Logic I: The Basics

A *statement* or *proposition* is an assertion which is either True, or False (though you may not know which). That is, a statement is something that has a *truth value*.

We usually use letters to denote statements. A good way to think of these letters is as variables that can take the values True and False (or 1 and 0).

Logical connectives. **Memorize!** Let p and q be statements:

• The *negation of* p (read: *not* p) is the statement $\neg p$ which is True if p is False, and False if p is True.

• The *conjunction of* p *and* q (read: p *and* q) is the statement $p \wedge q$ which is True if p is True and q is True, and False otherwise.

• The *disjunction of* p *and* q (read: p *or* q) is the statement $p \vee q$ which is True if p is True, or q is True, or both are True, and False otherwise.

• The *implication* $p \rightarrow q$ (read: p *implies* q, or *if* p *then* q) is the statement which is True if p is True and q is True, or if p is False (and q is True or False), and is False if p is True and q is False.

• The *biconditional* $p \leftrightarrow q$ (read: p *if and only if* q) is the statement which is True when p and q have the same truth value, and is False otherwise.

• The *exclusive or* of p and q is the statement $p \underline{\vee} q$ which is True when p and q have different truth values, and False otherwise. That is, $p \underline{\vee} q$ is True exactly when p is True, or q is True, but not both.

There are no prescedence rules, except that negations are done first. This means that $p \vee \neg q$ is $p \vee (\neg q)$, and that $p \vee q \rightarrow r$ is ambiguous. All expressions must therefore be properly bracketed.

You should make a truth table and check for yourself that $p \rightarrow q$ is not the same proposition as $q \rightarrow p$. **Memorize!** The implication $q \rightarrow p$ is called the *converse* of $p \rightarrow q$.

The truth values of a statement can be summarized in a *truth table*. You should be able to make one for any statement you are given, no matter how complex. To do this, start with columns corresponding to the statements represented by letters (if there are k letters you will need 2^k rows to list all possible combinations of truth values) and then, working with what's inside the brackets first (just like algebra!), add a new column for each connective in the expression, and fill in the truth values using the definitions above.

Memorize! A statement which is always True is called a *tautology*. A statement which is always False is called a *contradiction*.

For example, $p \wedge (\neg p)$ is a contradiction, while $p \vee (\neg p)$ is a tautology. Most statements are neither tautologies nor contradictions.

One way to determine if a statement is a tautology is to make its truth table and see if it is always True. Similarly, you can determine if a statement is a contradiction by making its truth table and seeing if it is always False.

Logic II: Logical Equivalence and Implication

Memorize! Two statements p and q are *logically equivalent* if $p \leftrightarrow q$ is a tautology. If p and q are logically equivalent, we write $p \Leftrightarrow q$.

There is a significant difference between $p \Leftrightarrow q$ and $p \leftrightarrow q$. The latter is a statement, so it has a truth value (i.e., $p \leftrightarrow q$ is either True or False). The former represents the fact that this statement is always True (it is a statement of fact, it does not have a truth value).

One way to check if two statements are logically equivalent is to make a truth table.

The Laws of Logic. You should **memorize** these, and be able to establish each of them with a truth table. In what follows, **T** denotes a statement that is always true, and **F** denotes a statement that is always False.

- $p \wedge \mathbf{T} \Leftrightarrow p, \quad p \wedge \mathbf{F} \Leftrightarrow \mathbf{F}$
- $p \vee \mathbf{T} \Leftrightarrow \mathbf{T}, \quad p \vee \mathbf{F} \Leftrightarrow p$
- $p \vee p \Leftrightarrow p, \quad p \wedge p \Leftrightarrow p$
- $p \wedge q \Leftrightarrow q \wedge p, \quad p \vee q \Leftrightarrow q \vee p$
- $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r), \quad (p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$
- Law of Double Negation: $\neg(\neg p) \Leftrightarrow p$
- Distributive Laws: $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r), \quad p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$
- DeMorgan's Laws: $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q, \quad \neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$

Some logical equivalences. The following are useful logical equivalences to know. You should **memorize** them, and also be able to to establish each with a truth table.

