First order optimality conditions for mathematical programs with second-order cone complementarity constraints

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Abstract

In this paper we consider a mathematical program with second-order cone complementarity constraints (SOCMPCC). The SOCMPCC generalizes the mathematical program with complementarity constraints (MPCC) in replacing the set of nonnegative reals by a second-order cone. We show that if the SOCMPCC is considered as an optimization problem with convex cone constraints, then Robinson’s constraint qualification never holds. We derive explicit formulas for the regular and limiting normal cone of the second-order complementarity cone. Using these formulas we derive explicit expressions for the strong-, Mordukhovich- and Clarke- (S-, M- and C-) stationary conditions and show that they are necessary optimality conditions under certain constraint qualifications. We have also shown that the classical KKT condition is in general not equivalent to the S-stationary condition unless the dimension of each second-order cone is not more than 2. Moreover we show that reformulating an MPCC as an SOCMPCC produces new and weaker necessary optimality conditions.

Key words: mathematical program with second-order cone complementarity constraints, necessary optimality conditions, constraint qualifications, S-stationary conditions, M-stationary conditions, C-stationary conditions.

AMS subject classification: 90C30, 90C33, 90C46.

1 Introduction

In this paper we consider the following mathematical program with second-order cone complementarity constraints (MPSOCC or SOCMPCC)

(SOCMPCC) \[ \min f(z) \]
\[ \text{s.t. } h(z) = 0, \quad g(z) \leq 0, \]
\[ K_i \ni G_i(z) \perp H_i(z) \in K_i, \quad i = 1, \ldots, J, \]

where \( a \perp b \) means that the vector \( a \) is perpendicular to vector \( b \). Throughout the paper we assume that \( f: \mathbb{R}^n \rightarrow \mathbb{R}, \ g: \mathbb{R}^n \rightarrow \mathbb{R}^p, \ h: \mathbb{R}^n \rightarrow \mathbb{R}^q, \ G_i: \mathbb{R}^n \rightarrow \mathbb{R}^{m_i}, \ H_i: \mathbb{R}^n \rightarrow \mathbb{R}^{m_i} \) are

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all continuously differentiable and $K_i$ is an $m_i$-dimensional second-order cone defined as

$$K_i := \{ x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{m_i-1} | x_1 \geq \|x_2\| \}$$

where $\| \cdot \|$ denotes the Euclidean norm and when $m_i = 1$, $K_i$ stands for the set of nonnegative reals $\mathbb{R}_+$.

SOCMPCC with all $m_i = 1$ for $i = 1, \cdots, J$ coincides with the mathematical program with complementarity constraints (MPCC) which has received a lot of attention in the last twenty years or so [10, 14]. The generalization from MPCC to SOCMPCC has many important applications. We briefly review two of them. In practice it is more realistic to assume that an optimization problem involves uncertainty. A recent approach to optimization under uncertainty is robust optimization. For example, it makes sense to consider a robust bilevel programming problem where for a fixed upper level decision variable $x$, the lower level problem is replaced by its robust counterpart:

$$P_x : \min_y \{ f(x, y, \zeta) : g(x, y, \zeta) \leq 0 \quad \forall \zeta \in U \},$$

where $U$ is some “uncertainty set” in the space of the data. It is well-known (see [2]) that if the uncertainty set $U$ is given by a system of conic quadratic inequalities, then the deterministic counterpart of the problem $P_x$ is a second-order cone program. If this second-order cone program can be equivalently replaced by its Karush-Kuhn-Tucker (KKT) condition, then it yields an SOCMPCC. Another application of SOCMPCC is in modelling an inverse quadratic programming problem over the second-order cone, in which the parameters in a given second-order cone quadratic programming problem need to be adjust as little as possible so that a known feasible solution becomes optimal (see [28] for details).

It is known that if an MPCC is treated as a nonlinear program with equality and inequality constraints, then Mangasarian-Fromovitz constraint qualification (MFCQ) fails to hold at each feasible point of the feasible region; see [26, Proposition 1.1]. This causes great difficulties in applying classical theories and algorithms in nonlinear programs directly to MPCCs. To remedy this problem, several variants of stationary conditions such as the strong (S-), Mordukhovich (M-), Clarke (C-) stationary conditions have been proposed and constraint qualifications under which a local minimizer is an S-, M-, C-stationary point have been studied; see e.g., [18, 24] for a detailed discussion. For SDCMPCC, the matrix analogue of the MPCC, it was shown in [6] that Robinson’s CQ, which is the usual constraint qualification for an optimization problem with convex cone constraints, fails to hold at each feasible point and the corresponding S-, M-, C-stationary conditions were proposed and the constraint qualifications under which a local minimizer is an S-, M-, C-stationary point have been studied.

The same difficulties exist for SOCMPCC. Notice that the cone complementarity constraint

$$K \ni G(z) \perp H(z) \in K,$$

where $G, H : \mathbb{R}^n \to \mathbb{R}^m$ and $K$ is the $m$-dimensional second-order cone, amounts to the following convex cone constraints:

$$\langle G(z), H(z) \rangle \leq 0, \quad G(z) \in K, \quad H(z) \in K.$$

In this paper we show that if SOCMPCC is regarded as an optimization problem with convex cone constraints, Robinson’s CQ fails to hold at each feasible point of SOCMPCC.
So far there are only a few papers devoted to the study of SOCMPCC \cite{9, 15, 19, 20, 21, 27, 28, 29} and \cite{19, 20, 21, 28, 29} mainly study numerical algorithms which are not the main purpose of this paper. To the best of our knowledge, the problem SOCMPCC was studied for the first time by Outrata and Sun in \cite{15}. The approach taken was to consider the cone complementarity constraint (1) as

\[(G(z) - H(z), G(z)) \in \text{gph}\Pi_K,\]

where \text{gph}\Pi_K is the graph of the metric projection operator onto the second order cone \(K\). By computing the limiting coderivative of the metric projection onto the second-order cone, a necessary optimality condition was derived. The same reformulation was further taken in Zhang, Zhang and Wu \cite{27} to define stationary conditions in terms of the regular, limiting and Clarke coderivative of the metric projection onto the second-order cone. However these optimality conditions are not in forms that are analogues to the S-, M-, C- stationary conditions for MPCCs and they are not explicit due to the existence of coderivatives of the metric projection onto the second order cone in these formulas.

Notice that the second-order cone complementarity constraint (1) can be reformulated as a nonconvex cone constraint:

\[(G(z), H(z)) \in \Omega,\]

where

\[\Omega := \{(x, y) | x \in K, \ y \in K, \ x^Ty = 0\}\]

is called the second-order complementarity cone. If the exact expression for the regular and the limiting normal cones of second-order complementarity cones can be derived, then the corresponding stationary conditions would be the suitable generalization of the S- and M-stationary conditions. The C-stationary condition can then be obtained by reformulating the cone complementarity constraint (1) as a nonsmooth equation constraint:

\[G(z) - \Pi_K(G(z) - H(z)) = 0\]

and applying the Clarke subdifferential calculus. The first attempt in this direction was initiated by Liang, Zhu and Lin in \cite{9} where they tried to derive exact expressions for the regular and the limiting normal cones of the second-order complementarity cone by using the relationships between the metric projection operator and the second-order complementary cone. Unfortunately, there are some gaps in their expressions of the regular and the limiting normal cones, mainly on the boundary points, which result in gaps in their proposed expressions for the S-, M-, and C-stationary conditions. In this paper we first fill in these gaps by deriving the correct exact expressions for the regular and limiting normal cone of the second-order complementary cone. Furthermore, we show that the regular and the proximal normal cones to the second-order complementary cone coincide. Using these exact expressions for the regular and the limiting normal cone of the second-order complementary cone, we propose the S-, M-, and C-stationary conditions for SOCMPCC.

It is well-known that for MPCC, the classical KKT condition is equivalent to the S-stationary condition see e.g. \cite{7}. For SDCMPCC it was shown in \cite{6} that in general the classical KKT condition is stronger than the S-stationary condition but these two conditions may not be equivalent. It is natural to ask the question whether or not the classical KKT condition is equivalent to the S-stationary condition for SOCMPCC. In this paper we show that for SOCMPCC, in general the classical KKT condition is a stronger condition than the S-stationary condition while these two concepts coincide when the dimension of each
second-order cone $K_i$ is not more than 2. Moreover we give an example to show that an S-stationary point may not be a classical KKT point when one of the second-order cone $K_i$ has dimensional greater than 2. Since in general the classical KKT condition and the S-stationary condition are different, we introduce a new stationary point concept called K-stationary point, which is equivalent to the classical KKT point. Moreover we have derived an exact expression for the set of all multipliers satisfying the K-stationary condition and shown that it is a subset of the regular normal cone of the second-order complementarity cone.

We summarize our main contributions as follows:

- We have presented the exact formula for the regular and limiting normal cone of the second-order complementarity cone and shown that the regular normal cone and the proximal normal cone of the second-order complementarity cone coincide with each other.

- We have proven that Robinson’s CQ fails to hold at every feasible point of SOCMPCC if the SOCMPCC is treated as an optimization problem with convex cone constraints.

- We have obtained the precise description for the S-, M-, and C-stationary conditions and derived necessary optimality conditions in terms of S-, M-, and C-stationary conditions. In particular we have shown that the S-stationary condition is a necessary optimality condition for a local minimum if SOCMPCC LICQ holds, i.e., if the gradients of all active constraints are linearly independent. We also shown that for the case where all mappings are affine and the dimension of each second-order cone is less or equal to 2, a local minimal solution of SOCMPCC must be an M-stationary point without any constraint qualification.

- We have shown that in general the K-stationary condition is stronger than the S-stationary condition but not equivalent and these two concepts coincide when the dimension of all $K_i$ is less or equal to 2.

We organize our paper as follows. Section 2 contains the preliminaries. In Section 3, we give the precise expression for the regular and the limiting normal cone for the second-order complementarity cone and shown that the regular and the proximal normal cone for the second-order complementarity cone are the same. In Section 4, we show that Robinson’s CQ never holds if SOCMPCC is considered as an optimization problem with convex cone constraints. The K-stationary condition is introduced and studied in this section. In Section 5, we give the explicit expression for the S-stationary condition. In Section 6, we give the explicit expression for the M-stationary condition and propose some constraint qualifications for the M-stationary condition. The explicit expression for the C-stationary condition is given in Section 7. Section 8 gives the connections among various stationary conditions. In Section 9 we reformulate MPCC as SOCMPCC and obtain some new and weaker necessary optimality conditions.

The following notations will be used throughout the paper. We denote by $I$ and $O$ the identity and zero matrix of appropriate dimensions respectively. For a matrix $A$, we denote by $A^T$ its transpose. The inner product of two vectors $x, y$ is denoted by $x^T y$ or $\langle x, y \rangle$. For any nonzero vector $z \in \mathbb{R}^m$, the notation $\bar{z}$ stands for the normalized vector $\frac{z}{\|z\|}$ if $z \neq 0$. For any $t \in \mathbb{R}$, define $t_+ := \max\{0, t\}$ and $t_- := \min\{0, t\}$. For $z = (z_1, z_2) \in \mathbb{R} \times \mathbb{R}^{m-1}$, we write its reflection about the $z_1$ axis as $\hat{z} := (z_1, -z_2)$. Given a vector $x$, denote by $\mathbb{R} x$ the set
2 Preliminaries

In this section we first review variational analysis and then specialize it to the second-order cone and the second-order complementarity cone.

