with an operation that reverses all arcs incident with a vertex, can be carried over to the more general situation of m edge sets and switching with respect to an arbitrary permutation group. Other parts of the theory are shown to carry over to switching with respect to an Abelian permutation group. Vertex colourings of these graphs are also considered. Adjacent vertices must be assigned different colours, and any pair of colours can appear on the ends of edges in at most one of the edge sets. We show that versions of Brooks' Theorem similar to those known for oriented graphs hold. The complexity of the problem of deciding whether a sequence of switches with respect to an Abelian group transforms a given graph into one which is k-colourable is determined.

1. Introduction

There is a weak duality between vertex colourings of oriented graphs and vertex colourings of 2-edge-coloured graphs: statements that hold for one family often also hold for the other with more or less the same proof. For examples, see [11, 16]. The Ph.D. Thesis of S. Sen [16] contains examples where results that hold for oriented graphs do not hold for 2-edge-coloured graphs. For example, the maximum value of the oriented chromatic number of an orientation of P_5 is three, while there are 2-edge-colourings of P_5 that have chromatic number four.

A connection betweenoriented graphs and 2-edge-coloured graphs arises through the (n, m)-*coloured mixed graphs* introduced by Nešetřil and Raspaud [13]. Informally, these are graphs with n arc sets and m edge sets, all of which are disjoint. Oriented graphs are $(1, 0)$-coloured

mixed graphs, and 2-edge-coloured graphs are $(0, 2)$-coloured mixed graphs. Theorems that hold for (n, m)-coloured mixed graphs hold for subfamilies, and methods which prove such results can be applied to subfamilies. See [10] for an example. Conversely, if a statement holds for both 2-edge-coloured graphs and oriented graphs with essentially the same proof, then there is some evidence that the method may apply to (n, m)-coloured mixed graphs. For example, results in [1, 15] are shown to hold for (n, m)-mixed graphs in [13].

In this paper, we consider $(0, m)$-coloured mixed graphs – graphs with m disjoint edge sets. These are more commonly called m-edge-coloured graphs; the edges belonging to the i-th edge set are referred to as the edges of colour i. One of our main goals is to extend results about vertex colourings that hold for oriented graphs, some of which are also known to hold for 2-edge-coloured graphs, into this more general framework. We show that results similar to those of Sopena [17], and of Kostochka, Sopena and Zhu [11], hold for m-edge-coloured graphs. The result similar to Sopena's bounds the minimum number of colours needed in a vertex colouring by demonstrating the existence of a homomorphism to a particular target m-edge-coloured graph which is a special case of those described by Nešetřil and Raspaud [13]. The result similar to Kostochka, Sopena and Zhu's is stronger, and is proved using the probabilistic method. Both of these can be regarded as versions of Brooks' Theorem.

A weak duality also exists between colouring results for *signed graphs*, 2-edge-coloured graphs with an operation that swaps the colours of all edges incident with a vertex (see [14]), and for oriented graphs with the *pushing* operation which reverses the orientation of all arcs incident with a vertex (see [9]). The edge-colour switching operation has been extended to m-edge-coloured graphs by Brewster and Graves [3]. In their work, the colours of all edges incident with a vertex are switched using a fixed cyclic permutation. Another of our main goals is to extend the results about switching to arbitrary permutation groups, so that switching at a vertex x means choosing a group element and using it to permute the colours of all edges incident with x.

Results similar to those of Klostermeyer and MacGillivray [9], and of Brewster and Graves [3], hold when switching is with respect to an Abelian group, and not when switching is with respect to a non-Abelian group [12]. These results make it easy to relate the minimum number of colours needed in a vertex colouring in the presence of switching to the number needed without switching. They also make it possible to

determine the complexity of deciding whether there exists a sequence of switches that transforms a given m-edge-coloured graph into one which is k-colourable for all Abelian groups.

The following is an outline of the paper. Definitions and other preliminaries are contained in the next section. In the two sections that follow, vertex colourings of m-edge-coloured graphs (without switching) are considered, and the results mentioned above are proved. Section 5 introduces the operation of switching at a vertex x with respect to an arbitrary finite permutation group Γ. We show that some some aspects of the theory of colourings and homomorphisms of signed graphs, and also oriented graphs with the pushing operation, carry over to switching with respect to arbitrary groups. In Section 6 we present the aspects of the theory which we are able to carry over to switching with respect to Abelian permutation groups, as well as a dichotomy theorem for the complexity of the vertex k-colouring problem.

2. Definitions and preliminaries

For basic definitions in graph theory, see the text by Bondy and Murty [2]. For results about graph colourings and homomorphisms, see the book by Hell and Nešetřil [8]. And for basic results in abstract algebra, see the text by Gallian [6].

Let $m \geq 1$ be an integer. An *m-edge-coloured graph* is an $(m+1)$-tuple

$$G = (V(G), E_0(G), E_1(G), \ldots, E_{m-1}(G)),$$

where

(i) $V(G)$ is a set of objects called vertices;
(ii) for $0 \leq i \leq m-1$, $E_i(G)$ is a set of unordered pairs of distinct vertices called the *edges of colour i*; and
(iii) $E_i(G) \cap E_j(G) = \emptyset$ when $i \neq j$.

An m-edge-coloured graph can be regarded as arising from a simple graph by assigning to each edge one of the m possible colours $0, 1, \ldots, m-1$. When the context is clear, the vertex set is referred to as V and the edge sets as $E_0, E_1, \ldots, E_{m-1}$.

If G is an m-edge-coloured graph, then the *underlying graph of G* is the graph *underlying*(G) with vertex set $V(G)$ and edge set $E_0(G) \cup E_1(G) \cup \cdots \cup E_{m-1}(G)$. One can think of obtaining it from G by removing the edge colours.