- $p \rightarrow q \Leftrightarrow \neg p \vee q$
- $p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$
- $p \underline{\vee} q \Leftrightarrow (p \wedge \neg q) \vee (\neg p \wedge q)$

Given any statement, you can use the Laws of Logic and the logical equivalences above above to write an equivalent statement that uses only the connectives $\wedge, \vee$, and $\neg$. For example $p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p) \Leftrightarrow (\neg p \vee q) \wedge (\neg q \vee p)$, so any statement involving $\leftrightarrow$ can be changed into one involving only $\wedge, \vee$, and $\neg$. If you use DeMorgan's Laws, then you really only need $\vee$ and $\neg$, or $\wedge$ and $\neg$. From before, $p \leftrightarrow q \Leftrightarrow (\neg p \vee q) \wedge (\neg q \vee p)$. By DeMorgan's Law, $(\neg p \vee q) \wedge (\neg q \vee p) \Leftrightarrow \neg\big(\neg(\neg p \vee q) \vee \neg(\neg q \vee p)\big)$, so $p \leftrightarrow q \Leftrightarrow \neg\big(\neg(\neg p \vee q) \vee \neg(\neg q \vee p)\big)$. The latter statement uses only $\vee$ and $\neg$. If you use DeMorgan's Law in a different way, then you can get an expression for $p \leftrightarrow q$ than involves only $\wedge$ and $\neg$. You should do it. It is a good exercise to do both for $p \underline{\vee} q$ and $p \rightarrow q$ too.

Memorize! $p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p$. The statement $\neg q \rightarrow \neg p$ is called *the contrapositive* of $p \rightarrow q$.

You should be able to use the logical equivalences above to establish other logical equivalences without making a truth table. You should also be able to use them to prove that a statement is a tautology (or contradiction) without making a truth table (this is the same as showing that it is logically equivalent to **T** (or **F**)). To do this, start with your statement and write a chain of logical equivalences (connected using $\Leftrightarrow$) that ends with $\Leftrightarrow$ **T** (or $\Leftrightarrow$ **F**).

Memorize! A statement p *logically implies* a statement q if q is true whenever p is true. (That is, $p \to q$ is a tautology.) If p logically implies q, we write $p \Rightarrow q$.

Note that in the definition of logically implies we don't care what happens if p is False. This is because of the truth table for implies: $p \to q$ is True when p is False.

There is a big difference between $p \to q$ and $p \Rightarrow q$. The former is a statement, and is therefore either True or False. The latter does not have a truth value, it represents the fact that $p \to q$ is a tautology.

The importance of logical implications arises from the fact that it is most often what we use in arguments. We usually want to say if "this" is true, then "that" must also be true. In other words, that "this" logically implies "that". Note the difference from $p \to q$, which is just True of False.

You should be able to explain why $p \Leftrightarrow q$ is the same as $p \Rightarrow q$ and $q \Rightarrow p$. The key is the logical equivalence $p \leftrightarrow q \Leftrightarrow (p \to q) \wedge (q \to p)$. Your proof needs two parts. In the first part you need to explain why if p is logically equivalent to q, then both p logically implies q and q logically implies p (i.e., if $p \leftrightarrow q$ is a tautology, then so are $p \to q$ and $q \to p$). In the second part you need to explain why if both p logically implies q and q logically implies p then p and q are logically equivalent (i.e., if $p \to q$ and $q \to p$ are tautologies, then so is $p \leftrightarrow q$).

Inference Rules. These are logical implications. You should **memorize** these (most are just common sense), and be able to establish each using a truth table.

- Modus Ponens: $(p \to q) \wedge p \Rightarrow q$
- Law of Syllogism: $(p \to q) \wedge (q \to r) \Rightarrow p \to r$
- Proof by Contradiction: $(\neg p \to \mathbf{F}) \Rightarrow p$
- Proof by Cases: $(p \to r) \wedge (q \to r) \Rightarrow (p \vee q) \to r$
- $p \Rightarrow p \vee q$
- $p \wedge q \Rightarrow p$

An *argument* is an implication $(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \to q$. The statements $p_1, p_2, \ldots, p_n$ are called *premises* (or hypotheses), and the statement q is called the *conclusion*. An argument is called *valid* if the implication is a tautology (i.e., if the conclusion is guaranteed to be true when all of the hypotheses are true), otherwise it is *invalid*.

To show an argument is invalid, you need to demonstrate that it is possible to make all of the premises True and the conclusion False. It is enough to give one example of truth values for the statements involved such that this happens. This is called a *counterexample* to the argument.

To show an argument is valid, you could make a truth table. But, if the number of variables involved is large, so is the table. Another way is to give a *proof*: a chain of logical equivalences and implications involving the premises (which are assumed to be true because the argument is an implication and is True otherwise). The idea is that every statement you write down is true, and is either a premise, or additional hypothesis, or is derives from statements know to be true via logical equivalences and implications.

Logic III: Quantifiers

The truth value of a statement like "x got 100% on her Math 122 exam" depends on x (it is a function of x; the values of the function are True and False). We denote such a statement by $P(x)$ and call it a *propositional function* of x, or an *open statement.*

For any allowed replacement for x (i.e, and x in the Universe), a propositional function $P(x)$ is either True or False, but you don't know which until you know x.

