EDGE-COLOURED GRAPHS AND SWITCHING WITH
$S_m, A_m$ AND $D_m$

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Abstract. We consider homomorphisms and vertex colourings of $m$-edge-coloured graphs that have a switching operation which permutes the colours of the edges incident with a specified vertex. The permutations considered arise from the action of the symmetric, alternating and dihedral groups on the set of edge colours. In all cases, after studying the equivalence classes of $m$-edge-coloured graphs determined by the switching operation, we describe dichotomy theorems for the complexity of the vertex $k$-colouring problem and the problem of deciding the existence of a homomorphism to a fixed target $H$.

1. Introduction

It is often true that a result which holds for 2-edge-coloured graphs (graphs with two disjoint edge sets) also holds for oriented graphs with essentially the same proof. On the other hand, Sen [13] has given examples of statements that hold for oriented graphs and not for 2-edge-coloured graphs. These examples may help to explain why a direct translation between the two families has not been found, and may not exist. Instead, Nešetřil and Raspaud [11] made a connection between them by defining $(n,m)$-mixed graphs, which have $n$ disjoint arc sets and $m$ disjoint edge sets. Oriented graphs are $(1,0)$-mixed graphs, and 2-edge-coloured graphs are $(0,2)$-mixed graphs. Results that hold for $(n,m)$-mixed graphs hold for both families just mentioned, and others. Conversely, if a theorem holds for both oriented graphs and 2-edge-coloured graphs, then there is evidence that perhaps a general theorem holds for $(n,m)$-mixed graphs.

The literature contains results about colourings and homomorphisms of signed graphs, 2-edge-coloured graphs that also have an operation that reverses the sets to which the edges incident with a specified vertex
belong, see [3, 12, 13]. These results are similar to results for oriented graphs with an operation that reverses the arcs incident with a vertex (see [8]). Hence it may be true that there are general theorems of which these results are special cases.

As a step in the quest for such theorems, MacGillivray and Warren [10] consider \(m\)-edge-coloured graphs \((0,m)\)-graphs together with a switching operation in which the elements of a group \(\Gamma \leq S_m\) are used to permute the sets to which the edges incident with a specified vertex belong. They showed that some theorems from [2] and [8] about homomorphisms and colourings hold in this more general setting, and others hold when the group \(\Gamma\) is Abelian.

In this paper we consider the case when the group \(\Gamma\) is one of several non-Abelian groups, specifically the symmetric group \(S_m\), where \(m \geq 3\), the alternating group \(A_m\), where \(m \geq 4\), and the dihedral group \(D_m\), where \(m \geq 3\) (here \(D_m\) is the group of symmetries of a regular \(m\)-gon). We prove results that specify when a given \(m\)-edge-coloured graph \(G\) can be transformed by a sequence of switching operations using one of these groups so that it becomes isomorphic to a given \(m\)-edge-coloured graph \(H\). We then consider the questions of when a given \(m\)-edge-coloured graph \(G\) can be transformed to that it has a homomorphism to a fixed \(m\)-edge-coloured graph \(H\), or a vertex \(k\)-colouring for some fixed integer \(k \geq 2\). In all cases we describe a dichotomy between the cases which are Polynomial and those which are NP-complete.

The colourings considered here are different than those considered by Zaslavsky [15]. His colourings are invariant under switching, whereas in our case switching may change the number of colours needed.

2. Preliminaries

For basic definitions in graph theory, see the text by Bondy and Murty [1]. We consider only finite, simple graphs and refer to them as graphs in the interest of brevity. For results about graph colourings and homomorphisms, see the book by Hell and Nešetřil [7]. For basic results in Abstract Algebra, see the text by Gallian [5].

Let \(m \geq 1\) be an integer. A \(m\)-edge-coloured graph is an \((m+1)\)-tuple

\[G = (V(G), E_0(G), E_1(G), \ldots, E_{m-1}(G)),\]

where

(i) \(V(G)\) is a set of objects call \textit{vertices};
(ii) for $0 \leq i \leq m - 1$, $E_i(G)$ is a set of unordered pairs of not-
necessarily distinct vertices called the \textit{edges of colour} $i$; and

(iii) $E_i(G) \cap E_j(G) = \emptyset$ when $i \neq j$.

When the context is clear, the vertex set is referred to as $V$ and the
edge sets as $E_0, E_1, \ldots, E_{m-1}$.

If $G$ is an $m$-edge-coloured graph, then the \textit{underlying graph} of $G$

is the graph with vertex set $V$ and edge set $E_0 \cup E_1 \cup \cdots \cup E_{m-1}$. An

$m$-edge-coloured graph $G$ can be regarded as being obtained from its

underlying graph by assigning each edge one of the $m$

available colours.

Let $G$ an $m$-edge-coloured graph, and $\Gamma \leq S_m$. For $x \in V(G)$, the

operation of \textit{switching at} $x$ \textit{with respect to} $\gamma \in \Gamma$ transforms $G$

into the $m$-edge-coloured graph $G^1 = G \cdot (x, \gamma)$ with vertex set $V(G)$ and edge

sets

$$E_i(G^1) = (E_i(G) - \{xy : xy \in E_i(G)\}) \cup \{xz : xz \in E_j(G) \text{ and } \gamma(j) = i\}.$$  

The permutation $\gamma$ acts on the colours of the edges incident with $x$: the

edge $xy$ belongs to $E_i(G^1)$ if and only if $xy \in E_i(G)$ and $\gamma(j) = i$.