Let G and H be m-edge-coloured graphs. A *homomorphism of G to H* is a function $h : V(G) \to V(H)$ such that $xy \in E_i(G)$ implies $f(x)f(y) \in E_i(H)$, that is, a mapping of the vertices of G to the vertices of H that preserves adjacency in each edge colour. We use the notation $h : G \to H$ to mean that h is a homomorphism of G to H. When the name of the function is unimportant, this is abbreviated as $G \to H$.

For an integer $k \geq 1$, a *vertex k-colouring* of an m-edge-coloured graph G is a homomorphism of G to some m-edge-coloured graph on k vertices. Since non-adjacent vertices of H can be joined by an edge of any colour without destroying the existence of such a mapping, *underlying*(H) can be assumed to be complete.

If $G \to H$ and the k vertices of H are regarded as colours, then k colours are being assigned to the vertices of G in such a way that vertices joined by an edge of a particular colour are assigned colours joined by an edge of that colour. Adjacent vertices are assigned different colours and the vertex colours respect the edge sets in the sense that each pair of different colours can appear on the ends of edges in at most one set. This is made precise below.

An equivalent definition of a vertex k-colouring is as a function $f : V \to \{1, 2, \ldots, k\}$ such that

(i) $f(x) \neq f(y)$ whenever there exists i such that $xy \in E_i$; and
(ii) if $c_1, c_2 \in \{1, 2, \ldots, k\}$ with $c_1 \neq c_2$ then there exists at most one subscript i for which there is an edge $xy \in E_i$ such that $f(x) = c_1$ and $f(y) = c_2$.

An m-edge-coloured graph is 1-colourable if and only if it has no edges, and 2-colourable if and only if it is *monochromatic* (only one edge set is non-empty) and the underlying graph is bipartite. It is clear that vertex k-colouring of m-edge-coloured graphs is Polynomial when $k \leq 2$ and NP-complete for each $k \geq 3$; NP-completeness follows from restricting the problem to monochromatic m-edge-coloured graphs.

The *chromatic number* of an m-edge-coloured graph G is the least k for which G is k-colourable, and is denoted by χ_e. The choice of the somewhat mysterious subscript e is related to switching operations, and is explained in Section 6.

The chromatic number of an m-edge-coloured graph can differ substantially from that of its underlying graph. For example, consider the 2-edge-coloured complete bipartite graph with vertex set $V = \{a_1, a_2, \ldots, a_n\} \cup \{b_1, b_2, \ldots, b_n\}$ and edge sets $E_0 = \{a_i b_j : i \geq j\}$

and $E_1 = \{a_i b_j : i < j\}$. The underlying graph has chromatic number two, but $\chi_e = 2n = |V|$ because every two non-adjacent vertices are joined by a path of length two that uses an edge from E_0 and an edge from E_1. Any two such vertices must be assigned different colours.

3. Brooks' Theorems

In this section we show that results similar to those of Sopena [17], for oriented graphs, and of Kostochka, Sopena and Zhu [11], for oriented graphs and 2-edge-coloured graphs, hold for m-edge-coloured graphs. Two bounds on χ_e are given. The first of these arises from demonstrating the existence of a homomorphism to a particular m-edge-coloured graph which is a special case of those described by Alon and Marshall [1] (also see [13]). The second bound proved using the probabilistic method, similarly as in [11].

Let G be an m-edge-coloured graph. It is known that $\chi_e(G) \leq \frac{m^t-1}{m-1}$, where t is the chromatic number of the square of *underlying*(G) [10]. Note that the integer t could be roughly the square of $\Delta(\textit{underlying}(G))$. Alon and Marshall have proved that if G is an m-edge-coloured graph for which the acyclic chromatic number of *underlying*(G) is at most k, then $\chi_e(G) \leq km^{k-1}$ [1] (also see [13, 15]). This, in turn, leads to the corollary that if G is planar, then $\chi_e(G) \leq 5m^4$.

Let $m, k \in \mathbb{Z}^+$. For $i = 1, 2, \ldots, k$, let

$$V_i = \{(i; c_1, c_2, \ldots, c_k) : c_i = \cdot, \text{ and } 0 \leq c_j \leq m-1, j \neq i)\}.$$

The m-edge-coloured graph $Z_{m,k}$ is defined to have vertex set

$$V(Z_{m,k}) = V_1 \cup V_2 \cup \cdots \cup V_k.$$

The integer i is called the *index* of the vertex $(i; c_1, c_2, \ldots, c_k)$. Vertices with the same index are not adjacent. For indices $i \neq j$ we have

$$(i; a_1, a_2, \ldots, a_k)(j; b_1, b_2, \ldots, b_k) \in E_{a_j + b_i (\text{mod } m)}.$$

Observe that *underlying*$(Z_{m,k})$ is a complete k-partite graph. Hence the subgraph of $Z_{m,k}$ induced by $A = \{v_1, v_2, \ldots, v_\ell\}$ is an m-edge-coloured complete graph if and only if $\ell \leq k$ and no two vertices in A have the same index.

Let $t \geq 0$ be an integer. An m-edge-coloured graph G is said to have *Property* $P_{m,t}$ if, for every m-edge-coloured complete subgraph with vertex set $\{v_1, v_2, \ldots, v_\ell\}$, where $0 \leq \ell \leq t$, and every ℓ-tuple $(i_1, i_2, \ldots, i_\ell) \in \mathbb{Z}_m^\ell$, there exists a vertex $x \in V(G)$ such that $xv_j \in$

E_{i_j}. Informally, G has Property $P_{m,t}$ if every m-edge-coloured complete subgraph of size at most t can be extended in all possible ways.

Theorem 3.1. *Let $m, k \in \mathbb{Z}^+$. The graph $Z_{m,k}$ has Property $P_{m,k-1}$.*

Proof. Let $A = \{v_1, v_2, \ldots, v_\ell\}$ be the vertex set of a complete subgraph of $Z_{m,k}$, where $\ell < k$. Then no two vertices in A have the same index. For $p = 1, 2, \ldots, \ell$ let the vertex $v_p \in A$ be $v_p = (i_p; a_{p_1}, a_{p_2}, \ldots, a_{p_k})$.