2.1 Background in variational analysis

In this subsection we summarize some background materials on variational analysis which will be used throughout the paper. Detailed discussions on these subjects can be found in [4, 5, 12, 13, 17].

Let \( C \) be a nonempty subset of \( \mathbb{R}^n \). Given \( x^* \in \text{cl} C \), the proximal normal cone of \( C \) at \( x^* \) is defined as

\[
N_C^p(x^*) := \{ v \in \mathbb{R}^n | \exists M > 0, \text{ such that } \langle v, x - x^* \rangle \leq M \| x - x^* \|^2 \ \forall x \in C \}
\]

and the regular/Fréchet normal cone is

\[
\hat{N}_C(x^*) := \{ v \in \mathbb{R}^n | \langle v, x - x^* \rangle \leq o(\| x - x^* \|) \ \forall x \in C \}.
\]

The limiting/Mordukhovich normal is defined as the outer limit of either the proximal normal cone or the regular normal cone, i.e.,

\[
N_C(x^*) := \{ \lim_{i \to \infty} \zeta_i | \zeta_i \in N_C^p(x_i), \ x_i \to x^*, \ x_i \in C \} = \{ \lim_{i \to \infty} \zeta_i | \zeta_i \in \hat{N}_C(x_i), \ x_i \to x^* \ x_i \in C \}
\]

and the Clarke normal cone is the closure of the convex hull of the limiting normal cone

\[
N_C^e(x^*) := \text{clco} N_C(x^*).
\]

When the set \( C \) is convex, all normal cones coincide with the normal cone in the sense of convex analysis defined by \( N_C(x^*) := \{ v \in \mathbb{R}^n | \langle v, x - x^* \rangle \leq 0, \ \forall x \in C \} \).

Proposition 2.1 (Change of coordinates) [17, Exercise 6.7] Let \( F : \mathbb{R}^n \to \mathbb{R}^m \) be smooth and set \( D \subset \mathbb{R}^m \). Suppose that \( \nabla F(x^*) \) has full column rank \( m \) at a point \( x^* \in F := \{ x \in \mathbb{R}^n | F(x) \in D \} \). Then

\[
\hat{N}_F(x^*) = \{ \nabla F(x^*)y | y \in \hat{N}_D(F(x^*)) \},
\]

\[
N_F(x^*) = \{ \nabla F(x^*)y | y \in N_D(F(x^*)) \}.
\]

Let \( \Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m \) be a set-valued map and \( (x^*, y^*) \in gph \Phi \). The regular coderivative and the limiting (Mordukhovich) coderivative of \( \Phi \) at \( (x^*, y^*) \) are the set-valued maps defined by

\[
\hat{D}^* \Phi(x^*, y^*)(v) := \{ u \in \mathbb{R}^n | (u, -v) \in \hat{gph} \Phi(x^*, y^*) \},
\]

\[
D^* \Phi(x^*, y^*)(v) := \{ u \in \mathbb{R}^n | (u, -v) \in gph \Phi(x^*, y^*) \}.
\]
respectively. We omit $y^*$ in the coderivative notation if the set-valued map $\Phi$ is single-valued at $x^*$.

For a single-valued Lipschitz continuous map $\Phi : \mathbb{R}^n \to \mathbb{R}^m$, the B(ouligand)-subdifferential $\partial_B \Phi$ is defined as

$$\partial_B \Phi(x) = \{ \lim_{k \to \infty} \mathcal{J} \Phi(x_k) | x_k \to x, \Phi \text{ is differentiable at } x_k \}. $$

It is known that $co \partial_B \Phi(x) = \partial^c \Phi(x)$, the Clarke generalized Jacobian of $\Phi$ at $x$ (see [4]). Moreover if $\Phi$ is a continuously differentiable single-valued map, then

$$\bar{D}^* \Phi(x^*) = D^* \Phi(x^*) = \mathcal{J} \Phi(x^*). $$

2.2 Background in variational analysis associated with the second-order cone

We first review some basic properties of a second-order cone. Let $\mathcal{K}$ be the $m$-dimensional second-order cone. The topological interior and the boundary of $\mathcal{K}$ are

$$\text{int}\mathcal{K} = \{(z_1, z_2) \in \mathbb{R} \times \mathbb{R}^{m-1} | z_1 > \|z_2\|\} \quad \text{and} \quad \text{bd}\mathcal{K} = \{(z_1, z_2) \in \mathbb{R} \times \mathbb{R}^{m-1} | z_1 = \|z_2\|\} ,$$

respectively.

Proposition 2.2 For any $x, y \in \text{bd}\mathcal{K}\backslash\{0\}$, the following equivalence holds:

$$x^T y = 0 \iff y = k\hat{x} \text{ with } k = y_1/x_1 > 0. $$

Proof. Suppose that $x, y \in \text{bd}\mathcal{K}\backslash\{0\}$ and $x^T y = 0$. Then

$$x_1 = \|x_2\| > 0, \quad y_1 = \|y_2\| > 0, \quad x^T y = x_1y_1 + x_2^Ty_2 = 0, \quad (2)$$

which implies that $-x_2^Ty_2 = x_1y_1 = \|x_2\\|\|y_2\|$. Hence there exists a positive constant $k$ such that $y_2 = -kx_2$. It follows from (2) that $k = y_1/x_1$ and hence $y = k\hat{x}$. The converse statement is obviously true.

For any given vector $z := (z_1, z_2) \in \mathbb{R} \times \mathbb{R}^{m-1}$, it can be decomposed as

$$z = \lambda_1(z)c_1(z) + \lambda_2(z)c_2(z),$$

where $\lambda_i(z)$ and $c_i(z)$ for $i = 1, 2$ are the spectral values and the associated spectral vectors of $z$ given by

$$\lambda_i(z) = z_1 + (-1)^i\|z_2\| \quad \text{and} \quad c_i(z) = \begin{cases} \frac{1}{2}(1, (−1)^i\bar{z}_2) & \text{if } z_2 \neq 0 \\ \frac{1}{2}(1, w) & \text{if } z_2 = 0 \end{cases}$$

with $w$ being any vector in $\mathbb{R}^{m-1}$ satisfying $\|w\| = 1$. For $z \in \mathbb{R}^m$, let $\Pi_\mathcal{K}(z)$ be the metric projection of $z$ onto $\mathcal{K}$. Then by [8], it can be calculated by

$$\Pi_\mathcal{K}(z) = (\lambda_1(z))_+c_1(z) + (\lambda_2(z))_+c_2(z). \quad (3)$$

It is well known that the projection operator $\Pi_\mathcal{K}(z)$ is $B$-differentiable for any given $x \in \mathbb{R}^n$, i.e., it is locally Lipschitz and has directional derivatives. Let $\Pi_\mathcal{K}'(z; h)$ denotes the directional derivative of $\Pi_\mathcal{K}$ at $z$ in direction $h$. The following proposition summarizes its formula (see [15, Lemma 2]).
**Proposition 2.3** Let $\mathcal{K}$ be the $m$-dimensional second-order cone. The mapping $\Pi_{\mathcal{K}}(\cdot)$ is directionally differentiable at any $z \in \mathbb{R}^m$ and for any $h \in \mathbb{R}^m$,

(i) if $z \in \text{int}\mathcal{K}$ or $z \in -\text{int}\mathcal{K}$ or $z \in (\mathcal{K} \cup \mathcal{K})^c$, then $\Pi'_{\mathcal{K}}(z; h) = \mathcal{J}\Pi_{\mathcal{K}}(z)h$;
(ii) if $z \in \text{bd}\mathcal{K}\setminus\{0\}$, then $\Pi'_{\mathcal{K}}(z; h) = h - 2(c_1(z)^T h)\cdot c_1(z)$;
(iii) if $z \in -\text{bd}\mathcal{K}\setminus\{0\}$, then $\Pi'_{\mathcal{K}}(z; h) = 2(c_2(z)^T h)\cdot c_2(z)$;
(iv) if $z = 0$, then $\Pi'_{\mathcal{K}}(z; h) = \Pi_{\mathcal{K}}(h)$.

The following proposition summarizes the regular and the limiting coderivatives of the metric projection operator (see [15, Lemma 1 and Theorems 1 and 2]).

**Proposition 2.4** Let $\mathcal{K}$ be the $m-$dimensional second-order cone.

(i) If $z \in \text{int}\mathcal{K}$, then $\Pi_{\mathcal{K}}$ is differentiable and $\mathcal{J}\Pi_{\mathcal{K}}(z) = I$.
(ii) If $z \in -\text{int}\mathcal{K}$, then $\Pi_{\mathcal{K}}$ is differentiable and $\mathcal{J}\Pi_{\mathcal{K}}(z) = \{0\}$.
(iii) If $z \in (\mathcal{K} \cup \mathcal{K})^c$, then $\Pi_{\mathcal{K}}$ is differentiable and

$$\mathcal{J}\Pi_{\mathcal{K}}(z) = \frac{1}{2}(1 + \frac{z_1}{\|z_2\|})I + \frac{1}{2}\left[\frac{-z_1}{\|z_2\|} \quad \frac{z_2^T}{z_2} \right].$$

(iv) If $z \in \text{bd}\mathcal{K}\setminus\{0\}$, then

$$\hat{D}^*\Pi_{\mathcal{K}}(z)(u^*) = \{x^* | u^* - x^* \in \mathbb{R}_+ c_1(z), \langle x^*, c_1(z) \rangle \geq 0\},$$

$$D^*\Pi_{\mathcal{K}}(z)(u^*) = \partial_B \Pi_{\mathcal{K}}(z)u^* \cup \{x^* | u^* - x^* \in \mathbb{R}_+ c_1(z), \langle x^*, c_1(z) \rangle \geq 0\},$$

and

$$\partial_B \Pi_{\mathcal{K}}(z) = \left\{I, I + \frac{1}{2}\left[-1 \quad \frac{z_2^T}{z_2} \right] \right\}.$$

(v) If $z \in -\text{bd}\mathcal{K}\setminus\{0\}$, then

$$\hat{D}^*\Pi_{\mathcal{K}}(z)(u^*) = \{x^* | x^* \in \mathbb{R}_+ c_2(z), \langle u^* - x^*, c_2(z) \rangle \geq 0\},$$

$$D^*\Pi_{\mathcal{K}}(z)(u^*) = \partial_B \Pi_{\mathcal{K}}(z)u^* \cup \{x^* | x^* \in \mathbb{R}_+ c_2(z), \langle u^* - x^*, c_2(z) \rangle \geq 0\},$$

and

$$\partial_B \Pi_{\mathcal{K}}(z) = \left\{O, \frac{1}{2}\left[1 \quad \frac{z_2^T}{z_2} \right] \right\}.$$  

(vi) If $z = 0$, then

$$\hat{D}^*\Pi_{\mathcal{K}}(z)(u^*) = \{x^* | x^* \in \mathcal{K}, u^* - x^* \in \mathcal{K}\},$$

$$D^*\Pi_{\mathcal{K}}(z)(u^*) = \partial_B \Pi_{\mathcal{K}}(0)u^* \cup \{x^* | x^* \in \mathcal{K}, u^* - x^* \in \mathcal{K}\}$$

$$\cup \bigcup_{\xi \in C} \{x^* | u^* - x^* \in \mathbb{R}_+ \xi, \langle x^*, \xi \rangle \geq 0\}$$

$$\cup \bigcup_{\eta \in C} \{x^* | x^* \in \mathbb{R}_+ \eta, \langle u^* - x^*, \eta \rangle \geq 0\},$$
where
\[ C := \{(1, w) \mid w \in \mathbb{R}^{m-1}, \|w\| = 1\} \] (4)

and
\[ \partial_B \Pi_K(0) = \{O, I\} \cup \left\{ \frac{1}{2} \begin{bmatrix} 1 & w^T \\ w & 2\alpha I + (1-2\alpha)ww^T \end{bmatrix} \mid w \in \mathbb{R}^{m-1}, \|w\| = 1, \alpha \in [0,1] \right\} . \]

By the definition of the normal cone and the projection operator, it is easy to show the following equivalence between the normal cone and the projection operator.

