RECONFIGURING $k$-COLOURINGS OF COMPLETE BIPARTITE GRAPHS

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Abstract. Let $H$ be a graph, and $k \geq \chi(H)$ an integer. We say that $H$ has a Gray code of $k$-colourings if and only if it is possible to list all its $k$-colourings in such a way that consecutive colourings (including the last and the first) agree on all vertices of $H$ except one. The Gray code number of $H$ is the least integer $k_0(H)$ such that $H$ has a Gray code of its $k$-colourings for all $k \geq k_0(H)$. For complete bipartite graphs, we prove that $k_0(K_{\ell,r}) = 3$ when both $\ell$ and $r$ are odd, and $k_0(K_{\ell,r}) = 4$ otherwise.

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1. Introduction

Let $H$ be a graph and $k$ a positive integer. The $k$-colouring graph of $H$, $G_k(H)$, has as its vertices the proper $k$-colourings of $H$, any two of which are joined by an edge if and only if they agree on all but one vertex of $H$. When this graph is connected, any given $k$-colouring can be reconfigured into any other via a sequence of recolourings which each change the colour of exactly one vertex. When it is hamiltonian, there is a cyclic list that contains all of the $k$-colourings of $G$ and consecutive elements of the list differ in the colour of exactly one vertex.

The Gray code number of $H$, denoted $k_0(H)$, is defined to be the smallest integer $k$ such that $G_k(H)$ has a Hamilton cycle for all $k \geq k_0(H)$. It is shown in [7] that for any simple graph $H$, $k_0(H)$ is well-defined; i.e., for $k \geq \text{col}(G) + 2$, where $\text{col}(G)$ denotes the colouring number of $G$, it is always possible to enumerate all proper $k$-colourings of $H$ in such a way that any two successive colourings, including the first and the last, differ on only one vertex. A discussion of the origins the Gray code number can be found in [7].

For our purposes, a proper $k$-colouring of a graph $H$ is a function $f : V(H) \rightarrow \{1, 2, \ldots, k\}$ such that if $xy \in E(H)$, $f(x) \neq f(y)$. We refer to the labels as the colours of the vertices, and for convenience use the term $k$-colouring (since we only consider proper $k$-colourings). This terminology is consistent the Bondy and Murty [2], and we refer the reader to that text for notation and terminology not defined here.

Choo and MacGillivray [7] establish Gray code numbers for various classes of graphs. For complete graphs, $k_0(K_1) = 3$ and $k_0(K_n) = n + 1$ when $n \geq 2$. For cycles, $k_0(C_n) = 4$ for $n \geq 3$. Any tree $T$ satisfies $k_0(T) = 3$, except if $T$ is a star with an odd number of (at least three) vertices, in which case $k_0(T) = 4$. The results here extend the work presented in [7] in that we determine the Gray code numbers of complete bipartite graphs, of which stars are a special case. The general case of bipartite graphs that are not complete remains
largely unexplored. Connectivity and hamiltonicity of the \( k \)-colourings graphs of complete multipartite graphs is addressed in [1].

Connectivity of \( k \)-colouring graphs arises in random sampling of \( k \)-colourings, and approximating the number of \( k \)-colourings (see [8, 12, 13]). Neither the 2-colouring graph of a bipartite graph nor the 3-colouring graph of a 3-chromatic graph is ever connected, but for each \( k \geq 4 \) there exist \( k \)-chromatic graphs for which the \( k \)-colouring graph is connected, and others for which it is disconnected [4]. On the other hand, for any graph \( H \), the \( k \)-colouring graph is connected for all \( k \geq \text{col}(H) + 1 \) [8]. While it is Polynomial to decide if the 3-colouring graph of a bipartite graph is connected [3], it is NP-complete to decide if two given colourings belong to the same component of such a graph [6]. In [3] it is shown that the diameter of any component of the 3-colouring graph of a bipartite graph is bounded by a quadratic function of the number of vertices, but for each \( k \geq 4 \) there exist bipartite graphs on \( n \) vertices for which the diameter of some component of the \( k \)-colouring graph is exponential in \( n \); for each \( k \geq 4 \) it is PSPACE complete to decide if two given \( k \)-colourings belong to the same component of the \( k \)-colouring graph.