Let $(c_1, c_2, \ldots, c_\ell) \in \mathbb{Z}_m^\ell$. Since $\ell < k$, there exists an integer $j \in \{1, 2, \ldots, k\}$ which is not an element of $\{i_1, i_2, \ldots, i_\ell\}$. The required vertex is then $x = (j; b_1, b_2, \ldots, b_k)$, where $b_r \equiv c_r - a_{r_j} (\text{mod } m)$, $1 \leq r \leq \ell$. □

The same arguments as in Sopena [17] now give the following results. For the sake of completeness, proof sketches are included.

Theorem 3.2 (cf. [17], Theorem 4.1)**.** *Let G be an m-edge-coloured graph for which underlying(G) has maximum degree at most k. Then*

(1) $G \to Z_{m,2k-1}$, *and*
(2) $\chi_e(G) \leq (2k-1)m^{2k-2}$.

Proof. Statement (ii) follows from (i), which is proved by induction on $|V|$. The statement is clear if G has one vertex. Suppose it holds for all m-edge-coloured graphs with fewer than n vertices, and consider an m-edge-coloured graph G with n vertices. Let $x \in V$ and denote the neighbours of x in *underlying*(G) by $v_1, v_2, \ldots, v_j$, where $j \leq n$.

By the induction hypothesis, $(G - x) \to Z_{m,2k-1}$. Further, since every vertex v_i has at most $k-1$ neighbours in *underlying*$(G - x)$, the vertices $v_1, v_2, \ldots, v_j$ can be mapped so that no two of them map to a vertex with the same index (there are vertices of at least k different indices available for each of them). Property $P_{m,2k-2}$ implies that the mapping can be extended to x. □

Kostochka, Sopena and Zhu [11] prove the two results below for oriented graphs, and then note that the same argument works for 2-edge-coloured graphs.

Let $I = \{v_1, v_2, \ldots, v_i\}$ be an i-subset of vertices of an m-edge-coloured graph G, and let x be a vertex of G which is adjacent to every vertex in I. Denote by $a_G(x, I)$ the element of $\mathbb{Z}_m^i$ whose j-th coordinate is the colour of the edge xv_j.

Lemma 3.3. *For all integers $k \geq 4$ and $m \geq 3$, there exists an m-edge-coloured graph $H = (V, E_0, E_1, \ldots, E_{m-1})$ on $t = k^2 m^{k+1}$ vertices for which the following statement is true:*

> *For every $i, 1 \leq i \leq k$, for all i-subsets $I \subseteq V$, and for every vector $b \in \mathbb{Z}_m^i$, there exist at least $(k-i)(k-1)+1$ vertices in $v \in V - I$ such that $a_H(v, I) = b$.*

Proof. We shall show that the probability that a random m-edge-coloured complete graph, H, on t vertices, in which each edge belongs to one of $E_0, E_1, \ldots, E_{m-1}$ independently with probability $1/m$, has this property with positive probability.

Let $1 \leq i \leq k$ and I be an i-subset of $V(H)$. Let $b \in \mathbb{Z}_m^i$, and $\mathcal{E}_{I,b}$ be the event that the number of vertices in $x \in V(H-I)$ with $a_H(x, I) = b$ is at most $(k-i)(k-1)$. We want to show that the probability of $\mathcal{E}_{I,b}$ is less than one.

For any fixed vertex $x \in V(H-I)$,e we have $[Pr(a_H(x, I) = b] = m^{-i}$. Since, for different vertices $x, y \in V - I$ the events $a_H(x, I) = b$ and $a_H(y, I) = b$ are independent (and recalling that $t = k^2 m^{k+1}$), we have

$$\begin{aligned}
Pr(\mathcal{E}_{I,b}) &= \sum_{j=0}^{(k-i)(k-1)} \binom{t-i}{j} (m^{-i})^j (1-m^{-i})^{((t-i)-j)} \\
&\leq (1-m^{-i})^t \sum_{j=0}^{(k-i)(k-1)} \frac{t^j}{j!} m^{-ij} (1-m^{-i})^{-(i+j)} \\
&< \left[\frac{m-1}{m-2}\right] e^{-tm^{-i}} \sum_{j=0}^{(k-i)(k-1)} t^j \\
&< e^{-tm^{-i}} t^{(k-i)(k-1)+1}
\end{aligned}$$

The probability that some event $\mathcal{E}_{I,b}$ occurs is at most

$$\begin{aligned}
Pr\left(\bigcup_{I,b} \mathcal{E}_{I,b}\right) &= \sum_{i=1}^{k} \sum_{|I|=i} \sum_{b \in \mathbb{Z}_m^i} Pr(a_T(x, I) = b) \\
&< \sum_{i=0}^{k} \binom{t}{i} m^i e^{-tm^{-i}} t^{(k-i)(k-1)+1} \\
&< \frac{m^m}{m!} \sum_{i=0}^{k} e^{-tm^{-i}} t^{[(k-i)(k-1)+1]+i}
\end{aligned}$$

$$< \quad e^m \sum_{i=0}^{k} e^{-tm^{-i}} t^{[(k-i)(k-1)+1]+i}$$

In the last sum, the ratio of the $(i+1)$-st term to the i-th term is

$$\frac{e^{tm^{-i}} t^{(k-(i+1))(k-1)+1+(i+1)}}{e^{tm^{-(i+1)}} t^{(k-i)(k-1)+1+i}} = \frac{e^{t(m-1)m^{-(i+1)}}}{t^{k-2}} \geq \frac{e^{t(m-1)m^{-k}}}{t^{k-2}}.$$

When $t = k^2 m^{k+1}$ we have

$$\frac{e^{t(m-1)m^{-k}}}{t^{k-2}} = \frac{e^{k^2(m-1)m}}{(k^2 m^{k+1})^{k-2}} = \frac{e^{k^2(m-1)m}}{k^{2k-4} m^{(k+1)(k-2)}}.$$

We show the ratio is greater than m, that is, that

$$e^{k^2(m-1)m} > m k^{2k-4} m^{(k+1)(k-2)}.$$

On taking logs of both sides, we see that this is true if and only if $k^2(m-1)m > (2k-4)\ln(k) + [(k+1)(k-2)+1]\ln(m)$, which holds for all integers m and k satisfying the hypothesis.