**Proposition 2.5** For the m-dimensional second-order complementarity cone \( \Omega \), the following equivalence holds:

\[ (x, y) \in \Omega \iff -y \in N_K(x) \iff x = \Pi_K(x - y) \iff -y = \Pi_K(x - y). \]

As we will show in the next proposition, the expressions of the regular and the limiting normal cone for the second-order complementarity cone can be derived from the expression for the coderivatives of the projection operators.

**Proposition 2.6** Let \( \Omega \) be the m-dimensional complementarity cone. Then

\[ \hat{N}_{\Omega}(x, y) = \left\{ (u, v) \mid -v \in \hat{D}\Pi_K(x - y)(-u - v) \right\} , \] (5)

\[ N_{\Omega}(x, y) = \left\{ (u, v) \mid -v \in D\Pi_K(x - y)(-u - v) \right\} . \] (6)

**Proof.** By Proposition 2.5, we have

\[ \Omega = \{(x, y) \mid -y \in N_K(x)\} = \{(x, y) \mid \Pi_K(x - y) = x\} \]
\[ = \{(x, y) \mid (x - y, x) \in \text{gph} \Pi_K\}. \]

It follows from the change of coordinate formula in Proposition 2.1 that

\[ \hat{N}_{\Omega}(x, y) = \begin{bmatrix} I & I \\ -I & 0 \end{bmatrix} \hat{N}_{\text{gph}\Pi_K}(x - y, x). \]

Hence

\[ (u, v) \in \hat{N}_{\Omega}(x, y) \iff \begin{bmatrix} 0 & -I \\ I & I \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \in \hat{N}_{\text{gph}\Pi_K}(x - y, x) \]
\[ \iff (-v, u + v) \in \hat{N}_{\text{gph}\Pi_K}(x - y, x) \]
\[ \iff -v \in \hat{D}\Pi_K(x - y)(-u - v), \]

from which (5) follows. The formula (6) comes from the same argument by replacing \( \hat{N}_{\Omega} \) by \( N_{\Omega} \) and \( \hat{D} \) by \( D \).

### 3 Expression of the regular and limiting normal cones

In order to characterize the S-stationary and M-stationary conditions, we need to give precise expressions for the regular/proximal and limiting normal cones of the second-order complementarity cone. The purpose of this section is to provide such formulas. The result is also of independent interest.
3.1 Expression of the regular normal cone

In [9, Proposition 2.2], Liang, Zhu and Lin gave a formula for the regular and limiting normal cone of the second-order complementarity cone. Their formula for the case where \((x,y) \in \Omega\) with \(x, y \in \text{bd}\mathcal{K}\setminus\{0\}\) is the following:

\[
\tilde{N}_\Omega(x,y) = \{(u,v) \in \mathbb{R}^m \times \mathbb{R}^m | u \in \mathbb{R}\hat{x}, v \in \mathbb{R}\hat{y}\}.
\]  

(7)

The following example shows that formula (7) is not correct when the dimension \(m\) is greater than 2. In the meantime, the example illustrates our new formula.

Example 3.1 Take \(x = (1,1/\sqrt{2},1/\sqrt{2})\) and \(y = (2,-\sqrt{2},-\sqrt{2})\). It is easy to see that \((x,y) \in \Omega\) with \(x, y \in \text{bd}\mathcal{K}\setminus\{0\}\), and \(y = 2\hat{x}\). Let \(u = (1/\sqrt{2},-1,0)\) and \(v = (1/(2\sqrt{2}),0,1/2)\). Since \(x - y \in \mathbb{R}^3\setminus(-\mathcal{K} \cup \mathcal{K})\), by Proposition 2.4(iii), we have

\[
\mathcal{J}\Pi\mathcal{K}(x-y) = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\
\frac{1}{2\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2\sqrt{2}} & \frac{1}{2} & \frac{1}{2}
\end{bmatrix}
\]

and hence

\[
\mathcal{D}^*\Pi\mathcal{K}(x-y)(-u-v) = \mathcal{J}\Pi\mathcal{K}(x-y)(-u-v) = (-\frac{1}{2\sqrt{2}},0,-\frac{1}{2}) = -v.
\]

By Proposition 2.6, it follows that \((u,v) \in \tilde{N}_\Omega(x,y)\). However since \(u \notin \mathbb{R}\hat{x}\) and \(v \notin \mathbb{R}\hat{y}\), formula (7) is incorrect. In fact, according to our formula to be derived in Proposition 3.1, \((u,v)\) is an element of the regular normal cone since \(u \perp x, v \perp y\) and \(x_1\hat{u} + y_1v = \sqrt{2}(1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \in \mathbb{R}x\).

In the following proposition, we revise the formula for the regular normal cone obtained in [9, Proposition 2.2] for the case where \(x, y \in \text{bd}\mathcal{K}\setminus\{0\}\), \(x^Ty = 0\). It is easy to see that when \(m = 2\), the condition \(u \perp x, v \perp y, x_1\hat{u} + y_1v \in \mathbb{R}x\) is equivalent to \(u \perp x, v \perp y\), which in turn is equivalent to \(u \in \mathbb{R}\hat{x}, v \in \mathbb{R}\hat{y}\). Hence when \(m \leq 2\), our regular normal cone formula is the same as the one given in [9, Proposition 2.2].

Proposition 3.1 Let \((x,y)\) be any element in the \(m\)-dimensional second-order complementarity cone \(\Omega\). Then

\[
\tilde{N}_\Omega(x,y) = \begin{cases}
\{(u,v) | u \in \mathbb{R}^m, v = 0\} & \text{if } x = 0, y \in \text{int}\mathcal{K}; \\
\{(u,v) | u = 0, v \in \mathbb{R}^m\} & \text{if } x \in \text{int}\mathcal{K} \text{ and } y = 0; \\
\{(u,v) | u \perp x, v \perp y, x_1\hat{u} + y_1v \in \mathbb{R}x\} & \text{if } x, y \in \text{bd}\mathcal{K}\setminus\{0\}, x^Ty = 0; \\
\{(u,v) | u \in \hat{y}, v \in \mathbb{R}^m - \hat{y}\} & \text{if } x = 0, y \in \text{bd}\mathcal{K}\setminus\{0\}; \\
\{(u,v) | u \in \mathbb{R}^m - \hat{x}, v \in \mathbb{R}^m - \hat{x}\} & \text{if } x \in \text{bd}\mathcal{K}\setminus\{0\}, y = 0; \\
\{(u,v) | u \in -\mathcal{K}, v \in -\mathcal{K}\} & \text{if } x = 0, y = 0.
\end{cases}
\]

Proof. See the “Appendix”. ■

3.2 Equivalence of the proximal and regular normal cones

In general the proximal normal cone is smaller than the regular normal cone for a nonconvex set. For the case where \(\Omega := \{(x,y) \in \mathbb{R}^m \times \mathbb{R}^m | 0 \leq x \perp y \geq 0\}\), the proximal normal cone and the regular normal cone of the complementarity cone obviously coincide. For the case
of the semidefinite matrix complementarity cone \( \Omega := \{(x, y) \in S \times S | S \ni x \perp y \in S_{+}\} \), where \( S \) and \( S_{+} \) denote the space of all symmetric matrices and the space of all positive semidefinite matrices respectively, it was shown in [6] that its proximal normal cone and regular normal cone coincide too. In this subsection we show that for the second-order complementarity cone, the proximal normal cone coincides with the regular normal cone as well.

In the following result, we show that the metric projection operator is not only \( B \)-differentiable but also calmly \( B \)-differentiable.

**Proposition 3.2** The metric projection operators \( \Pi_{K}(\cdot) \) and \( \Pi_{K^{o}}(\cdot) \) are calmly \( B \)-differentiable for any given \( x \in \mathbb{R}^{m} \), i.e., for any \( h \to 0 \),

\[
\Pi_{K}(x + h) - \Pi_{K}(x) - \Pi'_{K}(x; h) = O(h^{2}),
\]
\[
\Pi_{K^{o}}(x + h) - \Pi_{K^{o}}(x) - \Pi'_{K^{o}}(x; h) = O(h^{2}).
\]

**Proof.** See the “Appendix”. ■

In the following proposition, we establish the relationship between the proximal normal cone of the second order complementarity cone and the projection operation onto the second order cone.

**Proposition 3.3** Let \( (x, y) \in \Omega \), the \( m \)-dimensional second-order complementarity cone. Then \( (u, v) \in N_{\Omega}^{2}(x, y) \) if and only if

\[
\langle u + v, \Pi'_{K}(x - y; h) \rangle - \langle v, h \rangle \leq 0, \quad \forall h \in \mathbb{R}^{m}.
\]

**Proof.** “\( \Longleftarrow \)” Suppose that \( (u, v) \) satisfies the condition (8). By Propositions 2.5 and 3.2, there exist positive constants \( \delta, M \) such that for any \( (x', y') \in \Omega \) with \( \|(x', y') - (x, y)\| \leq \delta \), one has

\[
\langle (u, v), (x', y') - (x, y) \rangle
\]
\[
= \langle (u, v), (\Pi'_{K}(x' - y') - \Pi_{K}(x - y), -\Pi_{K^{o}}(x' - y') + \Pi'_{K^{o}}(x - y)) \rangle
\]
\[
\leq \langle (u, v), (\Pi'_{K}(x - y; x' - y' - x + y), -\Pi'_{K^{o}}(x - y; x' - y' - x + y)) \rangle + M\|(x', y') - (x, y)\|^2.
\]

Since \( x = \Pi_{K}(x) + \Pi_{K^{o}}(x) \) for all \( x \), taking the directional derivatives on both sides of the equation yields \( d = \Pi'_{K}(x; d) + \Pi'_{K^{o}}(x; d) \) for any \( d \). It then follows that

\[
\langle (u, v), (x', y') - (x, y) \rangle
\]
\[
\leq \langle (u, v), (\Pi'_{K}(x - y; x' - y' - x + y), -\Pi'_{K^{o}}(x - y; x' - y' - x + y)) \rangle + M\|(x', y') - (x, y)\|^2
\]
\[
= \langle (u + v), (\Pi'_{K}(x - y; x' - y' - x + y)) \rangle - \langle v, x' - y' - x + y \rangle + M\|(x', y') - (x, y)\|^2
\]
\[
\leq M\|(x', y') - (x, y)\|^2 \quad \text{by virtue of (8),}
\]

which implies that \( (u, v) \in N_{\Omega}^{2}(x, y) \) due to the local property of the proximal normal cone.