Other \( k \)-colouring graphs have also been considered. Viewing a \( k \)-colouring of \( H \) as a partition of \( V(H) \) with at most \( k \) cells leads to the \( k \)-Bell colour graph, while viewing it as a partition into exactly \( k \) parts leads to the \( k \)-Stirling colour graph. Every graph on \( n \) vertices has a hamiltonian \( n \)-Bell colour graph, and for each \( k \geq 4 \) the \( k \)-Stirling colour graph of a tree is hamiltonian [9]. The canonical \( k \)-colouring graph of \( H \) with respect to a fixed ordering \( \Pi \) of \( V(H) \) is the subgraph of \( G_k(H) \) obtained by first defining two \( k \)-colourings to be equivalent if they give rise to the same partition of \( V(H) \), and then taking the subgraph induced by the set of equivalence class representatives which are lexicographically least with respect to \( \Pi \). For every tree \( T \) there exists an ordering \( \Pi \) of the vertices such that the canonical \( k \)-colouring graph of \( T \) with respect to \( \Pi \) is Hamiltonian for all \( k \geq 3 \) [10]. For any graph \( H \) and any vertex ordering \( \Pi \), the canonical \( k \)-colouring graph of \( H \) with respect to \( \Pi \) is a spanning subgraph of the \( k \)-Bell colour graph of \( H \). Finally, connectivity of the graph of list-\( L(2,1) \)-labellings – proper colourings with some additional restrictions – has recently been studied in [11].

2. Gray code numbers of complete bipartite graphs

Let \( K_{\ell,r} \) be a complete bipartite graph with bipartition \((L,R)\), where \( L = \{p_1, p_2, \ldots, p_{\ell}\} \) and \( R = \{q_1, q_2, \ldots, q_r\} \). A colouring \( f \) of \( K_{\ell,r} \) with \( f(p_i) = a_i \), \( 1 \leq i \leq \ell \) and \( f(q_i) = b_i \), \( 1 \leq i \leq r \) is denoted \( \langle a_1a_2\ldots a_{\ell}|b_1b_2\ldots b_r \rangle \).

We begin by establishing a lower bound on \( k_0(K_{\ell,r}) \).

Theorem 1. For positive integers \( \ell \) and \( r \), \( G_2(K_{\ell,r}) \) is not hamiltonian, and \( G_3(K_{\ell,r}) \) is hamiltonian if and only if \( \ell, r \) are both odd.

Proof. A 2-colouring of \( K_{\ell,r} \) is completely determined by the colour of any one of its vertices, implying that \( |V(G_2(K_{\ell,r}))| = 2 \). Moreover, these two 2-colourings cannot be joined by an edge since the colours of all vertices of \( K_{\ell,r} \) must be changed to obtain one 2-colouring from the other. Since \( K_{\ell,r} \) has at least two vertices, \( G_2(K_{\ell,r}) \) is not connected and hence not hamiltonian.

Notice that every 3-colouring of \( K_{\ell,r} \) leaves at least one of \( L, R \) monochromatic, so for each \( j, 1 \leq j \leq 3 \), we define \( L_j \) to be the subgraph of \( G_3(H) \) induced by 3-colourings \( f \) in
which \( f(p) = j \) for all \( p \in L \); \( R_j \) is defined analogously. Thus every vertex of \( G_3(H) \) belongs to (at least) one of \( L_1, L_2, L_3, R_1, R_2, R_3 \).

The colourings in \( L_1 \) have all vertices of \( L \) coloured with 1 and the vertices of \( R \) coloured with 2 and 3. Thus each colouring in \( L_1 \) can be thought of as binary string of length \( r \) over \( \{2, 3\} \), implying that \( L_1 \) is isomorphic to the \( r \)-dimensional cube, \( Q_r \). It is routine to prove (and also follows from a result in [14]) that \( Q_r \) has a Hamilton path between \( 00\ldots0 \) and \( 11\ldots1 \) if and only if \( r \) is odd. Thus if \( r \) is odd, there is a Hamilton path \( P_{L,1} \) in \( L_1 \) between \( (1\ldots1|22\ldots2) \) and \( (1\ldots1|33\ldots3) \). If \( \ell \) is also odd, then \( R_3 \cong Q_\ell \), so \( R_3 \) has a Hamilton path \( P_{R,3} \) between \( (1\ldots1|33\ldots3) \) and \( (22\ldots2|33\ldots3) \). Analogously,

- \( L_2 \) has a Hamilton path \( P_{L,2} \) between \( (22\ldots2|33\ldots3) \) and \( (22\ldots2|11\ldots1) \);
- \( R_1 \) has a Hamilton path \( P_{R,1} \) between \( (22\ldots2|11\ldots1) \) and \( (33\ldots3|11\ldots1) \);
- \( L_3 \) has a Hamilton path \( P_{L,3} \) between \( (33\ldots3|11\ldots1) \) and \( (33\ldots3|22\ldots2) \);
- \( R_2 \) has a Hamilton path \( P_{R,2} \) between \( (33\ldots3|22\ldots2) \) and \( (11\ldots1|22\ldots2) \).