Hence,

$$\begin{aligned} Pr\left(\bigcup_{I,b} \mathcal{E}_{I,b}\right) &< e^m \sum_{i=0}^{k} e^{-tm^{-i}} t^{[(k-i)(k-1)+1]+i} \\ &< (m/(m-1))\, e^m e^{-tm^{-k}} t^{1+k} \\ &= (m/(m-1))\, e^m e^{-mk^2} (k^2 m^{k+1})^{1+k} \\ &= (m/(m-1))\, e^{m(1-k^2)} k^{2+2k} m^{(1+k)^2}. \end{aligned}$$

In order for the probability to be less than one, it is required that $(m-1)e^{m(k^2-1)} \geq m k^{2+2k} m^{(1+k)^2}$, or equivalently (after taking logs and rearranging) that

$$\frac{1}{k+1}[\ln(m) - \ln(m-1)] + 2[\ln(k) + \ln(m)] \leq (k-1)[m - \ln(m)],$$

which holds for $k \geq 4$ and $m \geq 3$.

This completes the proof. □

The desired bound then follows using a virtually identical argument as in [11]. A proof sketch is included for completeness.

Theorem 3.4. *Let G be an orientation of a graph with maximum degree Δ. Then $\chi_e \leq m\Delta^2 m^{\Delta}$.*

Proof. For $k \leq 3$, the statement follows from Theorem 3.2, so assume $k \geq 4$. Let H be the m-edge-coloured graph whose existence is asserted by Lemma 3.3.

Suppose $V(G) = \{v_1, v_2, \ldots, v_n\}$. For $t = 1, 2, \ldots, n$, let G_t be the subgraph of G induced by $\{v_1, v_2, \ldots, v_t\}$. Define a homomorphism $f_n : G \to H$ inductively, as follows. Suppose $f_t : G_t \to H$ has the property that if $v_j, v_k \in V(G_i)$ have a common neighbour in *underlying*(G), then $f_t(v_j) \neq f_t(v_k)$.

Suppose v_{t+1} is adjacent in *underlying*(G) to $w_1, w_2, \ldots, w_i \in V(G_t)$. Let $I_{t+1} = \{w_1, w_2, \ldots, w_i\}$, $a_H(v_{t+1}, I) = b$, and $X_{t+1} = \{x \in V(H) - I_{t+1} : a_H(x, I_{t+1}) = b\}$.

By Lemma 3.3, $|X_{t+1}| \geq 1 + (\Delta - i)(\Delta - 1)$. Let Y_{t+1} be the set of vertices in $\{v_{t+1}, v_{t+2}, \ldots, v_n\}$ which are adjacent in G to v_{t+1}. Then $|Y_{t+1}| \leq \Delta - i$. Let Z_{t+1} be the set of vertices G_{t+1} which are adjacent to a vertex in Y_{t+1}. Then $|Z_{t+1}| \leq (\Delta - 1)(\Delta - i)$. Thus there is a vertex $z_{t+1} \in X_{t+1} - f_t(Z_{t+1})$ (recall that vertices with a common neighbour have different images under f_t). Extend f_t to a homomorphism $f_{t+1} : G_{t+1} \to H$ by setting $f_{t+1}(v_{t+1}) = z_{t+1}$. The function f_{t+1} has the desired property. The result follows. □

4. Switching with respect to an arbitrary group

In this section we show that results of Klostermeyer and MacGillivray [9] for oriented graphs also hold for m-edge-coloured graphs (also see [3]).

Let G be an m-edge-coloured graph and let Γ be a group of permutations of $\{0, 1, \ldots, m-1\}$. For $\pi \in \Gamma$ and $x \in V(G)$, the operation of *switching at x with respect to π* transforms G into the m-edge-coloured graph G' with the same vertex set as G and edge sets $E_i(G') = (E_i(G) - \{xy : xy \in E_i(x)\}) \cup \{xz : xz \in E_j(x) \text{ and } \pi(j) = i\}, 1 \leq i \leq m$. The permutation π acts on the edges incident with x by possibly altering the sets to which they belong. For each i, it changes the edges of colour i incident with x to edges of colour $\pi(i)$.

Let G be an m-edge-coloured graph and Γ a group of permutations of $\{0, 1, \ldots, m-1\}$. Two m-edge-coloured graphs G and H on the same vertex set are *switch equivalent with respect to* Γ if there is a finite sequence of m-edge-coloured graphs, $G_0, G_1, G_2, \ldots, G_k$ where $G_0 = G$, $G_k \cong H$, and for $i = 1, 2, \ldots, k$, the graph G_i is obtained from G_{i-1} by switching at some vertex x_i with respect to a permutation

$\pi_i \in \Gamma$. Switch equivalence with respect to Γ is an equivalence relation on the set of all m-edge-coloured graphs. The equivalence class of an m-edge-coloured graph G with respect to switching with respect to Γ is denoted by $[G]_\Gamma$.

We note that, if Γ is non-Abelian, then the order in which switches are applied at vertices of G matters.

When Γ is Abelian, the order in which switches are applied at the vertices of G does not matter. We can associate a permutation $\pi_x \in \Gamma$ with each vertex $x \in V(G)$ and regard all switches as occurring simultaneously (for some vertices x, possibly π_x is the identity element). Let Π be the function that associates with each vertex of G a permutation $\pi_x \in \Gamma$. Define $\Pi(G)$ to be the graph with the same vertex set as G and edge sets defined by the rule: $x_i x_j \in E_\ell(G)$ if and only if $x_i x_j \in E_{\pi_i \pi_j(\ell)}(\Pi(G))$. We will show below that the $\Pi(G)$ encodes all graphs which are switch equivalent to G with respect to Γ, and thus the switching equivalence class of G with respect to Γ. Two m-edge-coloured graphs G and H are switch equivalent with respect to an Abelian group of permutations if and only if there exists $\Pi : V(G) \to \Gamma$ such that $H \cong \Pi(G)$.