Memorize! If $P(x)$ is a propositional function, the statement "for all x (in the universe), $P(x)$ is true" is called the *Universal quantification* of $P(x)$, and denoted $\forall x, P(x)$. This is a statement, and is either True or False.

The symbol $\forall$ is called a *universal quantifier.*

Memorize! If $P(x)$ is a propositional function, the statement "there exists an x (in the universe) such that $P(x)$ is true" is called the *Existential quantification* of $P(x)$, and denoted $\exists x, P(x)$. This is a statement, and is either True or False.

The symbol $\exists$ is called a *existential quantifier.*

In statements involving more than one quantifier, you must read them from left to right, and order is crucial. For example, let $P(x, y)$ be the propositional function "x is a parent of y". For the universe of all people, $\forall y \exists x, P(x, y)$ is the statement that for every person y there is a person x such that x is a parent of y (which is True), while $\exists x \forall y, P(x, y)$ is the statement that there is a person x such that for every person y, x is a parent of y (which is False).

You need to practice writing statements using quantifiers (translating into symbols), and determining the truth value of quantified statements.

You should be able to explain why:

- $\neg(\exists x, P(x)) \Leftrightarrow \forall x, \neg P(x)$
- $\neg(\forall x, P(x)) \Leftrightarrow \exists x, \neg P(x)$

You should check out which logical implications hold between the following statements, and which pairs are logically equivalent. For each implication or equivalence, you should be able to give an explanation of why, and otherwise you should be able to give an example of propositional functions $P(x)$ and $Q(x)$, and a Universe, which shows that the implication or equivalence does not hold. I'll do two as examples below. Use my solutions as models for what is expected of you.

- $\forall x, P(x) \vee Q(x)$ and $(\forall x, P(x)) \vee (\forall x, Q(x))$
- $\forall x, P(x) \wedge Q(x)$ and $(\forall x, P(x)) \wedge (\forall x, Q(x))$
- $\forall x, P(x) \rightarrow Q(x)$ and $(\forall x, P(x)) \rightarrow (\forall x, Q(x))$
- $\exists x, P(x) \vee Q(x)$ and $(\exists x, P(x)) \vee (\exists x, Q(x))$
- $\exists x, P(x) \wedge Q(x)$ and $(\exists x, P(x)) \wedge (\exists x, Q(x))$
- $\exists x, P(x) \rightarrow Q(x)$ and $(\exists x, P(x)) \rightarrow (\exists x, Q(x))$

$\forall x, P(x) \vee Q(x)$ and $(\forall x, P(x)) \vee (\forall x, Q(x))$ are not logically equivalent: Let $P(x)$ be "x is even" and $Q(x)$ be "x is odd", and consider the Universe of all integers. Then $\forall x, P(x) \vee Q(x)$ is True, as every integer is either even or odd, but $(\forall x, P(x))$ is False (because $P(1)$ is False), and $(\forall x, Q(x))$ is False (because $P(2)$ is False), so $(\forall x, P(x)) \vee (\forall x, Q(x))$ is False. Therefore the two statements are not logically equivalent.

$(\forall x, P(x)) \vee (\forall x, Q(x))$ logically implies $\forall x, P(x) \vee Q(x)$. To explain why this is true, you need to explain why if the first statement is True, then the second statement must be True. Suppose $(\forall x, P(x)) \vee (\forall x, Q(x))$ is true. Then, either $P(x)$ is True for every x in the universe, or $Q(x)$ is True for every x in the universe (or both). In either case, $P(x) \vee Q(x)$ is true for every x in the universe (because $P(x) \Rightarrow P(x) \vee Q(x)$, and $Q(x) \Rightarrow P(x) \vee Q(x)$) that is, $\forall x, P(x) \vee Q(x)$ is True.

Logic IV: Writing Proofs in Words - A Starting point

Suppose you want to write a proof in words for a statement of the form "if p then q". That is, you wish to establish the theorem $p \Rightarrow q$. There are many ways. The *direct method* to prove p logically implies q is to *assume p is true* and then *argue using defintions, known implications and equivalences that q must be true.* The reason for assuming p is true comes from the definition of logical implication. In this case the first line of the proof is "*Assume p.*" and the last is "*Therefore q*". What comes inbetween depends on p and q. Another way is to *prove the contrapositive.* That is, assume q is False, and argue using the same things as above that p must also be False. This works since $p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p$. In this case the first line of the proof is "*Assume $\neg q$.*" and the last is "*Therefore $\neg p$*". This method is sometimes called giving an *indirect proof.* Yet another way is to employ *proof by contradiction.* You begin by assuming q False and, again proceeding as above, derive a statement which is a (logical) contradiction. This enables you to conclude that q is true. In such a situation, the first line of the proof is "Assume p and $\neg q$." and the proof finished with " ... which is a contradiction. Therefore q."