“\( \Longrightarrow \)” Let \( (u, v) \in N_{\Omega}^{2}(x, y) \) be given. Then by definition of the proximal normal cone, there exists \( M > 0 \) such that for any \( (x', y') \in \Omega \),

\[
\langle (u, v), (x', y') - (x, y) \rangle \leq M\|(x', y') - (x, y)\|^2.
\]

(9)
In particular for any \( h \in \mathbb{R}^m \) and any \( t > 0 \), take
\[
x' := \Pi_K(x - y + th), \quad y' = -\Pi_{K^o}(x - y + th).
\]
By Moreau’s decomposition principle, \((x', y') \in \Omega\). Hence by (9) and Proposition 2.5, we have
\[
\langle (u, v), (\Pi_K(x - y + th) - \Pi_K(x - y), -\Pi_{K^o}(x - y + th) + \Pi_{K^o}(x - y)) \rangle \\
\leq M\| (\Pi_K(x - y + th) - \Pi_K(x - y), -\Pi_{K^o}(x - y + th) + \Pi_{K^o}(x - y)) \|^2 \\
\leq 2M^2\|h\|^2 \text{ by virtue of the global Lipschitz continuity of the projection operator.}
\]
Dividing the above inequality by \( t \) and taking the limits as \( t \downarrow 0 \), we have
\[
\langle u, \Pi_K'(x - y; h) \rangle + \langle v, -\Pi_{K^o}(x - y; h) \rangle \leq 0.
\]
Since \( \Pi_K'(x - y; h) + \Pi_{K^o}(x - y; h) = h \), the above inequality is the same as (8) and hence the proof of the proposition is complete. \( \blacksquare \)

Using Propositions 3.2 and 3.3 we can now show the equivalence between the regular and proximal normal cone.

**Proposition 3.4** For the second-order complementarity cone \( \Omega \), one has \( N_{\Omega}(x, y) = N_{\Omega}^p(x, y) \).

**Proof.** See the “Appendix”. \( \blacksquare \)

### 3.3 Expression of the limiting normal cone

Due to the mistake in the formula for the regular normal cone when \( x, y \in \text{bd}K \setminus \{0\} \), the limiting normal cone given in [9, Proposition 2.2] also contains mistakes for the case where \( x = 0, y \in \text{bd}K \setminus \{0\} \), the case where \( x \in \text{bd}K \setminus \{0\}, y = 0 \) and the case \( x = y = 0 \). This is not surprising since in these cases, the limiting normal cones contain elements which are limits of elements in the regular normal cone at points lying in \((\text{bd}K \setminus \{0\}) \times (\text{bd}K \setminus \{0\})\). The formulas of the limiting normal cone given in [9, Proposition 2.2] for these three cases are

\[
N_{\Omega}(x, y) = \{(u, v) | u \in \mathbb{R}^m, \ v = 0 \text{ or } u \in (\mathbb{R}_+\hat{y})^\circ, \ v \in \mathbb{R}_-\hat{y}\} \quad \text{if } x = 0, y \in \text{bd}K \setminus \{0\}, \quad (10)
\]

\[
N_{\Omega}(x, y) = \{(u, v) | u = 0, v \in \mathbb{R}^m \text{ or } u \in \mathbb{R}_-\hat{x}, \ v \in (\mathbb{R}_+\hat{x})^\circ\} \quad \text{if } x \in \text{bd}K \setminus \{0\}, y = 0, \quad (11)
\]

and if \( x = y = 0 \),

\[
N_{\Omega}(x, y) = \left\{ \begin{array}{ll}
(u, v) | u \in -\mathcal{K}, \ v \in -\mathcal{K} & \text{or } u \in \mathbb{R}^m, \ v = 0 \text{ or } u = 0, v \in \mathbb{R}^m \\
& \text{or } u \in \mathbb{R}_-\xi, \ v \in (\mathbb{R}_+\xi)^\circ \text{ for some } \xi \in C \\
& \text{or } u \in (\mathbb{R}_+\xi)^\circ, \ v \in \mathbb{R}_-\xi \text{ for some } \xi \in C \\
& \text{or } u \in \mathbb{R}_\xi, \ v \in \mathbb{R}_\xi \text{ for some } \xi \in C \end{array} \right\} (12)
\]

where \( C \) is defined as in (4). When \( m = 2 \), it is easy to see that for any \( \xi \in C \) and any \( \alpha \in [0, 1] \),

\[
u \perp \xi, \hat{v} \perp \hat{\xi} \iff u \in \mathbb{R}_\xi, v \in \mathbb{R}_\xi \iff u \perp \xi, v \perp \hat{\xi}, \alpha u + (1 - \alpha)\hat{v} \in \mathbb{R}_\hat{\xi}.
\]
Hence when \( m = 2 \), the limiting normal cone formula (12) at \((x, y) = (0, 0)\) is equivalent to our formula to be given in Proposition 3.5. The following example illustrates that the formulas (10) and (11) are not correct even when the dimension \( m = 2 \) and the formula (12) is incorrect when \( m \) is greater than 3.

**Example 3.2** 1) For \( x = (0, 0), y = (1, 1) \in \text{bd} K \setminus \{0\} \), let \( u = (1, 1) \) and \( v = (2, -2) \). Since \( x - y = (-1, -1) \in -\text{bd} K \setminus \{0\} \), by Proposition 2.4(iv)

\[
\frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \in \partial_B \Pi_K(x - y)
\]

and hence

\[
\frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} (-u - v) \in D^* \Pi_K(x - y)(-u - v).
\]

Note that

\[-v \in D^* \Pi_K(x - y)(-u - v).
\]

By Proposition 2.6, it follows that \((u, v) \in N_{\Omega}(x, y). But v \notin \mathbb{R}_- = \mathbb{R}_-(1, -1). Hence the formula (10) is incorrect. However \((u, v)\) satisfies the formula we proposed in Proposition 3.5 below, since

\[
u = (1, 1) \perp (1, -1) = \hat{y}, \quad v = (2, -2) = 2\hat{y} \in \mathbb{R}_\hat{y}.
\]

2) For \( x = (1, 1) \in \text{bd} K \setminus \{0\} \) and \( y = (0, 0) \), let \( u = (1, -1) \) and \( v = (1, 1) \). Since \( x - y = (1, 1) \in \text{bd} K \setminus \{0\} \) by Proposition 2.4(iv)

\[
\left( I + \frac{1}{2} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \right) \in \partial_B \Pi_K(x - y)
\]

and hence

\[
\left( I + \frac{1}{2} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \right) (-u - v) \in D^* \Pi_K(x - y)(-u - v).
\]

Note that

\[-v \in D^* \Pi_K(x - y)(-u - v).
\]

By Proposition 2.6, it follows that \((u, v) \in N_{\Omega}(x, y). But u = (1, -1) \notin \mathbb{R}_- = \mathbb{R}_\hat{x}. Hence the formula (11) is incorrect. However \((u, v)\) satisfies the formula proposed in Proposition 3.5 below, since

\[
v = (1, -1) \perp (1, -1) = \hat{x}, \quad u = \hat{x} \in \mathbb{R}_\hat{x}.
\]

3) For \( x = y = (0, 0, 0) \), let \( u = (1, -1, 1) \) and \( v = (0, 0, 1) \). Note that \( u \notin \mathbb{R}(1, -w) \) and \( v \notin \mathbb{R}(1, w) \) with \( \|w\| = 1 \), and hence \((u, v)\) does not belong to set proposed by the formula (12). However, by letting \( \alpha = 1/2 \) and \( w = (1, 0) \), we have

\[
\frac{1}{2} \begin{bmatrix} 1 & w^T \\ w & I \end{bmatrix} (-u - v) = -v.
\]

Hence \(-v \in D^* \Pi_K(x - y)(-u - v), i.e., \((u, v) \in N_{\Omega}(x, y) \) by Proposition 2.6. Take \( \xi = (1, 1, 0) \). Then \( u \perp \xi, v \perp \xi, \) and \( \frac{1}{2}u + \frac{1}{2}v = \frac{1}{2}(1, -1, 0) \in \mathbb{R}_\xi \), i.e., \((u, v)\) satisfies the formula proposed in Proposition 3.5 below.

We now give a correct formula for the limiting normal cone of the complementarity cone. Note the the conditions \( x_1 \hat{u} + y_1 \hat{v} \in \mathbb{R}x \) and \( \alpha u + (1 - \alpha) \hat{v} \in \mathbb{R}_\hat{x}, \alpha \in [0, 1] \) are redundant when \( m = 2 \).
Proposition 3.5 Let \((x, y) \in \Omega\) where \(\Omega\) is the \(m\)-dimensional second-order complementarity cone.

\[
N_\Omega(x, y) = \hat{N}_\Omega(x, y) = \begin{cases} 
\{(u, v)|u \in \mathbb{R}^m, \ v = 0\} \text{ if } x = 0, \ y \in \text{int}\mathbb{K}; \\
\{(u, v)|u = 0, \ v \in \mathbb{R}^m\} \text{ if } x \in \text{int}\mathbb{K}, \ y = 0; \\
\{(u, v)|u \perp x, \ v \perp y, \ x_1u + y_1v \in \mathbb{R}x\} \text{ if } x, y \in \text{bd}\mathbb{K}\{0\}. 
\end{cases}
\]

For \(x = 0, y \in \text{bd}\mathbb{K}\{0\}\),

\[
N_\Omega(x, y) = \{(u, v)|u \in \mathbb{R}^m, \ v = 0 \text{ or } u \perp \hat{y}, \ v \in \mathbb{R}\hat{y} \text{ or } \langle u, \hat{y} \rangle \leq 0, \ v \in \mathbb{R}_-\hat{y}\};
\]

for \(x \in \text{bd}\mathbb{K}\{0\}, y = 0\),

\[
N_\Omega(x, y) = \{(u, v)|u = 0, v \in \mathbb{R}^m \text{ or } u \in \mathbb{R}\hat{x}, v \perp \hat{x} \text{ or } u \in \mathbb{R}_-\hat{x}, \langle v, \hat{x} \rangle \leq 0\};
\]

for \(x = y = 0\),

\[
N_\Omega(x, y) = \{(u, v)|u \in -\mathbb{K}, v \in -\mathbb{K} \text{ or } u \in \mathbb{R}^m, v = 0 \text{ or } u = 0, v \in \mathbb{R}^m \\
\text{or } u \in \mathbb{R}_-\xi, v \in \xi^o \text{ or } u \in \xi^o, v \in \mathbb{R}_-\xi \\
\text{or } u \perp \xi, v \perp \hat{\xi}, \alpha u + (1 - \alpha)\hat{v} \in \mathbb{R}\hat{\xi}, \alpha \in [0, 1], \text{ for some } \xi \in C\}
\]

where \(C\) is defined as in (4).

Proof. See the “Appendix”. 

4 Failure of Robinson’s CQ and the classical KKT condition

Note that \(G_i(z), H_i(z) \in \mathbb{K}_i\) implies that \(G_i(z)^TH_i(z) \geq 0\) for \(i = 1, \ldots, m\). Hence SOCMPC can be rewritten as an optimization problem with a convex cone constraint:

\[
(\text{K-SOCMPC}) \quad \min f(z) \\
\text{s.t. } g(z) \leq 0, \ h(z) = 0, \\
\quad \langle G(z), H(z) \rangle \leq 0, \\
\quad (G(z), H(z)) \in \hat{\mathbb{K}} \times \mathbb{K},
\]

where \(G(z) := (G_1(z), \ldots, G_J(z)), H(z) := (H_1(z), \ldots, H_J(z))\), and \(\hat{\mathbb{K}} := \mathbb{K}_1 \times \mathbb{K}_2 \times \cdots \times \mathbb{K}_J\). We denote by \(\tau := \sum_{j=1}^J m_j\).