If $m = 2$ and $\Gamma = S_2$, then the equivalence classes under switching with respect to Γ correspond to signed graphs (e.g. see [14]).

Let G and H be m-edge-coloured graphs, and let Γ be a group of permutations of $\{0, 1, \ldots, m-1\}$. A Γ-*switchable homomorphism* of G to H is a homomorphism of some $G' \in [G]_\Gamma$ to H. If such a mapping exists, that is, if G can be switched so that the resulting m-edge-coloured graph has a homomorphism to H, then we say G is Γ-*switchably homomorphic to* H and write $G \to_\Gamma H$. It is possible that every switch is with respect to the identity element, so if $G \to H$, then $G \to_\Gamma H$.

The proof of the following proposition is clear, and omitted.

Proposition 4.1. *Let G and H be m-edge-coloured graphs, and let $\Gamma \leq S_m$. Then*

(1) *if $G \cong H$ then $G \in [H]_\Gamma$;*
(2) *if $G \in [H]_\Gamma$, then $G \to_\Gamma H$;*
(3) *if $G \to H$, then $G \to_\Gamma H$;*
(4) *If F is a subgraph of G and $G \to_\Gamma H$, then $F \to_\Gamma H$.*

We now show that switchable homomorphisms compose.

Proposition 4.2. *Let G, H, and F be m-edge-coloured graphs and $\Gamma \leq S_m$. If $G \to_\Gamma H$ and $H \to_\Gamma F$, then $G \to_\Gamma F$.*

Proof. Let $G' \in [G]_\Gamma$ be such that $g : G' \to H$ is a homomorphism, and $H' \in [H]_\Gamma$ be such that $h : H' \to F$ is a homomorphism. We show that there exists $G'' \in [G]_\Gamma$ such that $(h \circ g) : G'' \to F$.

By hypothesis, there is a sequence of permutations $\pi_1, \pi_2, \ldots, \pi_k$ which, when applied in turn at the not necessarily distinct vertices $x_1, x_2, \ldots, x_k$, transforms H into H'. Let G'' be the m-edge-coloured graph arising from applying π_1 to all elements of $g^{-1}(x_1)$ in $V(G')$, then π_2 to all elements of $g^{-1}(x_2)$ in $V(G')$, and so on. For any $x \in V(H)$, the set $g^{-1}(x)$ is independent in G', so the order in which the switches are applied to its elements does not matter.

We claim that $(h \circ g) : G'' \to H'$ is a homomorphism. Suppose $xy \in E_i(G')$ and $xy \in E_j(G'')$. We know $g(x)g(y) \in E_i(H)$. The edge xy is acted on by exactly the same permutations, in exactly the same order, in the formation of G'' from G' as is the edge $g(x)g(y)$ in the formation of H' from H. Therefore $g(x)g(y) \in E_j(H')$ and, since h is a homomorphism, $h(g(x))h(g(y)) \in E_j(F)$. so that $h \circ g$ is a homomorphism of G'' to F. □

Theorem 4.3. *Let G and H be m-edge-coloured graphs such that $G \to H$, and let Γ be a group of permutations of $\{0, 1, \ldots, m-1\}$.*

(1) *for any $G' \in [G]_\Gamma$ we have $G' \to_\Gamma H$; and*
(2) *for any $H' \in [H]_\Gamma$ there exists $G' \in [G]_\Gamma$ such that $G' \to H'$.*

Proof. We first prove (1). Since $G' \in [G]_\Gamma$, there exists a sequence of switches which transforms G' to G. By hypothesis, $G \to H$. The statement now follows from Propositions 4.1 and 4.2.

We now prove (2). Let $h : G \to H$ be a homomorphism. Since $H' \in [H]_\Gamma$, there exists a sequence of permutations $\pi_1, \pi_2, \ldots, \pi_k$ which, when applied in turn at the not necessarily distinct vertices $x_1, x_2, \ldots, x_k$, transforms H to H'. The result now follows from the same argument as in Proposition 4.2 by letting G' be the m-edge-coloured graph arising from applying π_1 to all elements of $h^{-1}(x_1)$, then π_2 to all elements of $h^{-1}(x_2)$, and so on; the mapping $h : G' \to H'$ is a homomorphism. □

Notice that the statement: *"if $G \to H$ is a homomorphism and $G' \in [G]_\Gamma$, then there exists $H' \in [H]_\Gamma$ such that $G' \to H'$."* is false. Consider the 2-edge coloured graph G consisting of four vertices and two edges, so that each vertex has degree 1, and each edge is in E_0. This graph clearly has a homomorphism to the m-edge coloured graph H consisting of two vertices joined by an edge in E_0. Let $\Gamma = S_2$. Let

G' be obtained from G by switching so that one of the edges is in E_1. Then $G' \in [G]_\Gamma$, but there is no $H' \in [H]_\Gamma$ such that $G' \to H'$.

Corollary 4.4. *Let G and H be m-edge-coloured graphs and $\Gamma \leq S_m$. If $G \to_\Gamma H$, then for all $G' \in [G]$ and $H' \in [H]$ we have $G' \to_\Gamma H'$.*

Recall that a homomorphism $f : G \to H$ is called *onto* if it maps $V(G)$ onto $V(H)$, and *complete* if for every i and every edge $uw \in E_i(H)$, there exists an edge $xy \in E_i(G)$ such that $f(x) = u$ and $f(y) = w$. A homomorphism $G \to G$ which is onto is also complete.