It follows that

\[
P_{L,1} \cup P_{R,3} \cup P_{L,2} \cup P_{R,1} \cup P_{L,3} \cup P_{R,2}
\]

is a Hamilton cycle of \( G_3(K_\ell,r) \).

Conversely, if \( r \) is even, then \( G_3(K_\ell,r) \) is not hamiltonian. The set \( \{(1\ldots1|22\ldots2), (1\ldots1|33\ldots3)\} \) forms a cut of \( G_3(K_\ell,r) \), since one must encounter at least one of these two vertices before leaving or entering \( L_1 \). Therefore, a Hamilton cycle of \( G_3(K_\ell,r) \) must contain a Hamilton path of \( L_1 \) that starts and ends at these two vertices. Since \( r \) is even, \( L_1 \cong Q_r \) contains no such Hamilton path, and thus \( G_3(K_\ell,r) \) is not hamiltonian.

Theorem 1 implies that if \( \ell, r \geq 1 \) and at least one of these is even, then \( k_0(K_\ell,r) \geq 4 \). It remains to show that this inequality is an equality.

Consider the complete graph \( K_n \) to have vertex set \( \{1, 2, \ldots, n\} \), and the cartesian product \( K_n \square K_n \) to have vertex set \( \{(i,j) \mid 1 \leq i, j \leq n\} \). Denote by \( J_n \) the graph obtained from \( K_n \square K_n \) by deleting the set of vertices \( \{(i,i) \mid 1 \leq i \leq n-1\} \).

**Lemma 2.** For \( n \geq 3 \), \( J_n \) has a Hamilton path between \( (n,n) \) and any vertex of \( J_n - (n,n) \).

**Proof.** Let \( v = (n,n) \). In Figure 2.1, we depict Hamilton paths between \( v \) and \( (1,2) \) when \( n \) is odd and when \( n \) is even, and Hamilton paths between \( v \) and \( (1,1) \) when \( n \) is even and when \( n \) is odd. The lemma is proved by showing that for every \( w \in V(J_n) \), \( w \neq v \), there is an automorphism of \( J_n \) that fixes \( v \) and maps \( w \) to either \( (1,2) \) or \( (1,1) \).

For any \( \pi \in S_n \), define \( \phi_\pi : V(J_n) \to V(J_n) \) by

\[
\phi_\pi(a,b) = (\pi(a), \pi(b)).
\]

If \( \pi(n) = n \), then it is straightforward to see that \( \phi_\pi \) is an automorphism of \( J_n \).

Suppose \( w = (w_1,w_2) \in V(J_n) \) is such that neither \( w_1 \) nor \( w_2 \) is equal to \( n \). Choose \( \pi = (1 \ w_1)(2 \ w_2) \), so that \( \phi_\pi \) is an automorphism of \( J_n \). Then

\[
\phi_\pi(w) = (\pi(w_1), \pi(w_2)) = (1,2),
\]

and hence \( J_n \) has a Hamilton path between \( v \) and \( w \). If \( w = (w_1,n) \), then choosing \( \pi = (1 \ w_1) \) again ensures that \( \phi_\pi \) is an automorphism of \( J_n \), and

\[
\phi_\pi(w) = (\pi(w_1), \pi(n)) = (1,n);
\]

\[
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\[
\phi_\pi(w) = (\pi(w_1), \pi(n)) = (1,n);
\]

and hence \( J_n \) has a Hamilton path between \( v \) and \( w \). If \( w = (w_1,n) \), then choosing \( \pi = (1 \ w_1) \) again ensures that \( \phi_\pi \) is an automorphism of \( J_n \), and
i.e., $J_n$ has a Hamilton path between $v$ and $w$. Finally, suppose $w = (n, w_2)$, and let \( \tau : V(J_n) \to V(J_n) \) be the automorphism of $J_n$ in which \( \tau(a, b) = (b, a) \).