For a general optimization problem with a cone constraint such as K-SOCMPC, the following Robinson’s CQ is considered to be a usual constraint qualification:

\[
\nabla h_i(z^*)(i = 1, \ldots, q) \text{ are linearly independent,}
\]

\[
\exists d \in \mathbb{R}^n \text{ such that } \begin{cases} \nabla h_i(z^*)^Td = 0, & i = 1, \ldots, q, \\
g(z^*) + \nabla g(z^*)^Td \in \text{int}\mathbb{R}_+^n, \\
(\nabla H(z^*)G(z^*) + \nabla G(z^*)H(z^*))^Td < 0, \\
G(z^*) + \nabla G(z^*)^Td \in \text{int}\hat{\mathbb{K}}, \\
H(z^*) + \nabla H(z^*)^Td \in \text{int}\hat{\mathbb{K}}.
\end{cases}
\]

It is well-known that the MFCQ never holds for MPCCs. We now show that Robinson’s CQ will never hold for K-SOCMPC.
**Proposition 4.1** For K-SOCMPCC, Robinson’s constraint qualification fails to hold at every feasible solution of SOCMPCC.

**Proof.** Any feasible solution \( z^* \) of SOCMPCC must be a solution to the following convex cone constrained program:

\[
\begin{align*}
\min & \quad \langle G(z), H(z) \rangle \\
\text{s.t.} & \quad G(z) \in \overline{K}, \quad H(z) \in \overline{K}.
\end{align*}
\]

By the Fritz John necessary optimality condition, there exist \( \lambda_0 \geq 0, \lambda^G \in \mathbb{R}^r, \lambda^H \in \mathbb{R}^r \) not all equal to zero such that

\[
0 = \lambda_0 \nabla \langle G, H \rangle(z^*) + \nabla G(z^*) \lambda^G + \nabla H(z^*) \lambda^H, \quad \lambda^G \in N_{\overline{K}}(G(z^*)), \quad \lambda^H \in N_{\overline{K}}(H(z^*)).
\]

It is clear that \((0, 0, 0, \lambda_0, \lambda^G, \lambda^H)\) is a singular Lagrange multiplier of K-SOCMPCC. By [3, Propositions 3.16 (ii) and 3.19(iii)], a singular Lagrange multiplier exists if and only if Robinson’s CQ does not hold. Therefore we conclude that the Robinson’s CQ does not hold at \( z^* \) for K-SOCMPCC.

For a feasible point \( z \) of SOCMPCC, define the following index sets

\[
I_g(z) := \{ i | g_i(z) = 0 \}, \\
I_G(z) := \{ i | G_i(z) = 0 \}, \quad I_G^+(z) := \{ i | G_i(z) = 0 \}, \quad B_G(z) := \{ i | G_i(z) \in \text{bd} \mathcal{K}_i \setminus \{0\} \}, \\
I_H(z) := \{ i | H_i(z) = 0 \}, \quad I_H^+(z) := \{ i | H_i(z) = 0 \}, \quad B_H(z) := \{ i | H_i(z) \in \text{bd} \mathcal{K}_i \setminus \{0\} \}.
\]

For simplicity of notations, we may omit the the dependence of \( z \) in the above index sets and denote \( G_i(z), H_i(z) \) by \( G_i, H_i \) and \( \hat{G}_i(z), \hat{H}_i(z) \) by \( \hat{G}_i, \hat{H}_i \), respectively if there is no confusion.

Now we introduce a new concept of stationary point for SOCMPCC, called K-stationary point and we show that the K-stationary condition (13) is equivalent to the classical KKT conditions (14).

**Definition 4.1 (K-stationary point)** Let \( z^* \) be a feasible solution of SOCMPCC. We say that \( z^* \) is an K-stationary point of SOCMPCC if there exists a multiplier \((\lambda^g, \lambda^h, \lambda^G, \lambda^H)\) such that the following K-stationary condition holds:

\[
\begin{cases}
\nabla f(z^*) + \nabla g(z^*) \lambda^g + \nabla h(z^*) \lambda^h + \sum_{i=1}^{J} \nabla G_i(z^*) \lambda^G_i + \sum_{i=1}^{J} \nabla H_i(z^*) \lambda^H_i = 0, \\
\lambda^g \geq 0, \quad g(z^*)^T \lambda^g = 0, \\
\lambda^G_i \in \mathbb{R}^{m_i}, \quad \lambda^H_i = 0 & \text{if} \quad i \in I_G \cap I_G^+, \\
\lambda^G_i = 0, \quad \lambda^H_i \in \mathbb{R}^{m_i} & \text{if} \quad i \in I_G^+ \cap I_H, \\
\lambda^G_i \in \mathbb{R}^\Delta G_i \quad \text{and} \quad \lambda^H_i \in \hat{H}_i & \text{if} \quad i \in B_G \cap B_H, \\
\lambda^G_i \in -\mathcal{K}_i + \mathbb{R}_+ H_i & \text{and} \quad \lambda^H_i \in \mathbb{R}_- \hat{H}_i & \text{if} \quad i \in I_G \cap I_H, \\
\lambda^G_i \in \mathbb{R}_- \hat{G}_i & \text{and} \quad \lambda^H_i \in -\mathcal{K}_i + \mathbb{R}_+ G_i & \text{if} \quad i \in B_G \cap I_H, \\
\lambda^G_i \in -\mathcal{K}_i & \text{if} \quad i \in I_G \cap I_H.
\end{cases}
\]

**Definition 4.2** We say that K-SOCMPCC is Clarke calm at a feasible solution \( z^* \) if there exist positive \( \varepsilon \) and \( \mu \) such that, for all \((r, s, t, p)\) in \( \varepsilon B \), for all \( z \in (z^* + \varepsilon B) \cap \mathcal{F}_K(r, t, s, p) \), one has

\[
f(z) - f(z^*) + \mu \|(r, s, t, p)\| \geq 0,
\]
\[ F_K(r, s, t, p) := \{ z \mid h(z) + r = 0, g(z) + s \leq 0, (G(z), H(z)) + t \leq 0, (G(z), H(z)) + p \in \tilde{K} \times \tilde{K} \}. \]

**Theorem 4.1** Let \( z^* \) be a local optimal solution of SOCMPCC. Suppose that the problem K-SOCMPCC is Clarke calm at \( z^* \). Then \( z^* \) is an \( K \)-stationary point.

**Proof.** Since the problem K-SOCMPCC is Clarke calm at \( z^* \), by the classical necessary optimality condition (see e.g. [6, Theorem 2.2]), there exists \( (\lambda^\theta, \lambda^h, a, b, \gamma) \) such that

\[
\begin{align*}
\nabla f(z^*) + \nabla g(z^*) \lambda^\theta + \nabla h(z^*) \lambda^h + \sum_{i=1}^J \nabla G_i(z^*) a_i + \sum_{i=1}^J \nabla H_i(z^*) b_i + \gamma \nabla (GT)(z^*) &= 0, \\
\lambda^\theta \geq 0, & g(z^*)^T \lambda^h = 0, \\
G_i(z^*) \in K_i, & -a_i \in K_i, \quad G_i(z^*)^T a_i = 0, \quad i = 1, \ldots, J, \\
H_i(z^*) \in K_i, & -b_i \in K_i, \quad H_i(z^*)^T b_i = 0, \quad i = 1, \ldots, J, \\
\gamma & \geq 0.
\end{align*}
\]

(14)

Let \( \lambda^G := a + \gamma H(z^*) \) and \( \lambda^H := b + \gamma G(z^*) \). We first show that \( (\lambda^\theta, \lambda^h, \lambda^G, \lambda^H) \) satisfies (13). We consider the following cases.

- \( i \in I_G \cap I^+_H \). Then \( G_i(z^*) = 0, H_i(x^*) \in \text{int}K_i \). By (14), \( b_i = 0 \) and hence \( \lambda^h_i = b_i + \gamma G_i(z^*) = 0 \).

- \( i \in I^+_G \cap I_H \). Similar to Case 1 we can show that \( \lambda^G_i = 0 \).

- \( i \in B_G \cap B_H \). Then \( H_i(z^*) = 0, G_i(z^*) \in \text{bd}K_i \setminus \{0\} \) and \( H_i(z^*) \perp G_i(z^*) \). Since \( -a_i \perp G_i(z^*) \) and \( -a_i \in K_i \) by (14), then \( -a_i \in \mathbb{R}_+ \tilde{G}_i(z^*) \). Similarly, \( -b_i \in \mathbb{R}_- \tilde{H}_i(z^*) \) by \( -b_i \perp \tilde{H}_i(z^*) \) and \( -b_i \in K_i \). It follows from Proposition 2.2 that \( \tilde{H}_i(z^*) \in \mathbb{R}_+ \tilde{G}_i(z^*) \) and \( \tilde{G}_i(z^*) \in \mathbb{R}_+ \tilde{H}_i(z^*) \). So \( \lambda^G_i = a_i + \gamma \tilde{H}_i(z^*) \in \mathbb{R}_+ \tilde{H}_i(z^*) \) and \( \lambda^H_i = b_i + \gamma \tilde{G}_i(z^*) \in \mathbb{R}_+ \tilde{H}_i(z^*) \).

- \( i \in I_G \cap B_H \). Then \( G_i(z^*) = 0, H_i(z^*) \in \text{bd}K_i \setminus \{0\} \). Since \( -a_i, -b_i \in K_i \) and \( -a_i \perp H_i(z^*) \) by (14), then \( -b_i \in \mathbb{R}_- \tilde{H}_i(z^*) \). Hence \( \lambda^G_i = a_i + \gamma H_i(z^*) \in K_i + \mathbb{R}_+ \tilde{H}_i(z^*) \) and \( \lambda^H_i = b_i + \gamma G_i(z^*) \in \mathbb{R}_- \tilde{H}_i(z^*) \).

- \( i \in B_G \cap I_H \). Similarly to the above case, we have \( \lambda^G_i = a_i + \gamma H_i(z^*) = a_i \in \mathbb{R}_- \tilde{G}_i(z^*) \) and \( \lambda^H_i = b_i + \gamma G_i(z^*) \in K_i + \mathbb{R}_+ \tilde{G}_i(z^*) \).

- \( i \in I_G \cap I_H \). Then \( G_i(z^*) = H_i(z^*) = 0 \). By (14), we have \( \lambda^G_i = a_i \in -K_i \) and \( \lambda^H_i = b_i \in -K_i \).

Hence \( (\lambda^\theta, \lambda^h, \lambda^G, \lambda^H) \) satisfies (13). Conversely, let \( (\lambda^\theta, \lambda^h, \lambda^G, \lambda^H) \) satisfying (13). Let \( a := \lambda^G - \gamma H(z^*) \) and \( b := \lambda^H - \gamma G(z^*) \) where \( \gamma > 0 \). We now show that \( (\lambda^\theta, \lambda^h, a, b, \gamma) \) satisfies (14) if \( \gamma \) is sufficiently large. Consider the following cases.