Let $\Gamma \leq S_m$. Define the relation $\sim_\Gamma$ on the set of all $m$-edge-

coloured graphs by $G \sim_\Gamma H$ if and only if there is a finite sequence

$(x_1, \gamma_1), (x_2, \gamma_2), \ldots, (x_t, \gamma_t)$ where $x_i \in V$ and $\gamma_i \in \Gamma$, $1 \leq i \leq t$, such

that if $G^0 = G$ and $G^i = G^{i-1} \cdot (x_i, \gamma_i)$ for $i = 1, 2, \ldots, t$, then $G^t \cong H$.

It is easy to see that $\sim_\Gamma$ is a equivalence relation. If $G \sim_\Gamma H$, then

we say $G$ is \textit{switch equivalent to} $H$ \textit{with respect to} $\Gamma$, or that $G$

is $\Gamma$-\textit{switchable to} $H$.

Some authors consider switch equivalence as being with respect to la-

belled graphs (see [15]). In this case only $m$-edge-coloured graphs with

the same underlying graph can be switch equivalent. For example, if

both $G$ and $H$ have underlying graph $P_3$, but $\{1, 3\}$ is independent in $G$

while $\{2, 3\}$ is independent in $H$, then $G$ and $H$ can not be switch equiva-

lent. Our definition coincides with this one, up to automorphisms of

the underlying graph. Since our interest is in homomorphisms, taking

automorphisms of the underlying graph into account is better suited

to our purposes. Under our definition any two $m$-edge-coloured graphs

with underlying graph $P_3$ are switch equivalent with respect to any

group $\Gamma \leq S_m$ which acts transitively on $\{0, 1, \ldots, m - 1\}$.

Let $G$ and $H$ be $m$-edge-coloured graphs. A \textit{homomorphism of} $G$ \textit{to}

$H$ is a mapping of the vertices of $G$ to the vertices of $H$ that preserves

the edges and edge sets. Formally, it is a function $h : V(G) \to V(H)$
such that $xy \in E_i(G)$ implies $f(x)f(y) \in E_i(H)$. We sometimes use
the notation \( h : G \rightarrow H \) to mean that \( h \) is a homomorphism of \( G \) to \( H \); when the name of the function is unimportant the existence of a homomorphism of \( G \) to \( H \) is abbreviated to \( G \rightarrow H \).

For \( \Gamma \leq S_m \), a \( \Gamma \)-switchable homomorphism of \( G \) to \( H \) is a homomorphism \( G' \rightarrow H \) for some \( G' \) which is switch equivalent to \( G \) with respect to \( \Gamma \). If such a mapping exists then we say \( G \) is \( \Gamma \)-switchably homomorphic to \( H \) and write \( G \rightarrow_{\Gamma} H \).

While the definition of a \( \Gamma \)-switchable homomorphism of \( G \) to \( H \) involves only switch equivalence to \( G \), it transpires that switching at vertices of \( H \) can transform the question of whether \( G \rightarrow_{\Gamma} H \) into a question about homomorphisms of \( m \)-edge-coloured graphs (without switching).

**Theorem 2.1** ([10]). Let \( G \) and \( H \) be \( m \)-edge-coloured graphs and \( \Gamma \leq S_m \). Then \( G \rightarrow_{\Gamma} H \) if and only if there exists \( G' \sim_{\Gamma} G \) and \( H' \sim_{\Gamma} H \) such that \( G' \rightarrow H' \).

In Section 5 we will determine, for all fixed \( m \)-edge-coloured graphs \( H \) and all \( \Gamma \in \{S_m : m \geq 3\} \cup \{A_m : m \geq 4\} \cup \{D_m : m \geq 3\} \), the complexity deciding the existence of a \( \Gamma \)-switchable homomorphism from a given \( m \)-edge-coloured graph \( G \) to \( H \). This is made possible by our work on switch-equivalence with respect to these groups in Sections 3 and 4.

A vertex \( k \)-colouring of a graph \( G \) can be regarded as a homomorphism of \( G \) to a complete graph \( K_k \) with vertex set \( \{1, 2, \ldots, k\} \). More generally, a homomorphism of \( G \) to any \( k \)-vertex graph \( H \) can be regarded as a \( k \)-colouring of \( G \): adjacent vertices in \( G \) are assigned adjacent, thus different, colours from \( V(H) \).