Proposition 4.5. *Let G, and H be m-edge-coloured graphs and $\Gamma \leq S_m$. If there are homomorphisms $G \to_\Gamma H$ and $H \to_\Gamma G$, which are both onto, then G and H are switch equivalent with respect to Γ.*

Proof. Let $G' \in [G]_\Gamma$ be such that $g : G' \to H$ is onto, and let $H' \in [H]_\Gamma$ be such that $h : H' \to G$ is onto. It is clear that $|V(G)| = |V(H)|$. Thus, since g and h are onto, each edge of H is the image under g of at most one edge of G', and similarly each edge of G is the image under h of at most one edge of H'. Therefore, the graphs *underlying*(G) and *underlying*(H) have the same number of edges. It follows that both g and h are complete. Therefore, they are isomorphisms. □

5. Switching with respect to an Abelian group

In this section we show that results of Brewster and Graves [3] about switching m-edge-coloured graphs with respect to a cyclic group extend to switching with respect to an Abelian group.

Let G be an m-edge-coloured graph, and $\Gamma \leq S_m$ be Abelian. Let $P_\Gamma(G)$ be the m-edge-coloured graph with vertex set $V(P_\Gamma(G)) = V(G) \times \Gamma$ and edge sets defined by the rule: for each $xy \in E_i(G)$ and $\alpha, \beta \in \Gamma$, the edge $(x, \alpha)(y, \beta) \in E_{\alpha\beta(i)}(P_\Gamma(G))$. We call $P_\Gamma(G)$ the *switch graph of G with respect to Γ*.

If follows from the definition that the subgraph of $P_\Gamma(G)$ induced by the vertex set $\{(x, e) : x \in V(G)\}$ is isomorphic to G, and that if $\pi_x \in \Gamma$ is the permutation associated with $x \in V(G)$, then the subgraph of $P_\Gamma(G)$ induced by $\{(x, \pi_x) : x \in V(G)\}$ is isomorphic to $\Pi(G)$. Thus every element of $[G]_\Gamma$ is an induced subgraph of $P_\Gamma(G)$.

The next proposition follows immediately from the definitions.

Proposition 5.1. *Let $\Gamma \leq S_m$ be Abelian. If G is a subgraph of an m-edge-coloured graph H, then $P_\Gamma(G)$ is a subgraph of $P_\Gamma(H)$.*

Proposition 5.2. *Let G be an m-edge-coloured graph, and $\Gamma \leq S_m$ be Abelian. Then $P_\Gamma(G) \to_\Gamma G$, and at least one such mapping is onto.*

Proof. For each vertex $(w, \gamma) \in V(P_\Gamma(G))$, switch with respect to the γ^{-1}. Suppose $xy \in E_i(G)$. After switching, an edge $(x, \alpha)(y, \beta)$ belongs to the edge set $E_{\alpha\alpha^{-1}\beta\beta^{-1}(i)} = E_i$. Since for each $w \in V(G)$, the set of vertices $\{(w, \gamma) : \gamma \in \Gamma\}$ is independent in $P_\Gamma(G)$, these sets imply a natural onto Γ-switchable homomorphism $P_\Gamma(G) \to_\Gamma G$. □

Brewster and Graves [3] generalized and proved a conjecture in [9] by establishing a result similar to the following in the case where Γ is a cyclic group. It establishes the graph $P_\Gamma(G)$ as a sort of representative of $[G]_\Gamma$ in the sense that each equivalence class gives rise to one well-defined graph, and different equivalence classes give rise to different graphs.

Theorem 5.3. *Let G and H be m-edge-coloured graphs and let $\Gamma \leq S_m$ be Abelian. Then $G \in [H]_\Gamma$ if and only if $P_\Gamma(G)$ is isomorphic to $P_\Gamma(H)$.*

Proof. Suppose $P_\Gamma(G)$ is isomorphic to $P_\Gamma(H)$. A switchable homomorphism of G onto H is obtained by composiing an embedding $G \to P_\Gamma(G)$, an isomorphism $P_\Gamma(G) \to P_\Gamma(H)$, and an onto homomorphism $P_\Gamma(H) \to H$. Similarly, there is a switchable homomorphism of H onto G. Proposition 4.5 now implies that G and H are switch equivalent with respect to Γ.

Now suppose $G \in [H]_\Gamma$. Then there exists $\Pi : V(G) \to \Gamma$ such that $H \cong \Pi(G)$. For simplicity of notation, assume $H = \Pi(G)$ and use π_x to denote the permutation $\Pi(x)$. Recall that if $xy \in E_i(G)$, then $xy \in E_j(H)$, where $j = (\pi_x\pi_y)(i)$

Define $f : V(P_\Gamma(G)) \to V(P_\Gamma(H))$ by $f((x, \alpha)) = (x, \alpha\pi_x^{-1})$. Then f is a bijection between $V(P_\Gamma(G))$ and $V(P_\Gamma(H))$. Since $P_\Gamma(G)$ and $P_\Gamma(H)$ have the same number of edges, it suffices to prove that f is a homomorphism.

Suppose $(x, \alpha)(y, \beta) \in E_i(P_\Gamma(G))$. Then $xy \in E_k(G)$, where $k = (\alpha^{-1}\beta^{-1})(i)$, so that $xy \in E_j(H)$, where $j = (\pi_x\pi_y\alpha^{-1}\beta^{-1})(i)$. Thus $f((x, \alpha))f((y, \beta)) = (x, \alpha\pi_x^{-1})(y, \beta\pi_y^{-1}) \in E_\ell((P_\Gamma(H))$, where $\ell = (\pi_x\pi_y$ $\alpha^{-1}\beta^{-1}\alpha\pi_x^{-1}\beta\pi_y^{-1})(i) = i$. The result follows. □

Theorem 5.4. *Let G and H be m-edge-coloured graphs, and $\Gamma \leq S_m$ be Abelian. Then $G \to_\Gamma H$ if and only if $P_\Gamma(G) \to P_\Gamma(H)$.*

Proof. Suppose $G \to_\Gamma H$, and let $G' \in [G]_\Gamma$ be such that there is a homomorphism $g : G' \to H$. Since $P_\Gamma(G) \cong P_\Gamma(G')$, it is enough to show $P_\Gamma(G') \to P_\Gamma(H)$.