- \( i \in I_G \cap I^+_H \). Then \( a_i = \lambda^G_i - \gamma H_i(z^*) = \gamma \left[ \lambda^G_i - H_i(z^*) / \gamma \right] \) and \( b_i = \lambda^h_i - \gamma G_i(z^*) = \lambda^H_i = 0 \). Since \( H_i(z^*) \in \text{int}K_i \), \( \lambda^G_i / \gamma - H_i(z^*) \in -K_i \) when \( \gamma \) is sufficiently large. Hence \( a_i \in -K_i \) and \( b_i \in 0 \in -K_i \).

- \( i \in I^+_G \cap I_H \). Similar to Case 1, we can show that \( a_i = 0 \in -K_i \) and \( b_i \in -K_i \).
- \( i \in B_G \cap B_H \). Then \( G_i(z^*) \), \( H_i(z^*) \) \( i \in b\mathcal{K}_i \setminus \{0\} \). By Proposition 2.2, \( G_i(z^*) = k\hat{H}_i(z^*) \) for some \( k > 0 \), which in turn implies that \( H_i(z^*) = \hat{G}_i(z^*)/k \). By (13), \( \lambda_i^G = t_1\hat{G}_i(z^*) \) and \( \lambda_i^H = t_2\hat{H}_i(z^*) \) for some \( t_1, t_2 \in \mathbb{R} \). Hence \( -a_i = \gamma H_i(z^*) - \lambda_i^G = (\gamma/k - t_1)\hat{G}_i(z^*) \in \mathcal{K}_i \) and \( -b_i = \gamma G_i(z^*) - \lambda_i^H = (\gamma k - t_2)\hat{H}_i(z^*) \in \mathcal{K}_i \) provided that \( \gamma \) is sufficiently large. In addition, \( \langle a_i, G_i(z^*) \rangle = (t_1 - \gamma/k)\langle \hat{G}_i(z^*), G_i(z^*) \rangle = 0 \) and \( \langle b_i, H_i(z^*) \rangle = (t_2 - \gamma k)\langle \hat{H}_i(z^*), H_i(z^*) \rangle = 0 \).

- \( i \in I_G \cap B_H \). It follows from (13) that \( \lambda_i^G = t_1 H_i(z^*) - \xi_i \) and \( \lambda_i^H = t_2\hat{H}_i(z^*) \) for some \( t_1, t_2 \geq 0 \) and \( \xi_i \in \mathcal{K}_i \). Hence \( -a_i = \gamma H_i(z^*) - \lambda_i^G = (\gamma - t_1)H_i(z^*) + \xi_i \in \mathcal{K}_i \) as \( \gamma \geq t_1 \). Similarly, \( b_i = \lambda_i^H - \gamma G_i(z^*) = \lambda_i^H = -t_2\hat{H}_i(z^*) \in -\mathcal{K}_i \). In addition, \( \langle a_i, G_i(z^*) \rangle = 0 \) since \( G_i(z^*) = 0 \) and \( \langle b_i, H_i(z^*) \rangle = -t_2\hat{H}_i(z^*), H_i(z^*) \rangle = 0 \).

- \( i \in B_G \cap I_H \). The argument is similar to the above case.

- \( i \in I_G \cap I_H \). Then \( G_i(z^*) = H_i(z^*) = 0 \) and \( a_i = \lambda_i^G \in -\mathcal{K}_i \) and \( b_i = \lambda_i^H \in -\mathcal{K}_i \).

Hence \( (\lambda^G, \lambda^H, a, b, \gamma) \) satisfies (14).

## 5 S-stationary conditions

For MPCC, it is known (see Ye [22, Theorem 3.2]) that the S-stationary condition is equivalent to the stationary condition derived by using the proximal normal cone of the complementarity cone. In this vector case, the regular normal cone is the same as the proximal normal cone. For SDCMPCC, it was shown that the regular normal cone is the same as the proximal normal cone and the S-stationary condition is defined by using the proximal normal cone [6]. Similarly, in this paper we have verified that the regular normal cone coincides with the proximal normal cone for the second-order complementarity cone and hence we can define the S-stationary condition as follows. First we introduce the concept of weak (W-) stationary points. Note that when the dimension \( m_i = 2 \), the condition \( (G_i(z^*))_1\hat{\lambda}_G^G + (H_i(z^*))_1\hat{\lambda}_H^H \in \mathbb{R}G(z^*) \) is redundant and can be omitted.

**Definition 5.1 (W-stationary point)** Let \( z^* \) be a feasible solution of SOCMPCC. We say that \( z^* \) is a weak stationary point of SOCMPCC if there exist a multiplier \( (\lambda^G, \lambda^H, \hat{\lambda}_G^G, \hat{\lambda}_H^H) \) such that

\[
\begin{align*}
\nabla f(z^*) + \nabla g(z^*)\lambda^G + \nabla h(z^*)\lambda^H + \sum_{i=1}^{J} \nabla G_i(z^*)\lambda_i^G + \sum_{i=1}^{J} \nabla H_i(z^*)\lambda_i^H &= 0, \\
\lambda^G &\geq 0, \quad g(z^*)^T\lambda^G = 0, \\
\lambda_i^G &\in \mathbb{R}^{m_i}, \quad \lambda_i^H = 0 \quad \text{if} \quad i \in I_G \cap I_H^+, \\
\lambda_i^G &\geq 0, \quad \lambda_i^H \in \mathbb{R}^{m_i} \quad \text{if} \quad i \in I_G^+ \cap I_H, \\
\lambda_G^G \perp G_i(z^*), \quad \lambda_H^H \perp H_i(z^*), \quad (G_i(z^*))_1\hat{\lambda}_G^G + (H_i(z^*))_1\hat{\lambda}_H^H &\in \mathbb{R}G(z^*) \quad \text{if} \quad i \in B_G \cap B_H.
\end{align*}
\]

**Definition 5.2 (S-stationary point)** Let \( z^* \) be a feasible solution of SOCMPCC. We say that \( z^* \) is a strong stationary point of SOCMPCC if there exist a multiplier \( (\lambda^G, \lambda^H, \hat{\lambda}_G^G, \hat{\lambda}_H^H) \) such that

\[
\begin{align*}
0 &= \nabla f(z^*) + \nabla g(z^*)\lambda^G + \nabla h(z^*)\lambda^H + \sum_{i=1}^{J} \nabla G_i(z^*)\lambda_i^G + \sum_{i=1}^{J} \nabla H_i(z^*)\lambda_i^H, \\
\lambda^G &\geq 0, \quad g(z^*)^T\lambda^G = 0, \\
(\lambda_G^G, \lambda_H^H) &\in \hat{N}_{\Omega_i}(G_i(z^*), H_i(z^*)), \quad i = 1, \ldots, J.
\end{align*}
\]

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or equivalently such that (15) and the following condition hold:

\[
\begin{align*}
\lambda_i^H & \in \mathbb{R}_-\bar{H}_i(z^*), \quad \langle \lambda_i^G, \bar{G}_i(z^*) \rangle \leq 0 \quad \text{if } i \in I_G(z^*) \cap B_H(z^*), \\
\lambda_i^G & \in \mathbb{R}_-\bar{G}_i(z^*), \quad \langle \lambda_i^H, \bar{H}_i(z^*) \rangle \leq 0 \quad \text{if } i \in B_G(z^*) \cap I_H(z^*), \\
\lambda_i^G & \in -\mathcal{K}_i, \quad \lambda_i^H \in -\mathcal{K}_i \quad \text{if } i \in I_G(z^*) \cap I_H(z^*).
\end{align*}
\]

**Definition 5.3** Let \( z^* \) be a a feasible solution of SOCP. We say that SOCP-LICQ holds at \( z^* \) provided that the gradient vectors

\[
\nabla g_i(z^*), \quad i \in I_g(z^*); \quad \nabla h_i(z^*), \quad i = 1, \ldots, q; \quad \nabla G_i(z^*), \quad i \in I_2^c; \quad \nabla H_i(z^*), \quad i \in I_1^c
\]

with \( I_1 := I_G(z^*) \cap I_H^+(z^*), I_2 := I_G^+(z^*) \cap I_H(z^*) \) are linearly independent.

In the following theorem we show that under SOCP-LICQ, a local optimal solution of SOCMPC must be an S-stationary point.

**Theorem 5.1** Let \( z^* \) be a local optimal solution of SOCP. If SOCP-LICQ holds at \( z^* \), then \( z^* \) is an S-stationary point.

**Proof.** Since \( z^* \) is a local optimal solution, it is also a local optimal solution of the problem with the same objective function and with the inactive constraints \( g_i(z) < 0 \ i \notin I_g(z^*), H_i(z) \in \text{int} \mathcal{K} \ i \in I_1, G_i(z) \in \text{int} \mathcal{K} \ i \in I_2 \) deleted from the feasible region, i.e., \( z^* \) is a local optimal solution to the problem:

\[
\min \quad f(z) \\
\text{s.t.} \quad h(z) = 0, \ g_i(z) \leq 0, \ i \in I_g(z^*), \\
\quad \quad G_i(z) = 0, \ i \in I_1, \ H_i(z) = 0, \ i \in I_2 \\
\quad \quad \mathcal{K}_i \ni G_i(z) \perp H_i(z) \in \mathcal{K}_i, \ i \in (I_1 \cup I_2)^c.
\]

Then

\[
0 \in \nabla f(z^*) + \tilde{N}_F(z^*),
\]

where \( F := \{ z | F(z) \in D \} \) is the feasible region of the above problem with

\[
F(z) := (h(z), g_{I_1}(z), G_{I_1}(z), H_{I_2}(z), G_{(I_1 \cup I_2)^c}(z), H_{(I_1 \cup I_2)^c}(z)) \\
D := \{ 0 \} \times \mathbb{R}^{I_g} \times \{ 0 \}^{I_1} \times \{ 0 \}^{I_2} \times \Omega_{(I_1 \cup I_2)^c}.
\]

and

\[
\Omega_{(I_1 \cup I_2)^c} := \{ (u_i, v_i) | \mathcal{K}_i \ni u_i \perp v_i \in \mathcal{K}_i, \ i \in (I_1 \cup I_2)^c \}.
\]

By the SOCP-LICQ, \( \nabla F(z^*) \) has a full column rank. The desired result follows from Propositions 2.1 and 3.1 by setting \( \lambda_i^g = 0 \) for \( i \notin I_g(z^*) \), \( \lambda_i^H = 0 \) for \( i \in I_1 \) and \( \lambda_i^G = 0 \) for \( i \in I_2 \), i.e., letting the multiplies corresponding to the deleted constraints be zero. \( \blacksquare \)

### 6 M-stationary conditions

In this section we study the M-stationary conditon for SOCMPC. For this purpose we rewrite the SOCP as an optimization problem with the nonconvex complementarity
cone constraint:
(M-SOCMPCC) \[
\begin{align*}
\min & \quad f(z) \\
\mathrm{s.t.} & \quad h(z) = 0, \\
& \quad g(z) \leq 0, \\
& \quad (G_i(z), H_i(z)) \in \Omega_i, \ i = 1, \ldots, J.
\end{align*}
\]

where \( \Omega_i := \{(x, y) | x \in \mathcal{K}_i, \ y \in \mathcal{K}_i, \ x \perp y \} \).