For an integer \( k \geq 1 \), a vertex \( k \)-colouring of an \( m \)-edge-coloured graph \( G \) is a homomorphism of \( G \) to some \( m \)-edge coloured graph with \( k \) vertices. We note that, since non-adjacent vertices of \( H \) can be joined by an edge of any colour with destroying the existence of such a mapping, the underlying graph of \( H \) can be assumed to be complete. Put differently, a vertex \( k \)-colouring of an \( m \)-edge-coloured graph \( G \) is an assignment of \( k \) colours, say \( 1, 2, \ldots, k \), to the vertices of \( G \) such that adjacent vertices get different colours and the vertex colours respect the edge sets in the sense that each pair of different colours can appear on the ends of edges in at most one set \( E_i \). It follows from the definition that vertices joined by a path of length two containing edges from different edge sets must be assigned different colours.
For $\Gamma \leq S_m$, a $\Gamma$-switchable vertex $k$-colouring of $G$ is a $\Gamma$-switchable homomorphism $G \to H$, where $|V(H)| = k$. The complexity of deciding the existence of such a colouring has been determined for all Abelian groups:

**Theorem 2.2** ([10]). Let $\Gamma \leq S_m$ be Abelian. Then, the problem of deciding whether a given $m$-edge-coloured graph $G$ has a $\Gamma$-switchable vertex $k$-colouring is Polynomial if $k = 2$, and NP-complete if $k \geq 3$.

In Section 5 we will determine the complexity of deciding the existence of a $\Gamma$-switchable $k$-colouring when $\Gamma \in \{S_m : m \geq 3\} \cup \{A_m : m \geq 4\} \cup \{D_m : m \geq 3\}$.

The chromatic number of an $m$-edge-coloured graph $G$ is the least $k$ for which $G$ has a vertex $k$-colouring, and is denoted by $\chi_e(G)$, or just $\chi_e$ when the context is clear. The $\Gamma$-switchable chromatic number of $G$ is the least $k$ for which $G$ has a $\Gamma$-switchable vertex $k$-colouring, and is denoted by $\chi_\Gamma(G)$, or simply $\chi_\Gamma$.

As an aside, we note that the chromatic number of an $m$-edge-coloured graph can differ substantially from that of its underlying graph. For example, consider the 2-edge-coloured complete bipartite graph with vertex set $V = \{a_1, a_2, \ldots, a_n\} \cup \{b_1, b_2, \ldots, b_n\}$ and edge sets $E_0 = \{a_ib_j : i \geq j\}$ and $E_1 = \{a_ib_j : i < j\}$. The underlying graph has chromatic number two, but $\chi_e = 2n = |V|$ because every two non-adjacent vertices are joined by a path of length two that uses an edge from $E_0$ and an edge from $E_1$.

When $\Gamma$ is Abelian, there is simple relationship between $\chi_e$ and $\chi_\Gamma$:

**Theorem 2.3** ([10]). Let $\Gamma \leq S_m$ be Abelian. For any $m$-edge-coloured graph $G$, $\chi_\Gamma(G) \leq 2\chi_e(G)$, and $\chi_e(G) \leq |\Gamma|\chi_\Gamma(G)$.

When $m = 2$ switching occurs with respect to $S_2$. The upper bound on $\chi_e(G)$ is $2\chi_\Gamma(G)$, a result which appears in [13] (see [8] for the corresponding theorem for oriented graphs with the vertex pushing operation). In Section 5 we shall give evidence that no such simple relationship is likely to hold when $\Gamma$ is non-Abelian.

### 3. Switching using groups with Property $\mathcal{T}_j$

In this section we describe a group property which implies the existence a sequence of switches which transforms the colour of given edge $xy$ from $i$ to $j$, and leaves the colour of all other edges unchanged.
When there exists \( j \) such that this property holds for all \( i \), any \( m \)-edge-coloured graph \( G \) is \( \Gamma \)-switchable to one which is \textit{monochromatic of colour} \( j \) (all edges have colour \( j \)). If \( \Gamma \) has this property for all \( i \) and \( j \), then any two graphs with isomorphic underlying graphs are switch equivalent with respect to \( \Gamma \). There is a sense in which the results in this section say that groups with property \( T_j \) are less interesting.

Let \( \Gamma \) be a group of permutations of \( \{0, 1, \ldots, m - 1\} \). For \( i, j \in \{0, 1, \ldots, m - 1\} \), we say \( \Gamma \) has \textit{Property} \( T_{i,j} \) if it acts transitively on \( \{0, 1, \ldots, m - 1\} \) and there exists \( k \in \{0, 1, \ldots, m - 1\} \) and a permutation in \( \alpha \in \text{Stabilizer}(k) \) such that \( \alpha(i) = j \).

If \( i \neq j \), then property \( T_{i,j} \) requires that some \( k \in \{0, 1, \ldots, m - 1\} \) has a non-trivial stabilizer (and so, by the Orbit-Stabilizer Theorem, every \( k \in \{0, 1, \ldots, m - 1\} \) does). Hence, and group with property \( T_{i,j} \) for some \( i \neq j \), is necessarily non-Abelian. Further, it is implicit in the definition that any such group has order at least three.

**Lemma 3.1.** Let \( G \) be a \( m \)-edge-coloured graph and \( \Gamma \leq S_m \). If \( \Gamma \) has property \( T_{i,j} \) and the edge \( xy \) has colour \( i \) in \( G \), then \( G \) is \( \Gamma \)-switchable to a \( m \)-edge-coloured graph \( G' \) in which \( xy \) has colour \( j \) and all other edges have the same colour as in \( G \).