Choosing \( \pi = (1 \ w_2) \) ensures that \( \phi_\pi \circ \tau \) is an automorphism of $J_n$ in which \( \phi_\pi \circ \tau(n, w_2) = \phi_\pi(w_2, n) = (\pi(w_2), \pi(n)) = (1, n) \).

Again, there is a Hamilton path in $J_n$ between $v$ and $w$. \( \square \)

We now use Lemma 2 to prove our main theorem.

**Theorem 3.** Let $1 \leq \ell \leq r$ and let $k \geq 4$. Then $G_k(K_{\ell,r})$ is hamiltonian.

**Proof.** The proof is by induction on $\ell$. When $\ell = 1$, the graph $K_{\ell,r}$ is a star, and it is known [7, Corollary 5.6] that $G_k(K_{1,r})$ is hamiltonian for $k \geq 4$.

For $\ell \geq 2$, let $K_{\ell,r}$ have bipartition $(L, R)$ with $u \in L$ and $v \in R$, and let $H$ denote the graph obtained from $K_{\ell,r}$ by deleting $u$ and $v$. Then $H \cong K_{\ell-1,r-1}$, and has bipartition $(L', R')$ where $L' = L \setminus \{u\}$ and $R' = R \setminus \{v\}$. Suppose $f_0, f_1, \ldots, f_{N-1}, f_0$ is a Hamilton cycle in $G_k(H)$. For $0 \leq i \leq N - 1$, define $F_i$ to be the subgraph of $G_k(K_{\ell,r})$ induced by the colourings that agree with $f_i$ on $H$. In what follows, the subscripts of $f_i$ and $F_i$ are taken modulo $N$. Let $[F_i, F_{i+1}]$ denote the set of edges that have one end in $F_i$ and the other end in $F_{i+1}$.

Suppose $i \in \{0, 1, \ldots, N - 1\}$. A colouring $t_i \in V(F_i)$ is called a sink if it is incident to an edge in $[F_i, F_{i+1}]$. If $t_i$ is a sink, then it is adjacent to exactly one colouring in $V(F_{i+1})$. 
Claim. For any $s_i \in V(F_i)$, there exists a sink $t_i \neq s_i$, and a Hamilton path in $F_i$ between $s_i$ and $t_i$.

Proof. Assume that the set of all colours is $C := \{1, 2, \ldots, k\}$. Let $U_r(i)$ and $U_r(i)$ be the sets of colours used in $L'$ and $R'$, respectively, under the colouring $f_i$. Then $A_r(i) := C \setminus U_r(i)$ and $A_r(i) := C \setminus U_r(i)$ are the sets of colours available for $u$ and $v$, respectively, to extend $f_i$ to a colouring in $F_i$.

Since only one vertex of $H$ changes colour between $f_i$ and $f_{i+1}$, at least one of the equalities $U_r(i+1) = U_r(i)$ or $U_r(i+1) = U_r(i)$ holds, implying that $A_r(i+1) = A_r(i)$ or $A_r(i+1) = A_r(i)$, respectively. Without loss of generality, assume that $A_r(i+1) = A_r(i)$.

Write $A_r(i) = \{x_1, x_2, \ldots, x_{\alpha_i}\}$ and $A_r(i) = \{y_1, y_2, \ldots, y_{\beta_i}\}$. If $A_r(i+1) \nsubseteq A_r(i)$, then the colour change from $f_i$ to $f_{i+1}$ introduces a new colour to $R'$, i.e., there exists a colour $x_j \in U_r(i+1) \setminus U_r(i)$. Since only one vertex of $H$ changes colour between $f_i$ and $f_{i+1}$, $x_j$ is unique and we may assume, without loss of generality, that $A_r(i+1) \setminus A_r(i+1) = \{x_1\}$, and hence $x_1 \in U_r(i+1) \setminus U_r(i)$. It follows that if a colouring $t_i \in V(F_i)$ is not a sink, then $t_i(u) = x_1$.

Let $d_i := |A_r(i) \cap A_r(i)|$ be the number of colours available to both $u$ and $v$ when extending $f_i$ to a colouring in $F_i$. Then $d_i < \min\{\alpha_i, \beta_i\}$ since $A_r(i), A_r(i)$ each contain colours not found in the other, namely, the colours used in $U_r(i), U_r(i)$, respectively. Assume $x_j = y_j$ for all $j$, $1 \leq j \leq d_i$.