As in the MPCC case, we will show that the M-stationary condition introduced below is the KKT condition of M-SOCMPCC by using the limiting normal cone. Note that when the dimension \( m_i = 2 \), the condition \( \alpha_i \lambda_i^G + (1 - \alpha_i) \hat{\lambda}_i^H \in \mathbb{R} \xi_i \) for some \( \alpha_i \in [0, 1] \) is redundant and can be omitted.

**Definition 6.1 (M-stationary point)** Let \( z^* \) be a feasible solution of SOCMPCC. We say that \( z^* \) is a M-stationary point of SOCMPCC if there exist multipliers \( (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \) such that

\[
\begin{align*}
0 &= \nabla f(z^*) + \nabla g(z^*) \lambda^g + \nabla h(z^*) \lambda^h + \sum_{i=1}^{J} \nabla G_i(z^*) \lambda_i^G + \sum_{i=1}^{J} \nabla H_i(z^*) \lambda_i^H, \\
\lambda^g &\geq 0, \ g(z^*)^T \lambda^g = 0, \\
(\lambda_i^G, \lambda_i^H) &\in \mathcal{N}_0(G_i(z^*), H_i(z^*)), \ i = 1, \ldots, J,
\end{align*}
\]

or equivalently such that (15) and the following condition hold:

\[
\begin{align*}
\lambda_i^G &\in \mathbb{R}^{m_i}, \ \lambda_i^H = 0 \text{ or } \lambda_i^G \perp \hat{H}_i(z^*), \ \lambda_i^H \in \mathbb{R} \hat{H}_i(z^*) \text{ or } \\
\lambda_i^H &\in \mathbb{R} \hat{H}_i(z^*), \ \lambda_i^G \hat{H}(z^*) \leq 0 \quad \text{if } i \in I_G \cap B_H, \\
\lambda_i^G &\in \mathbb{R} \hat{H}_i(z^*), \ (\lambda_i^G, \hat{H}_i(z^*)) \leq 0 \quad \text{if } i \in B_G \cap I_H, \\
\lambda_i^G &\in \mathbb{R} \xi_i, \ \lambda_i^H \in \mathbb{R} \xi_i \text{ or } \lambda_i^H = 0, \ \lambda_i^G \in \mathbb{R}^{m_i} \text{ or } \lambda_i^G = 0, \ \lambda_i^H \in \mathbb{R}^{m_i}, \text{ or } \\
\lambda_i^G &\in \mathbb{R} \xi_i, \ \lambda_i^H \in \mathbb{R} \xi_i \text{ or } \lambda_i^H = 0, \ \lambda_i^G \in \mathbb{R}^{m_i} \text{ or } \lambda_i^G = 0, \ \lambda_i^H \in \mathbb{R}^{m_i}, \text{ or } \\
\lambda_i^G &\perp \xi_i, \ \lambda_i^H \perp \xi_i, \ \alpha_i \lambda_i^G + (1 - \alpha_i) \hat{\lambda}_i^H \in \mathbb{R} \xi_i \quad \text{for some } \alpha_i \in [0, 1] \text{ and some } \xi_i \in \{(1, w) \in \mathbb{R} \times \mathbb{R}^{m_i-1} | \|w\| = 1\} \\
&\quad \text{if } i \in I_G \cap I_H.
\end{align*}
\]

**Definition 6.2** We say that M-SOCMPCC is Clarke calm at a feasible solution \( z^* \) if there exist positive \( \varepsilon \) and \( \mu \) such that, for all \((r, s, t) \) in \( \varepsilon B \), for all \( z \in (z^* + \varepsilon B) \cap F_M(r, s, t) \), one has

\[
f(z) - f(z^*) + \mu \|(r, s, t)\| \geq 0,
\]

where

\[
F_M(r, s, t) := \{z | h(z) + r = 0, g(z) + s \leq 0, (G_i(z), H_i(z)) + t_i \in \Omega_i, i = 1, \ldots, J\}.
\]

**Theorem 6.1** Let \( z^* \) be a local optimal solution of SOCMPCC. Suppose that the problem M-SOCMPCC is Clarke calm at \( z^* \). Then \( z^* \) is an M-stationary point of SOCMPCC.

**Proof.** By Theorem [6, Theorem 2.2], there exists a multiplier \( (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \) such that

\[
0 = \nabla f(z^*) + \nabla g(z^*) \lambda^g + \nabla h(z^*) \lambda^h + \sum_{i=1}^{J} \nabla G_i(z^*) \lambda_i^G + \sum_{i=1}^{J} \nabla H_i(z^*) \lambda_i^H,
\]

\[
\lambda^g \geq 0, \ g(z^*)^T \lambda^g = 0, \ (\lambda_i^G, \lambda_i^H) \in \mathcal{N}_0(G_i(z^*), H_i(z^*)), \ i = 1, \ldots, J,
\]

and so the desired result follows from using the expression of the limiting normal cone in Proposition 3.5.
**Definition 6.3** Let $z^*$ be a feasible solution of SOCMPCC. We say that the constraint system of M-SOCMPCC has a local error bound at $z^*$ if there exist $\mu, \varepsilon > 0$ such that
\[
\text{dist}(z, F_M(0, 0, 0)) \leq \mu \| (r, s, t) \|, \quad (r, s, t) \in \varepsilon B \text{ and } z \in F(r, s, t) \cap B_{\varepsilon}(z^*).
\]

Note that the constraint system of M-SOCMPCC has a local error bound at $z^*$ if and only if the set-valued mapping $F_M(r, s, t)$ is calm (or pseudoupper-Lipschitz continuous using the terminology of [25]) around $(0, 0, 0, z^*)$. $F_M(r, s, t)$ being either pseudo-Lipschitz continuous ([1]) or upper-Lipschitz continuous ([16]) at $z^*$ implies that the constraint system of M-SOCMPCC has a local error bound at $z^*$.

The proposition below is an easy consequence of Clarke’s exact penalty principle [4, Proposition 2.4.3] and the calmness of the constraint system. See [23, Proposition 4.2] for a proof.

**Proposition 6.1** If the objective function is Lipschitz near $z^*$ and $F_M(r, s, t)$ is calm at $(0, 0, 0, z^*)$, then the problem M-SOCMPCC is calm at $z^*$.

**Definition 6.4 (SOCMPCC-NNAMCQ)** Let $z^*$ be a local optimal solution of SOCMPCC. We say that SOCMPCC-No Nonezero Abnormal Multiplier Constraint Qualification (SOCMPCC NNAMCQ) holds at $z^*$ if there is no nonzero vector $(\lambda^g, \lambda^h, \lambda^G, \lambda^H)$ such that the following conditions hold:
\[
\begin{cases}
0 = \nabla g(z^*)\lambda^g + \nabla h(z^*)\lambda^h + \sum_{i=1}^{J} \nabla G_i(z^*)\lambda^G_i + \sum_{i=1}^{J} \nabla H_i(z^*)\lambda^H_i, \\
\lambda^g \geq 0, \quad (\lambda^G_i, \lambda^H_i) \in N_{\Omega_i}(G_i(z^*), H_i(z^*)), \quad i = 1, \ldots, J.
\end{cases}
\]

**Theorem 6.2** Let $z^*$ be a local optimal solution of SOCMPCC. Then $z^*$ is an M-stationary point under one of the following constraint qualifications:

(i) The SOCMPCC-NNAMCQ holds at $z^*$.

(ii) All mappings $h, g, G, H$ are affine and $m_i \leq 2$ for $i = 1, \ldots, J$.

**Proof.** By Theorem 6.1 and Proposition 6.1, it suffices to show the calmness of $F_M$.

(i) Similarly as in [23, Theorem 4.4], we can show that under SOCMPCC-NNAMCQ, the constraint system of M-SOCMPCC is pseudo-Lipschitz continuous around $(0, 0, 0, z^*)$ and hence has a local error bound at $z^*$.

(ii) Since when $m_i \leq 2$, the second order cone $K_i$ is polyhedral and hence the second order complementarity cone $\Omega_i$ is a union of finitely many polyhedral convex sets. Since all mappings $h, g, G, H$ are affine, the graph of the set-valued mapping $F_M$ is a union of polyhedral convex sets and hence $F_M$ is a polyhedral set-valued mapping. By [16, Proposition 1], $F_M$ is upper-Lipschitz and hence the local error bound condition holds at $z^*$.

\[\blacksquare\]
7 C-stationary conditions

In this section, we consider the C-stationary condition by reformulating SDCMPCC as a nonsmooth problem:

(C-SOCMPCC) \[
\min \ f(z) \\
\text{s.t. } h(z) = 0,
\]
\[
g(z) \leq 0, \\
G_i(z) - \Pi_{K_i}(G_i(z) - H_i(z)) = 0, \ i = 1, \ldots, J.
\]

As in the MPCC case, the C-stationary condition introduced below is the nonsmooth KKT condition of C-SOCMPCC by using the Clarke subdifferential.

Definition 7.1 (C-stationary point) Let \(z^*\) be a feasible solution of SOCMPCC. We say that \(z^*\) is a C-stationary point of SOCMPCC if there exists a multiplier \((\lambda^h, \lambda^g, \lambda^H)\) such that (15) and the following conditions hold:

\[
\begin{align*}
\lambda^H_i &\in R^{\hat{H}_i(z^*)} & \text{if } i \in I_G \cap B_H, \\
\lambda^G_i &\in R^{\hat{G}_i(z^*)} & \text{if } i \in B_G \cap I_H, \\
\langle \lambda^G_i, \lambda^H_i \rangle &\geq 0 & \text{for all } i = 1, \ldots, J.
\end{align*}
\]

We present the first order optimality condition of SOCMPCC in terms of C-stationary conditions in the following result.

Definition 7.2 We say that C-SOCMPCC is Clarke calm at a feasible solution \(z^*\) if there exist positive \(\epsilon\) and \(\mu\) such that, for all \((r, s, t)\) in \(\epsilon B\), for all \(z \in (z^* + \epsilon B) \cap F_C(r, t, s)\), one has

\[
f(z) - f(z^*) + \mu \| (r, s, t) \| \geq 0,
\]

where

\[
F_C(r, t, s) := \{z | h(z) + r = 0, g(z) + s \leq 0, G_i(z) - \Pi_{K_i}(G_i(z) - H_i(z)) + t_i = 0, i = 1, \ldots, J\}.
\]

Theorem 7.1 Let \(z^*\) be a local optimal solution of SOCMPCC. Suppose that the problem C-SOCMPCC is Clarke calm at \(z^*\). Then \(z^*\) is a C-stationary point of SOCMPCC.