**Proof.** Since \( \Gamma \) has Property \( T_{i,j} \), there exists \( k \in \{0, 1, \ldots, m - 1\} \) and \( \alpha \in \text{Stabilizer}(k) \) such that \( \alpha(i) = j \), and \( \beta \in \Gamma \) such that \( \beta(j) = k \). Consider the sequence of switches \( (\alpha, x), (\beta, y), (\alpha^{-1}, x), (\beta^{-1}, y) \), which transform \( G \) into \( G' \).

The only edges which change colour during this sequence of switches are those incident with \( x \) or \( y \). After the first, second, third and fourth switch in the sequence, the edge \( xy \) has colour \( j, k, j, k \), respectively. Any edge \( e \) incident with \( x \) and not \( y \) changes from its colour \( c_e \) to \( \alpha(c_e) \) and then back to \( c_e \). Similarly, any edge incident with \( y \) and not \( x \) changes from its colour and then back to it. The result follows. \( \square \)

For \( j \in \{0, 1, \ldots, m - 1\} \), we say that \( \Gamma \) has \textit{Property} \( T_j \) if it has Property \( T_{i,j} \) for every \( i \in \{0, 1, \ldots, m - 1\} \).

**Theorem 3.2.** If \( \Gamma \) has Property \( T_j \) for all \( j \in \{0, 1, \ldots, m - 1\} \), then any two \( m \)-edge-coloured graphs with isomorphic underlying graphs are switch equivalent with respect to \( \Gamma \).

**Proof.** Suppose \( G \) and \( H \) are \( m \)-edge-coloured graphs with isomorphic underlying graphs. Assume that the vertices of \( G \) have been relabelled
so that the underlying graph of $G$ equals the underlying graph of $H$. Define the distance between $G$ and $H$ to be the number of edges of $G$ which have a different colour than the corresponding edge in $H$.

The proof is by induction on the distance between $G$ and $H$. If the distance between $G$ and $H$ is zero, then they are equal and hence switch equivalent with respect to $\Gamma$. Suppose that any two $m$-edge-coloured graphs with isomorphic underlying graphs which are at distance less than $d$ are switch equivalent with respect to $\Gamma$. Let $G$ and $H$ be $m$-edge-coloured graphs with isomorphic underlying graphs which are at distance $d$.

By Lemma 3.1, there is a sequence of switches that transform $G$ into an $m$-edge-coloured graph $G'$ which is at distance $d - 1$ from $H$. By the induction hypothesis, $G'$ is $\Gamma$-switchable to $H$. Therefore so is $G$. The result now follows by induction. \[\square\]

Essentially the same argument as above proves the following.

**Proposition 3.3.** Let $G$ be a $m$-edge-coloured graph and $\Gamma \leq S_m$. If $\Gamma$ has Property $T_j$, then $G$ is $\Gamma$-switchable to a $m$-edge-coloured graph $G'$ which is monochromatic of colour $j$.

**Corollary 3.4.** Let $G$ be a $m$-edge-coloured graph such that the underlying graph of $G$ is bipartite. If $\Gamma \leq S_m$ has Property $T_j$, then $G$ is $\Gamma$-switchable to a $m$-edge-coloured graph $G'$ which is monochromatic of colour $i$ for any $i \in \{0, 1, \ldots, m - 1\}$.

**Proof.** Let $(X, Y)$ be a bipartition of the underlying graph of $G$. By Proposition 3.3, $G$ is $\Gamma$-switchable to a $m$-edge-coloured graph $G'$ which is monochromatic of colour $j$. Since $\Gamma$ acts transitively on $\{0, 1, \ldots, m - 1\}$, for any $i \in \{0, 1, \ldots, m - 1\}$, there is a permutation $\alpha \in \Gamma$ such that $\alpha(j) = i$. Switching with respect to $\alpha$ at each vertex in $X$ transforms $G'$ to a $m$-edge-coloured graph in which every edge has colour $i$. \[\square\]

We observe that any 2-transitive group of permutations of $\{0, 1, \ldots, m - 1\}$ has Property $T_j$ for every $j \in \{0, 1, \ldots, m - 1\}$. In particular, the symmetric group $S_m$ is 2-transitive for all $m \geq 3$, and the Alternating group $A_m$ is 2-transitive for all $m \geq 4$. By contrast, $S_2$, $A_2$ and $A_3$ are all Abelian. None of these have property $T_{i,j}$ for $i \neq j$, and none can be used to switch any $m$-edge coloured graph to one where all edges have the same colour.

The following is now an easy consequence of Proposition 3.3.
Corollary 3.5. Let $\Gamma \in \{S_m : m \geq 3\} \cup \{A_m : m \geq 4\}$. Any $m$-edge-coloured graph $G$ is $\Gamma$-switchable to a $m$-edge-coloured graph $G'$ which is monochromatic of colour $j$, for any $j \in \{0, 1, \ldots, m - 1\}$.

For $S_m$, $m \geq 3$, the corollary was independently proved by Christopher Duffy [private communication] using essentially the same argument as given above.

4. Switching with respect to Dihedral groups

Recall that we denote by $D_m$ the group of symmetries of the regular $m$-gon with vertex set $\{0, 1, \ldots, m - 1\}$. It transpires that the cases of $m$ odd and $m$ even are different. We consider the case of odd $m$ first.