Define $f : V(P_\Gamma(G')) \to V(P_\Gamma(H))$ by $f((w, \phi)) = (g(w), \phi)$. Suppose $(u, \alpha)(v, \beta) \in E_j(P_\Gamma(G'))$. Then $uv \in E_i(G')$, where $i = \alpha^{-1}\beta^{-1}(j)$. Since g is a homomorphism, $g(u)g(v) \in E_i(H)$. Thus, $f((u, \alpha))f((v, \beta)) \in E_j(P_\Gamma(H))$.

Suppose $P_\Gamma(G) \to P_\Gamma(H)$. Then, since $G \to P_\Gamma(G)$ and $P_\Gamma(H) \to_\Gamma H$, we have $G \to_\Gamma H$ by Proposition 4.2. □

Corollary 5.5. *Let $\Gamma \leq S_m$ be Abelian. Then, $G \to_\Gamma H$ if and only if $G \to P_\Gamma(H)$.*

Proof. Suppose $G \to_\Gamma H$. Then by Theorem 5.4 $P_\Gamma(G) \to P_\Gamma(H)$. The inclusion map gives $G \to P_\Gamma(G)$, so by Proposition 4.2, we have $G \to P_\Gamma(H)$.

Suppose $G \to P_\Gamma(H)$. By Proposition 5.2, $P_\Gamma(H) \to_\Gamma H$. The statement now follows from Propositions 4.1 and 4.2. □

A slightly different m-edge-coloured graph plays the role of P_Γ in the paper of Brewster and Graves (they call it the *permutation graph*). Their graph is a subgraph. We will show below that it is, in fact, a retract of P_Γ. While all of the results above can be seen to hold for this subgraph, we have chosen to present them for P_Γ because it is easier to understand and work with.

For each $x \in V(G)$, define a relation $\sim_x$ on Γ by $\alpha \sim_x \beta$ if and only if $\alpha(i) = \beta(i)$ for all i such that there is an edge in E_i incident with x in G. Then each $\sim_x$ is an equivalence relation. Let $S_\Gamma(G)$ be any subgraph of $P_\Gamma(G)$ induced by a set of vertices consisting of a representative of each equivalence class of $\sim_x$ for each $x \in V(G)$. It follows from the definition of the relations $\sim_x$ that any two such subgraphs are isomorphic. We can therefore think of the vertices of $S_\Gamma(G)$ as being labelled by the equivalence classes they represent. When the group Γ is the cyclic group generated by a permutation π, the graph $S_\Gamma(G)$ coincides with the permutation graph defined by Brewster and Graves [3].

Let G and H be m-edge-coloured graphs such that H is a subgraph of G. A *retraction* of G to H is a homomorphism $r : G \to H$ such that $r(x) = x$ for each $x \in V(H)$. If there is a retraction of G to H, then H is called a *retract* of G.

Proposition 5.6. *Let G be an m-edge-coloured graph and $\Gamma \leq S_m$ be Abelian. Then $S_\Gamma(G)$ is a retract of $P_\Gamma(G)$.*

Proof. Let $r : V(P_\Gamma(G)) \to V(S_\Gamma(G))$ be defined by $r((x,\alpha)) = [\alpha]_{\sim_x}$. If $(x,\alpha)(y,\beta) \in E_i(P_\Gamma(G))$, then by the definitions of $S_\Gamma(G)$ and the relations $\sim_x$ we have $[\alpha]_{\sim_x}[\beta]_{\sim_y} \in E_i(S_\Gamma(G))$. □

The m-edge-coloured graph $S_\Gamma(G)$ represents the equivalence classes for switching with respect to an Abelian group $\Gamma \leq S_m$ in the same way as does $P_\Gamma(G)$, that is, as in Theorem 5.3 and the results which follow from it.

Corollary 5.7. *Let G be an m-edge-coloured graph and $\Gamma \leq S_m$ be Abelian. Then every element of $[G]_\Gamma$ is an induced subgraph of $S_\Gamma(G)$.*

6. Switchable vertex k-colouring with respect to an Abelian group

Let G be an m-edge-coloured graph and $\Gamma \leq S_m$. Recall that a Γ-switchable vertex k-colouring of G is a Γ-switchable homomorphism of G to some m-edge-coloured graph on k vertices. The Γ-switchable chromatic number of G is the least k such that G has a Γ-switchable vertex k-colouring, that is,

$$\chi_\Gamma(G) = \min_{G' \in [G]_\Gamma} \chi_e(G').$$

If Γ consists of only the identity element, then $\chi_\Gamma = \chi_{\{e\}} = \chi_e$. This underlies the choice of the subscript e.

Theorem 6.1. *Let Γ be an Abelian group of permutations of $\{0, 1, \ldots, m\}$. For any m-edge-coloured graph G, there is a constant $c \leq |\Gamma|$ such that*

$$\chi_\Gamma \leq \chi_e \leq c \cdot \chi_\Gamma.$$

Proof. By hypothesis, there is a Γ-switchable homomorphism $G \to_\Gamma K$ for some m-edge-coloured complete graph K on $\chi_\Gamma(G)$ vertices. By Corollary 5.5, there is a homomorphism $G \to P_\Gamma(K)$. Since $P_\Gamma(K))$ has $|\Gamma| \cdot \chi_\Gamma$ vertices, $\chi_e \leq |\Gamma| \cdot \chi_\Gamma$. □

In the case where $m = 2$ and $\Gamma = S_2$, we have the following; A similar result for oriented graphs with the pushing operation appears in [9].

Corollary 6.2 ([16]). *For any 2-edge-coloured graph G,*

$$\chi_{S_2} \leq \chi_e \leq 2\chi_{S_2}.$$

In the remainder of this section we turn our attention to complexity matters.

