16 Function Composition

Suppose f and g are functions, and that the domain of g contains all real numbers in the range of f. We can then 'chain' the two functions together and get a new function by first evaluating f at x, and then evaluating g at $f(x)$. This new function is called the **composition** of f and g, and is denoted by $(g \circ f)$. The formal definition is

$$(g \circ f)(x) = g(f(x)).$$

As with other functions, the domain of $(g \circ f)$ is taken to be the largest set of real numbers for which the function is defined. This is the set of all x such that x is in the domain of f and $f(x)$ is in the domain of g (because first f is evaluated at x, and then g is evaluated at $f(x)$).

Example 1 *Let $f(x) = 3x - 1$ and $g(x) = x^2 + 1$. Then:*

- $f(2) = 3(2) - 1 = 6 - 1 = 5$ and $g(2) = 2^2 + 1 = 4 + 1 = 5$.
- $(g \circ f)(2) = g(f(2)) = g(5) = 26$.
- $(f \circ g)(2) = f(g(2)) = f(5) = 14$.
- Note: the previous two bullet points imply that $(g \circ f)(x)$ is not necessarily equal to $(f \circ g)(x)$. The order in which functions are composed matters!
- $(g \circ f)(y) = g(f(y)) = g(3y - 1) = (3y - 1)^2 + 1 = 9y^2 - 6y + 1 + 1 = 9y^2 - 6y + 2$
- $(f \circ g)(x) = f(g(x)) = f(x^2 + 1) = 3(x^2 + 1) - 1 = 3x^2 + 3 - 1 = 3x^2 + 2$
- Both $(g \circ f)$ and $(f \circ g)$ have domain equal to the set of all real numbers.

Example 2 *Let: $f(x) = \sqrt{x + 2}$ and $g(x) = 3 - x$. Determine the domain of $(g \circ f)$, the domain of $(f \circ g)$ and an expression for each of these functions.*

First, note that the domain of f is $[-2, \infty)$, and the domain of g is the set of all real numbers.

The domain of $(g \circ f)$ is the set of all x such that x is in the domain of f and $f(x)$ is in the domain of g. This is $[-2, \infty)$ because x must belong to the domain of f, and since the domain of g is the set of all real numbers, for any such x the number $f(x)$ is in the domain of g.

We have

$$(g \circ f)(x) = g(f(x)) = g(\sqrt{x + 2}) = 3 - \sqrt{x + 2}.$$

The domain of $(f \circ g)$ is the set of all x such that x is in the domain of g and $g(x)$ is in the domain of g. The domain of g is the set of all real numbers, so now we need to determine which of these are such that $g(x) = 3 - x$ is in $[-2, \infty)$, the domain of f. This happens when $3 - x \geq -2$, that is, when $x \leq 5$. Therefore the domain of $(f \circ g)$ is $(-\infty, 5]$.

We have

$$(f \circ g)(x) = f(g(x)) = f(3 - x) = \sqrt{(3 - x) + 2} = \sqrt{5 - x}.$$

A skill that is useful in calculus is recognizing when a function can be expressed as a composition of two other functions.

Example 3 *Suppose* $h(x) = (x^3 - 1)^7 + 2$. *Find functions* f *and* g *such that* $h = (g \circ f)$.

Looking at the expression of $h(x)$, it is an expression raised to the power 7, plus 2. Thus it makes sense to take $f(x)$ to be that expression, i.e., $f(x) = x^3 - 1$, and $g(x)$ to be a function that raises its input to the seventh power and then adds two, i.e., $g(x) = x^7 + 2$.

As a check, we can compute

$$(g \circ f)(x) = g(f(x)) = g(x^3 - 1) = (x^3 - 1)^7 + 2 = h(x).$$

16.1 Practice Problems

1. Let $f(x) = 3x + 5$, $g(x) = \frac{x^2+1x+2}{x}$ and $h(x) = \sqrt{x+7}$.
   (a) Find a formula for $(f \circ g)(x)$. What is the domain of $(f \circ g)(x)$?
   (b) Find a formula for $(g \circ f)(x)$. What is the domain of $(g \circ f)(x)$?
   (c) Find a formula for $(f \circ h)(x)$ .What is the domain of $(f \circ h)(x)$?
   (d) Find a formula for $(h \circ f)(x)$. What is the domain of $(h \circ f)(x)$?
   (e) Find a formula for $(h \circ g)(x)$ What is the domain of $(h \circ g)(x)$?
2. Let $f(x) = \sqrt{x}$ and $g(x) = x^2$.
   (a) Find a formula for $(f \circ g)(x)$. What us the domain of $(f \circ g)(x)$?
   (b) Find a formula for $(g \circ f)(x)$. What us the domain of $(g \circ f)(x)$.
   (c) Do the functions in (a) and (b) have the same domain? What does this tell you?
3. Fund functions $f(x)$ and $g(x)$ so that $(g \circ f)(x) = 3e^{2x-4} + x$.

16.2 Solutions

1. (a) $(f \circ g)(x) = \frac{3x^2+8x+6}{x}$, Domain: $\{x : x \neq 0\}$.
   (b) $(g \circ f)(x) = \frac{9x^2+33x+32}{3x+5}$, Domain: $\{x : x \neq -\frac{5}{3}\}$.
   (c) $(f \circ h)(x) = 3\sqrt{x+7} + 5$, Domain: $\{x : x \geq -7\}$.
   (d) $(h \circ f)(x) = \sqrt{3x+12}$, Domain: $\{x : x \geq -4\}$.
   (e) $(h \circ g)(x) = \sqrt{\frac{x^2+1x+2}{x} + 7}$, Domain: $\{x : -4 - \sqrt{14} \leq x \leq \sqrt{14} - 4$ or $x > 0\}$.

   (a) $(f \circ g)(x) = \sqrt{x^2}$, Domain:$(-\infty, \infty)$.
   (b) $(g \circ f)(x) = (\sqrt{x})^2$, Domain:$[0, \infty)$.
   (c) They are not the same, and therefore the two functions are not the same.
2. $f(x) = 2x - 4$ and $g(x) = 3e^x + \frac{x+4}{2}$.