Proposition 4.1. For any odd integer $m \geq 3$, the group $D_m$ has Property $T_j$ for every $j \in \{0, 1, \ldots, m - 1\}$.

Proof. Let $i \in \{0, 1, \ldots, m - 1\}$. Since $m$ is odd, either the least residue of $i - j$ modulo $m$ is even, or the least residue of $j - i$ modulo $m$ is even. Without loss of generality, the latter holds. Then there exists $k \in \{0, 1, \ldots, m - 1\}$ such that $j - k \equiv k - i \pmod m$. Let $\alpha$ be the permutation of $\{0, 1, \ldots, m - 1\}$ which corresponds to flipping the $m$-gon about vertex $k$. Then $\alpha \in \text{Stabilizer}(k)$ and $\alpha(i) = j$, so $D_m$ has property $T_{i,j}$. This completes the proof. $\square$

Corollary 4.2. If $m \geq 3$ is odd, then any two $m$-edge-coloured graphs with isomorphic underlying graphs are switch equivalent with respect to $D_m$. In particular, any $m$-edge-coloured graph is $D_m$-switchable to one which is monochromatic of colour $j$, for any $j \in \{0, 1, \ldots, m - 1\}$.

We now turn to the case where $m$ is even. Some basic results from group theory are needed. Note that if $m \geq 2$ is even and we let $\mathcal{E} = \{0, 2, \ldots, m - 2\}$ and $\mathcal{O} = \{1, 3, \ldots, m - 1\}$, then the partition $\{\mathcal{O}, \mathcal{E}\}$ is a block system for the action of $D_m$ on $\{0, 1, \ldots, m - 1\}$.

Proposition 4.3. Suppose $m \geq 2$ is even, and let $\mathcal{E} = \{0, 2, \ldots, m - 2\}$ and $\mathcal{O} = \{1, 3, \ldots, m - 1\}$. Then,

1. $\text{Stabilizer}(\mathcal{E}) = \text{Stabilizer}(\mathcal{O})$ is a normal subgroup of $D_m$,
2. $D_m/\text{Stabilizer}(\mathcal{E}) \cong S_2$,
3. $\text{Stabilizer}(\mathcal{E})$ has Property $T_{i,j}$ for all $i, j \in \mathcal{E}$, and
4. $\text{Stabilizer}(\mathcal{O})$ has Property $T_{i,j}$ for all $i, j \in \mathcal{O}$.
Proof. We prove only statement (3). The proof of (4) is similar. Let \( i, j \in \mathcal{E} \). Without loss of generality, \( i < j \). Since \( i \) and \( j \) are even, \( k = (j - i)/2 \) is an integer. Let \( \alpha \) be the permutation corresponding to a flip of the regular \( m \)-gon about vertex \( k \). Then \( \alpha \in \text{Stabilizer}(k) \) and \( \alpha(i) = j \). The result follows. \( \square \)

Let \( G \) be an \( m \)-edge coloured graph, where \( m \geq 2 \) is an even integer. The 2-edge coloured graph \( G_2 \) is obtained from \( G \) by assigning each edge its colour in \( G \) modulo 2.

Theorem 4.4. Let \( G \) and \( H \) be \( m \)-edge-coloured graphs, where \( m \geq 2 \) is an even integer. Then \( G \) and \( H \) are switch equivalent with respect to \( D_m \) if and only if \( G_2 \) and \( H_2 \) are switch equivalent with respect to \( S_2 \).

Proof. Suppose \( G \) and \( H \) are switch equivalent with respect to \( D_m \). Then there is a sequence of switches \( S = (\pi_1, x_1), (\pi_2, x_2), \ldots, (\pi_t, x_t) \) that transforms \( G \) so it is isomorphic to \( H \). Each permutation \( \pi_i \in D_m \) either fixes both \( \mathcal{E} \) and \( \mathcal{O} \), or exchanges them. Let \( S' \) be the subsequence of \( S \) consisting of the permutations that exchange \( \mathcal{E} \) and \( \mathcal{O} \). Replacing each of the permutations in this subsequence by the transposition \((0 1)\) gives a sequence of switches that transforms \( G_2 \) to a graph \( H_2 \).

Now suppose \( G_2 \) and \( H_2 \) are switch equivalent with respect to \( S_2 \). Then there is a sequence of switches \( A = (\sigma_1, x_1), (\sigma_2, x_2), \ldots, (\sigma_p, x_p) \) that transforms \( G_2 \) to a 2-edge coloured graph \( G'_2 \) which is isomorphic to \( H_2 \). Replacing each permutation \( \sigma_i \in S_2 \) by \((0 1 \cdots m - 1)\) in \( D_m \) gives a sequence of switches that transforms \( G \) to a graph \( G' \) in which edge edge colour has the same parity as in \( G'_2 \). Since both \( \text{Stabilizer}(\mathcal{E}) \) and \( \text{Stabilizer}(\mathcal{O}) \) have Property \( T_{i,j} \) for all \( i, j \in \mathcal{E} \), and all \( i, j \in \mathcal{O} \), respectively, the \( m \)-edge coloured graph \( G' \) is \( D_m \)-switchable to \( H \). \( \square \)

Let \( G_2 \) be a 2-edge-coloured graph. Define the graph \( P_{S_2}(G_2) \) to have vertex set \( V(G_2) \times S_2 \), with \((x, \pi_1)(y, \pi_2) \in E_i(P_{S_2}(G_2)) \) if and only if \( xy \in E_j(G_2) \) and \( \pi_1\pi_2(j) = i \).