Proposition 6.3. *For any group Γ and any integer $k \geq 3$, the problem of deciding if an m-edge-coloured graph G has a Γ-switchable vertex k-colouring is NP-complete.*

Proof. The transformation is from k-colouring. Let G' be an instance of k-colouring, that is, a graph. The transformed instance of Γ-switchable vertex k-colouring is the monochromatic m-edge-coloured graph G derived from $G' \cup K_k$ by letting all edges have colour 0. The transformation can clearly be carried out in Polynomial time. Any Γ-switchable vertex k-colouring of G gives a k-colouring of G' as adjacent vertices have different colours. On the other hand, if G' is k-colourable, then there is a homomorphism $G' \to K_k$ and consequently a (Γ-switchable) homomorphism of G to the m-edge coloured complete graph on k vertices which is monochromatic of colour 0. □

Since an m-edge-coloured graph G has a vertex 1-colouring if and only it has no edges, we focus our attention on the question of deciding the existence of a Γ-switchable 2-colouring. By Theorem 5.4, the m-edge-coloured graph G has 2-colouring if and only if there is a homomorphism $G \to P_\Gamma(K_2(i))$ for some $i \in \{0, 1, \ldots, m-1\}$, where $K_2(i)$ is a complete graph on two vertices which is monochromatic of colour i. Observe that $P_\Gamma(K_2(i))$ is an m-edge-coloured complete bipartite graph.

Let $i \in \{0, 1, \ldots, m-1\}$. We use $Orbit_\Gamma(i)$ and $Stabilizer_\Gamma(i)$ to denote the orbit and stabilizer of the integer i with respect to $\Gamma \leq S_m$, respectively.

Proposition 6.4. *Let $\Gamma \leq S_m$. Then, for any $i \in \{0, 1, \ldots, m-1\}$, the graph $P_\Gamma(K_2(i))$ is vertex-transitive. Further, $P_\Gamma(K_2(i)) \cong P_\Gamma(K_2(j))$ if and only if $Orbit_\Gamma(i) = Orbit_\Gamma(j)$.*

Proof. Let $V(K_2(i)) = \{x, y\}$. Set $X = \{(x, \gamma) : \gamma \in \Gamma\}$, and $Y = \{(y, \gamma) : \gamma \in \Gamma\}$. Then the vertex set of $P_\Gamma(K_2(i))$ is the disjoint union $X \cup Y$. Since there is an automorphism of $P_\Gamma(K_2(i))$ that exchanges X and Y, to show that $P_\Gamma(K_2(i))$ is vertex-transitive, it suffices to show that, for each $(x_1, \alpha), (x_2, \beta) \in X$, there is an automorphism that maps

(x_1, α) to (x_2, β). Define $f_{\alpha\beta} : V(P_\Gamma(K_2(i)) \to V(P_\Gamma(K_2(i))$ by

$$f_{\alpha\beta}(u, \pi) = \begin{cases} (u, \pi\alpha^{-1}\beta) \text{ if } u = x; \text{ and} \\ (u, \pi\alpha\beta^{-1}) \text{ if } u = y. \end{cases}$$

Then f is a bijection, and $f((x, \alpha)) = (x, \beta)$. We must show that f preserves edges and edge colours. Let $(x, \gamma)(y, \sigma) \in E_j(P_\Gamma(K_2(i))$. Note that $j = \gamma\sigma(i)$. Then $f((x, \gamma))f((y, \sigma)) \in E_\ell(P_\Gamma(K_2(i))$, where $\ell = \gamma\alpha^{-1}\beta\sigma\alpha\beta^{-1}(i) = j$.

Suppose $P_\Gamma(K_2(i)) \cong P_\Gamma(K_2(j))$. Then $K_2(i) \in [K_2(j)]_\Gamma$, so that $i \in Orbit_\Gamma(j)$. Conversely, if $i \in Orbit_\Gamma(j)$, then $K_2(i) \in [K_2(j)]_\Gamma$ and consequently $P_\Gamma(K_2(i)) \cong P_\Gamma(K_2(j))$. □

For an m-edge-coloured graph G, let $\mathcal{E}_G = \{j : E_j \neq \emptyset\}$.

Proposition 6.5. *Let G be an m-edge-coloured graph and $\Gamma \leq S_m$ be Abelian. If $G \to K_2(i)$, then $\mathcal{E}_G \subseteq Orbit_\Gamma(i)$.*

Corollary 6.6. *Let G be an m-edge-coloured graph and $\Gamma \leq S_m$. If $\mathcal{E}_G \not\subseteq Orbit_\Gamma(i)$, then G is not Γ-switchably 2-colourable.*

Proposition 6.7. *Let G be an m-edge-coloured graph and $\Gamma \leq S_m$ be Abelian and act transitively. The existence of a Γ-switchable 2-colouring of G can be determined in polynomial time.*

Proof. First, observe that an m-edge-coloured graph H has a homomorphism to $P_\Gamma(K_2(i))$ if and only if has a homomorphism to $S_\Gamma(K_2(i))$ (from Proposition 5.6). By definition of $S_\Gamma(K_2(i))$ and Proposition 6.4, every vertex is incident with exactly one edge of each colour. By Proposition 6.4, for each component of G, one vertex can be mapped to an arbitrary vertex of $S_\Gamma(K_2(i))$ and then the image of every other vertex in the component is uniquely determined. □

The work in this section is summarized in the following theorem.

Theorem 6.8. *Let G be an m-edge-coloured graph, $\Gamma \leq$ be Abelian, and k be a fixed positive integer. If $k \geq 3$, then deciding the existence of a Γ-switchable vertex k-colouring of G is NP-complete. If $k \leq 2$, then deciding the existence of a Γ-switchable vertex k-colouring of G is Polynomial.*

Proof. The first statement follow from Proposition 6.3. We prove the last statement. There must exist i such that $\mathcal{E}_G \subseteq Orbit_\Gamma(i)$, otherwise G is not Γ-switchably 2-colourable. Replace Γ by $Orbit_\Gamma(i)$. The statement now follows from Proposition 6.7. □

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