If $d_i = 0$, then all colours of $C$ are used in $f_i$ and $\{U_r(i), U_r(i)\}$ is a partition of $C$. It follows that $U_r(i+1) \subseteq U_r(i)$, and hence $A_r(i) \subseteq A_r(i+1)$. Since $A_r(i) = A_r(i+1)$, every colouring in $V(F_i)$ is a sink. In this case, $F_i \cong K_{\alpha_i \Delta K_{\beta_i}}$; since $\alpha_i + \beta_i \geq 4$, $F_i$ is hamiltonian. We obtain a Hamilton path with $s_i \in V(F_i)$ as one end by deleting an edge incident to $s_i$ in an arbitrary Hamilton cycle of $F_i$.

Now suppose $d_i \geq 1$; then $\alpha_i \geq 2$ and $\beta_i \geq 2$. Let $s_i \in V(F_i)$. In what follows, we construct a Hamilton cycle in $F_i$ so that on the Hamilton cycle, $s_i$ is adjacent to a sink $t_i$. The subsequent deletion of the edge $s_i t_i$ results in the required Hamilton path.
First consider the case when $\alpha_i = 2$. Then $d_i = 1$, $x_1 = y_1$, and $\beta_i \geq 3$ (since $k \geq 4$ and $A_r(i) \cup A_t(i) = \{1, 2, \ldots, k\}$). If $s_i(v) = y_1$, choose $y_2$ is arbitrarily; otherwise, let $y_2 = s_i(v)$. Figure 2.2 shows a Hamilton cycle in $F_i$ when $\alpha_i = 2$ and $\beta_i = 7$, where the hollow vertices are sinks. This Hamilton cycle generalizes to arbitrary $\beta_i \geq 3$. Notice that if $s_i(v) = y_1$, then $s_i(u) = x_2$; otherwise, $s_i(v) = y_2$. In either case, $s_i$ is adjacent to a hollow vertex (sink) $t_i$ on the Hamilton cycle.

Now suppose $\alpha_i \geq 3$. Figures 2.3 and 2.4 show Hamilton cycles in $F_i$ when $\alpha_i = 4$ and $\beta_i = 7, 6$, respectively; again, the hollow vertices are sinks, and the Hamilton cycles generalize to arbitrary $\alpha_i$ and $\beta_i$ odd/even, respectively. Notice that any $s_i \in V(F_i)$ is adjacent to a hollow vertex (sink) $t_i$ on the Hamilton cycle.

We now describe a Hamilton cycle of $G_k(K_{\ell,r})$ for $r \geq l \geq 2$. Choose $f_0 = (11 \ldots 1|22 \ldots 2)$; since $r \geq l \geq 2$, $1 \not\in U_r(1)$ and $2 \not\in U_l(1)$. Thus $(11 \ldots 1|22 \ldots 2)$ is a sink in $V(F_0)$, so we define $t_0 = (11 \ldots 1|22 \ldots 2)$.

For $1 \leq i \leq N - 2$, define $s_i \in V(F_i)$ to be the vertex adjacent to $t_{i-1}$. By our earlier claim, there is a Hamilton path in $F_i$ between $s_i$ and a sink $t_i$. Suppose $s_{N-1}$ is the colouring in $F_{N-1}$ adjacent to $t_{N-2}$. Observe that all vertices of $F_{N-1}$ are sinks since the colours used in $f_0$ are used in $f_{N-1}$. Thus the Hamilton cycle in $F_{N-1}$ (whose existence is guaranteed in the proof of the claim) offers two choices for $t_{N-1}$: the two colourings adjacent to $s_{N-1}$ in the Hamilton cycle. Choose $t_{N-1}$ so that it is not adjacent to $t_0$, and let $s_0$ be the colouring in $F_0$ adjacent to $t_{N-1}$. This choice guarantees that $s_0 \neq t_0$. Since $F_0$ is isomorphic to the graph $G$ in Lemma 2 with $n = k - 1$, it follows from that lemma that $F_0$ contains a Hamilton path between $s_0$ and $t_0$. The union of the Hamilton paths from the $F_i$s, $0 \leq i \leq n - 1$, along with the edges $t_is_{i+1}$, $0 \leq i \leq n - 1$, yields the required Hamilton cycle.

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