Brewster and Graves [2] (also see [8]) proved that the question of whether two \( m \)-edge coloured graphs are switch equivalent with respect to \( \mathbb{Z}_m \) can be translated into a question about isomorphism of \( m \)-edge coloured graphs. A similar theorem holds for any Abelian group \( \Gamma \) [10]. We state the result only for \( S_2 = \mathbb{Z}_2 \).
Theorem 4.5 ([2]). Let $G_2$ and $H_2$ be 2-edge-coloured graphs. Then $G_2$ and $H_2$ are switch equivalent with respect to $S_2$ if and only if $P_{S_2}(G_2) \cong P_{S_2}(H_2)$.

Corollary 4.6. Let $G$ and $H$ be $m$-edge-coloured graphs, where $m \geq 2$ is an even integer. Then $G$ and $H$ are switch equivalent with respect to $D_m$ if and only if $P_{S_2}(G_2) \cong P_{S_2}(H_2)$.

An analogue of the following theorem of Zaslavsky is immediate.

Theorem 4.7 ([14]). The 2-edge-coloured graphs $G$ and $H$ with the same underlying graph are switch equivalent with respect to $Z_2$ if and only if some automorphism of the underlying graph transforms $G$ to a 2-edge-coloured graph which the same collection of cycles with an odd number of edges in $E_0$ as $H$.

Corollary 4.8. Suppose $m \geq 2$ is even. The $m$-edge-coloured graphs $G$ and $H$ are switch equivalent with respect to $D_m$ if and only if some automorphism of the underlying graph transforms $G$ to an $m$-edge-coloured graph with the same collection of cycles with an odd number of edges whose colour is in $E$.

Proof. Since switching at a vertex preserves the parity of the number of edges on each cycle whose colour is in $E$, any two $m$-edge-coloured graphs with the same underlying graph which are switch equivalent with respect to $D_m$ have the same collection of cycles with an odd number of edges whose colour is in $E$. Hence if $G$ and $H$ are switch equivalent with respect to $D_m$ then some automorphism of the underlying graph transforms $G$ to an $m$-edge-coloured graph with the same collection of cycles with an odd number of edges whose colour is in $E$.

Suppose some automorphism of the underlying graph transforms $G$ to an $m$-edge-coloured graph with the same collection of cycles with an odd number of edges whose colour is in $E$. Then, by Theorem 4.7, the corresponding 2-edge-coloured graphs $G_2$ and $H_2$ are switch equivalent with respect to $S_2$. The result now follows from Theorem 4.4. □

Corollary 4.9. Suppose $m \geq 2$ is even. Then,

1. for any $i \in E$, the $m$-edge-coloured graph $G$ is $D_m$-switchable to a $m$-edge-coloured graph $G'$ which is monochromatic of colour $i$ if and only if every cycle in $G$ has an even number of edges whose colour is in $O$, and
2. for any $j \in O$, the $m$-edge-coloured graph $G$ is $D_m$-switchable to a $m$-edge-coloured graph $G'$ which is monochromatic of colour
Corollary 4.10. Suppose \( m \geq 2 \) is even. If the underlying graph of the \( m \)-edge-coloured graph \( G \) is bipartite then, for any \( j \in \{0, 1, \ldots, m-1\} \), \( G \) is \( D_m \)-switchable to a graph which is monochromatic of colour \( j \) if and only if every cycle has an even number of edges whose colour is in \( E \).

5. Switchable homomorphisms and vertex colourings

For a fixed \( m \)-edge-coloured graph \( H \) and \( \Gamma \leq S_m \), we define \( \Gamma \)-switchable \( \text{Hom}_H \) to be the problem of deciding whether there is a \( \Gamma \)-switchable homomorphism of \( G \) to \( H \). In what follows we will determine the complexity of \( \Gamma \)-switchable \( \text{Hom}_H \) for all groups \( \Gamma \) with property \( T_j \), and then \( D_m \) for even \( m \geq 2 \).

We will make use of the following.

Theorem 5.1 ([6]). Let \( H \) be a finite, undirected graph. If \( H \) is bipartite, then \( \text{Hom}_H \) is polynomial. If \( H \) is not bipartite, then \( \text{Hom}_H \) is \( \text{NP-complete} \).

Theorem 5.2. Let \( \Gamma \leq S_m \) have Property \( T_j \), and let \( H \) be an \( m \)-edge-coloured graph. If the underlying graph of \( H \) is bipartite, then \( \Gamma \)-switchable \( \text{Hom}_H \) is Polynomial. If the underlying graph of \( H \) is not bipartite, then \( \Gamma \)-switchable \( \text{Hom}_H \) is \( \text{NP-complete} \).

Proof. Let \( H' \) be the underlying graph of \( H \). For a given \( m \)-edge-coloured graph \( G \), we claim that \( G \rightarrow_{\Gamma} H \) if and only if \( G' \rightarrow H' \). It is clear that if \( G \rightarrow_{\Gamma} H \), then \( G' \rightarrow H' \). Suppose \( G' \rightarrow H' \). Both \( G \) and \( H \) are \( \Gamma \)-switchable to \( m \)-edge-coloured graphs \( G_j \) and \( H_j \), respectively, which are monochromatic of colour \( j \). Since \( G_j \rightarrow H_j \), the claim follows from Theorem 2.1.

Suppose that \( H' \) is bipartite. By the claim, if \( G \) is an \( m \)-edge-coloured graph, then \( G \rightarrow_{\Gamma} H \) if and only if the underlying graph of \( G \) is bipartite. It follows that \( \Gamma \)-switchable \( \text{Hom}_H \) is Polynomial.

Now suppose that \( H' \) is not bipartite. The transformation is from \( \text{Hom}'_H \). Let \( G' \) be an instance of \( \text{Hom}'_H \), that is, an undirected graph. The transformed instance of \( \Gamma \)-switchable \( \text{Hom}_H \) is the \( m \)-edge-coloured graph \( G \) which is monochromatic of colour \( j \) and has underlying graph \( G' \). The transformation can clearly be carried out in polynomial time. The result now follows from the claim, and Theorem 5.1. \( \square \)
Corollary 5.3. Let $\Gamma \in \{S_m : m \geq 3\} \cup \{A_m : m \geq 4\} \cup \{D_m : \text{odd } m \geq 3\}$, and let $H$ be a $m$-edge-coloured graph. If the underlying graph of $H$ is bipartite, then $\Gamma$-switchable $\text{Hom}_H$ is Polynomial. If the underlying graph of $H$ is not bipartite, then $\Gamma$-switchable $\text{Hom}_H$ is NP-complete.

It remains to consider the dihedral groups $D_m$ where $m \geq 2$ is even. We will use the following special case of a theorem of Brewster et al. [3]. Note that the property that every cycle has an even number of edges of each colour can hold only if the underlying graph is bipartite.

Theorem 5.4 ([3]). Let $H$ be a 2-edge-coloured-graph. If every cycle has an even number of edges of each colour, then $S_2$-switchable $\text{Hom}_H$ is Polynomial. If $H$ has a cycle with an odd number of edges of some colour, then $S_2$-switchable $\text{Hom}_H$ is NP-complete.

The existence of a $S_2$-switchable homomorphism between two 2-edge-coloured graphs can be transformed into one about the existence of a homomorphism without switching. A similar statement holds for all Abelian groups, see [10].

Theorem 5.5 ([2, 8]). Let $G_2$ and $H_2$ be 2-edge-coloured graphs. Then $G_2 \to_{S_2} H_2$ if and only if $P_{S_2}(G_2) \to P_{S_2}(H_2)$.

Corollary 5.6. Let $H$ be a $m$-edge-coloured graph, where $m \geq 2$ is even. If every cycle has an even number of edges whose colour is in $E$ and an even number of edges whose colour is in $O$, then $D_m$-switchable $\text{Hom}_H$ is Polynomial. If $H$ has a cycle with an odd number of edges whose colour is in $E$ or an odd number of edges whose colour is in $O$, then $D_m$-switchable $\text{Hom}_H$ is NP-complete.

Proof. Suppose If every cycle in $H$ has an even number of edges whose colour is in $E$ and an even number of edges whose colour is in $O$. Then every cycle in $H_2$ has an even number of edges of each colour. By Theorem 5.4, $S_2$-switchable $\text{Hom}_{H_2}$ is Polynomial. It follows from Theorem 4.4 that $D_m$-switchable $\text{Hom}_H$ is Polynomial.

Now suppose $H$ has a cycle with an odd number of edges whose colour is in $E$ or an odd number of edges whose colour is in $O$. The transformation is from $S_2$-switchable $\text{Hom}_{H_2}$. Let $G_2$ be an instance of $S_2$-switchable $\text{Hom}_{H_2}$. The transformed instance of $D_m$-switchable $\text{Hom}_H$ is the $m$-edge-coloured graph $G$ which is identical to $G_2$ (the edge sets $E_2, E_3, \ldots, E_{m-1}$ are all empty). The transformation can clearly be carried out in Polynomial time. The result follows from Theorem 4.4. \qed
We now determine the complexity of deciding whether a given $m$ edge-coloured graph $G$ has a $\Gamma$-switchable vertex $k$-colouring for the groups $\Gamma$ we have been considering.

**Theorem 5.7.** Let $k \geq 2$ be an integer and $\Gamma \leq S_m$. If $\Gamma \in \{S_m : m \geq 2\} \cup \{A_m : m \geq 2\} \cup \{D_m : m \geq 2\}$ or if $\Gamma$ has property $T_j$ for some $j$, then the problem of deciding whether a given $m$-edge-coloured graph $G$ is $\Gamma$-switchably $k$-colourable is Polynomial if $k = 2$, and NP-complete if $k \geq 3$.

**Proof.** If $\Gamma \in \{S_2, S_2, A_3, D_2\}$ then the statement follows from Theorem 2.2. Hence suppose $\Gamma \in \{S_m : m \geq 3\} \cup \{A_m : m \geq 4\} \cup \{D_m : m \geq 3\}$.

An $m$-edge-coloured graph $G$ has a vertex 2-colouring if and only if it is bipartite and $\Gamma$-switchable to be monochromatic of some colour. That is, if there is a $\Gamma$-switchable homomorphism to a monochromatic $K_2$. By Theorem 5.2 and Corollary 5.6, this can be decided in Polynomial time.

Suppose $k \geq 3$ and $\Gamma \leq S_m$ has Property $T_j$. The transformation is from vertex $k$-colouring of undirected graphs. Recall that a vertex $k$-colouring of an undirected graph $G$ is a homomorphism $G \rightarrow K_k$. Suppose an undirected graph $G$ is given. The transformed instance is the $m$-edge-coloured graph $G_j$ which is monochromatic of colour $j$. The transformation can be accomplished in Polynomial time. A vertex $k$-colouring of $G$ is clearly a $\Gamma$-switchable vertex $k$-colouring of $G_j$, and vice-versa.

Finally, suppose $k \geq 3$ and $\Gamma \in \{D_m : m \geq 4 \text{ even}\}$. The transformation is from $S_2$-switchable vertex $k$-colouring. Let $G_2$ be an instance of $S_2$-switchable vertex $k$-colouring (which is NP-complete by Theorem 2.2). The transformed instance of $D_m$-switchable vertex $k$-colouring is the $m$-edge-coloured graph $G$ which is identical to $G_2$ (the edge sets $E_2, E_3, \ldots, E_{m-1}$ are all empty). The transformation can clearly be carried out in Polynomial time. The result follows from Theorem 4.4. \qed

If $\Gamma$ has property $T_j$ for some $j \in \{0, 1, \ldots, m - 1\}$, then a certificate that a given $m$-edge-coloured graph $G$ does not have a vertex 2-colouring is an odd cycle in the underlying graph of $G$. If $\Gamma$ is $D_m$, where $m \geq 2$ is even, then it is a cycle with an odd number of edges in $\mathcal{E}$ or an odd number of edges in $\mathcal{O}$.

We conclude the paper by considering whether there might be a version of Theorem 2.3 for non-Abelian groups. Our goal is to suggest
that the constant in the upper bound is likely to be a fairly complicated function of the group. We will make use of a result for $m$-edge-coloured graphs which is similar to Observations 1 and 4 in [9] (for oriented graphs).

**Proposition 5.8.** Let $G$ be a graph, and $k$ be a positive integer. If $\chi_e \leq k$ for every $m$-edge-colouring of $G$, then $m^\binom{k}{2} k^{|V|} \geq m^{|E|}$.

**Proof.** Regard the vertices of $G$ as being labelled. Then there are $m^{|E|}$ labelled $m$-edge-coloured graphs with underlying graph $G$.

Any partition of the vertex set of $G$ into at most $k$ independent sets is a vertex $k$-colouring of at most $m^\binom{k}{2}$ of the $m^{|E|}$ labelled $m$-edge-coloured graphs with underlying graph $G$. Since the number of possible vertex $k$-colourings is less than $k^{|V|}$, it follows that $m^\binom{k}{2} k^{|V|} \geq m^{|E|}$. □

**Proposition 5.9.** For every integer $\Delta \geq 2$, there exists an $m$-edge-coloured graph for which the underlying graph is $\Delta$-regular and $\chi_e > m^{\Delta/2}$.

**Proof.** Let $G$ be a $\Delta$-regular graph. Then $G$ has $\Delta |V|/2$ edges. By Proposition 5.8, if every $m$-edge-colouring of $G$ has $\chi_e \leq m^{\Delta/2}$, then $\log_m(\chi_e) \geq \Delta/2 - \binom{m^{\Delta/2}}{2}/|V|$, or equivalently, $\chi_e \geq m^{\Delta/2} \cdot m^{|V|/\binom{m^{\Delta/2}}{2}}$.

Since there are $\Delta$-regular graphs with arbitrarily many vertices, and $m$ and $\Delta$ are constants, the graph $G$ can be chosen to have enough vertices so that

$$m^{|V|/\binom{m^{\Delta/2}}{2}} > 1.$$  

It follows that some $m$-edge-colouring of $G$ has $\chi_e > m^{\Delta/2}$. □

Suppose we seek a version of Theorem 2.3 for $D_m$, where $m$ is odd. Since $\chi_{D_m}(G) = \chi(U)$, where $U$ is the underlying graph of $G$, we would need a constant $c$ such that

$$m^{\Delta/2} < \chi_e \leq c \cdot \chi(U) \leq c(1 + \Delta(U)) \leq m \Delta(U)m^{\Delta+1},$$

where the Brooks-like upper bound is from [10]. Such a constant $c$ must be exponential in the maximum degree of the underlying graph, whereas $|D_m|$ is linear in $m$.

We suggest that the underlying reason is that for Abelian groups it can be assumed that there is one switch at each vertex (possibly with respect to the identity element), whereas for non-Abelian groups the order in which the switches are made is important.
References


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\(^\dagger\) Research supported by NSERC