Proof. Since the problem is calm, by the Clarke nonsmooth KKT condition ([4, Proposition 6.4.4]), there exist \(\lambda^h \in R^q, \lambda^g \in R^p\) and \(\beta_i \in R^{m_i}(i = 1, \ldots, J)\) such that

\[
0 \in \partial^c_L(z^*, \lambda^h, \lambda^g, \beta), \quad \lambda^g \geq 0 \quad \text{and} \quad \langle \lambda^g, g(z^*) \rangle = 0,
\]

where \(\partial^c_L\) denotes the Clarke generalized gradient with respect to \(z\) and

\[
L(z, \lambda^h, \lambda^g, \beta) := f(z) + \langle \lambda^h, h(z) \rangle + \langle \lambda^g, g(z) \rangle + \sum_{i=1}^{J} \langle \beta_i, G_i(z) - \Pi_{K_i}(G_i(z) - H_i(z)) \rangle.
\]

Consider the Clarke generalized gradient of the nonsmooth function

\[
S_i(z) := \langle \beta_i, \Pi_{K_i}(G_i(z) - H_i(z)) \rangle.
\]
Applying the Jacobian chain rule [4, Theorem 2.6.6] twice yields
\[ \partial^c S_i(z^*) \subseteq \beta^T \partial^c \Pi_{K_i}(G_i(z^*) - H_i(z^*)) (JG_i(z^*) - JH_i(z^*)). \]

Therefore, since any element of the Clarke subdifferential of the metric projection operator to a close convex set is self-adjoint (see e.g., [11, Proposition 1(a)]), we know from (16) that there exists \( A_i \in \partial^c \Pi_{K_i}(G_i(z^*) - H_i(z^*)) \) such that
\[
\nabla f(z^*) + \nabla h(z^*) \lambda^h + \nabla g(\bar{z}) \lambda^g + \sum_{i=1}^J \nabla G_i(z^*) \beta_i - \sum_{i=1}^J \left( \nabla G_i(z^*) - \nabla H_i(z^*) \right) A_i \beta_i = 0. \tag{17}
\]

Define \( \lambda^i_G := \beta_i - A_i \beta_i \) and \( \lambda^i_H := A_i \beta_i \). Then it follows from (16) and (17) that
\[
0 = \nabla f(\hat{z}) + \nabla h(\hat{z}) \lambda^h + \nabla g(\hat{z}) \lambda^g + \sum_{i=1}^J \nabla G_i(\hat{z}) \lambda^i_G + \sum_{i=1}^J \nabla H_i(\hat{z}) \lambda^i_H,
\]
\[
\lambda^g \geq 0, \quad \langle \lambda^g, g(\bar{z}) \rangle = 0.
\]

We now continue to show that (15) holds. Notice that for \( i \in (I_G \cap I^-_H) \cup (I^+_G \cap I_H) \cup (B_G \cap B_H) \), \( \Pi_{K_i}(z) \) is continuously differentiable at \( z^* \). Hence \( A_i = J \Pi_{K_i}(G_i(z^*) - H_i(z^*)) \). Since
\[
\lambda^i_H = A_i \beta_i = A_i \left[ (I_i - A_i) \beta_i + A_i \beta_i \right] = A_i (\lambda^i_G + \lambda^i_H),
\]

where \( I_i \) denotes the \( m_i \)-dimensional identity matrix, it follows that
\[
-\lambda^i_H = \hat{D}^* \Pi_{K_i}(G_i(z^*) - H_i(z^*)) (-\lambda^i_G - \lambda^i_H).
\]

Hence \( (\lambda^i_G, \lambda^i_H) \in \hat{N}_{K_i}(G_i(z^*), H_i(z^*)) \) by (5) for \( i \in (I_G \cap I^-_H) \cup (I^+_G \cap I_H) \cup (B_G \cap B_H) \).

Using the formula of the regular normal cone given in Proposition 3.1 yields
\[
\begin{cases}
\lambda^i_G \in \mathbb{R}^{m_i}, \lambda^i_H = 0 & \text{if } i \in I_G \cap I^-_H, \\
\lambda^i_G = 0, \lambda^i_H \in \mathbb{R}^{m_i} & \text{if } i \in I^+_G \cap I_H, \\
\lambda^i_G \perp G_i(z^*), \lambda^i_H \perp H_i(z^*), G_1(z^*) \lambda^i_G + H_1(z^*) \lambda^i_H \in \mathbb{R} G(z^*) & \text{if } i \in B_G \cap B_H,
\end{cases}
\]

which implies that (15) holds. Now consider the case where \( i \in I_G \cap B_H \). In this case we have \( \lambda^i_H = A_i \beta_i \) and
\[
A_i \in \co \partial_B \Pi_{K_i}(G_i(z^*) - H_i(z^*)) = \co \left\{ O, \frac{1}{2(H_i)^T(z^*)} \tilde{H}_i(z^*) \tilde{H}_i(z^*)^T \right\},
\]

which implies \( \lambda^i_H \in \hat{H}_i(z^*) \). In the case where \( i \in B_G \cap I_H \) we have \( \lambda^i_G = (I_i - A_i) \beta_i \) and
\[
A_i \in \co \partial_B \Pi_{K_i}(G_i(z^*) - H_i(z^*)) = \co \left\{ I, I - \frac{1}{2(G_i)^T(z^*)} \tilde{G}_i(z^*) \tilde{G}_i(z^*)^T \right\}.
\]

It follows that \( \lambda^i_G \in \mathbb{R}\tilde{G}_i(z^*) \).

Moreover, from [11, Proposition 1(c)], we know that
\[
\langle A_i \beta_i, \beta_i - A_i \beta_i \rangle \geq 0,
\]

which implies \( \langle \lambda^i_G, \lambda^i_H \rangle \geq 0 \) for all \( i = 1, \ldots, J \). The proof of the theorem is complete. ■
8 Connections between various stationary points

In this section, we discuss the relationships among various stationary points given in the previous sections. First, we give the following result.

**Proposition 8.1** Let \( (x, y) \in \Omega \) with \( \Omega \) being the m-dimensional second-order complementarity cone. Then

\[
(N_K(x) + \mathbb{R}_+ y, N_K(y) + \mathbb{R}_+ x) = \begin{cases}
(\mathbb{R}^m, 0) & \text{if } x = 0, y \in \text{int} K; \\
(0, \mathbb{R}^m) & \text{if } x \in \text{int} K, y = 0; \\
(\mathbb{R}^\hat{x}, \mathbb{R}^\hat{y}) & \text{if } x, y \in \text{bd} K \setminus \{0\}, x^T y = 0; \\
(-\mathbb{K} + \mathbb{R}_+ y, \mathbb{R}_- \hat{y}) & \text{if } x = 0, y \in \text{bd} K \setminus \{0\}; \\
(\mathbb{R}_- \hat{x}, -\mathbb{K} + \mathbb{R}_+ x) & \text{if } x \in \text{bd} K \setminus \{0\}, y = 0; \\
(-\mathbb{K}, -\mathbb{K}) & \text{if } x = 0, y = 0,
\end{cases}
\]

and

\[
(N_K(x) + \mathbb{R}_+ y, N_K(y) + \mathbb{R}_+ x) \subset \widehat{N}_\Omega(x, y).
\]

**Proof.** Consider the following cases.

- Let \( x = 0 \) and \( y \in \text{int} K \). For any \( z \in \mathbb{R}^m \), since \( y \in \text{int} K \), there exists \( t > 0 \) such that \( y - tz \in K \). Hence \( z \in \frac{-K + y}{t} \subset -K + \mathbb{R}_+ y \). Because \( z \in \mathbb{R}^m \) is arbitrary, we have \(-K + \mathbb{R}_+ y = \mathbb{R}^m \). Thus

\[
(N_K(x) + \mathbb{R}_+ y, N_K(y) + \mathbb{R}_+ x) = (-\mathbb{K} + \mathbb{R}_+ y, 0) = (\mathbb{R}^m, 0) = \widehat{N}_\Omega(x, y).
\]

- Let \( x \in \text{int} K \) and \( y = 0 \). Then similar to the above case we can show that

\[
(N_K(x) + \mathbb{R}_+ y, N_K(y) + \mathbb{R}_+ x) = (0, -\mathbb{K} + \mathbb{R}_+ x) = (0, \mathbb{R}^m) = \widehat{N}_\Omega(x, y).
\]

- Let \( x, y \in \text{bd} K \setminus \{0\} \) and \( x^T y = 0 \). It follows from Proposition 2.2 that \( y \in \mathbb{R}_+ \hat{x} \) and \( x \in \mathbb{R}_- \hat{y} \). Note that \( N_K(x) = \mathbb{R}_- \hat{x} \) and \( N_K(y) = \mathbb{R}_- \hat{y} \). This implies

\[
(N_K(x) + \mathbb{R}_+ y, N_K(y) + \mathbb{R}_+ x) = (\mathbb{R}_- \hat{x} + \mathbb{R}_+ \hat{x}, \mathbb{R}_- \hat{y} + \mathbb{R}_+ \hat{y}) = (\mathbb{R}^\hat{x}, \mathbb{R}^\hat{y}),
\]

since \( \mathbb{R} = \mathbb{R}_- + \mathbb{R}_+ \). For \( (u, v) \in (\mathbb{R}^\hat{x}, \mathbb{R}^\hat{y}) \), we have \( u \perp x, v \perp y \), and \( x_1 \hat{u} + y_1 v \in \mathbb{R}x \) due to \( \hat{y} \in \mathbb{R}_+ x \). Comparing the formula given in (20) and Proposition 3.1 yields

\[
(N_K(x) + \mathbb{R}_+ y, N_K(y) + \mathbb{R}_+ x) \subset \widehat{N}_\Omega(x, y).
\]

- Let \( x = 0 \) and \( y \in \text{bd} K \setminus \{0\} \), then by Proposition 3.1 we have

\[
(N_K(x) + \mathbb{R}_+ y, N_K(y) + \mathbb{R}_+ x) = (-\mathbb{K} + \mathbb{R}_+ y, \mathbb{R}_- \hat{y}) \subset \widehat{N}_\Omega(x, y),
\]

since \( \langle w + \beta y, \hat{y} \rangle = \langle w, \hat{y} \rangle \leq 0 \) for all \( w \in K \) and \( \beta \in \mathbb{R}_+ \).

- Let \( x \in \text{bd} K \setminus \{0\} \) and \( y = 0 \). Similarly as the above case we can show that

\[
(N_K(x) + \mathbb{R}_+ y, N_K(y) + \mathbb{R}_+ x) = (\mathbb{R}_- \hat{x}, -\mathbb{K} + \mathbb{R}_+ x) \subset \widehat{N}_\Omega(x, y).
\]
Let \((x, y) = (0, 0)\). Then by Proposition 3.1 we have
\[
(N_K(x) + \mathbb{R}_+ y, N_K(y) + \mathbb{R}_+ x) = (-K, -K) = \hat{N}_\Omega(x, y).
\]

Comparing Proposition 8.1 and (13) leads to the following expression of the K-stationary condition immediately.

**Corollary 8.1** A feasible solution \(z^*\) is an K-stationary point of SOCMPCC if and only if there exist a multiplier \((\lambda^0, \lambda^h, \lambda^G, \lambda^H)\) such that
\[
\left\{
\begin{array}{l}
\nabla f(z^*) + \nabla g(z^*)\lambda^0 + \nabla h(z^*)\lambda^h + \sum_{i=1}^J \nabla G_i(z^*)\lambda^G_i + \sum_{i=1}^J \nabla H_i(z^*)\lambda^H_i = 0, \\
\lambda^0 \geq 0, \ g(z^*)^T \lambda^0 = 0, \\
(\lambda^G_i, \lambda^H_i) \in (N_{K_i}(G_i(z^*)) + \mathbb{R}_+ H_i(z^*), N_{K_i}(H_i(z^*)) + \mathbb{R}_+ G_i(z^*)), \ i = 1, \ldots, J.
\end{array}
\right.
\]

In the following proposition we show that (19) becomes an equality when the dimension of \(K\) is less or equal to 2.

**Proposition 8.2** If \(K\) is the \(m\)-dimensional second-order cone with \(m \leq 2\), then for \((x, y) \in \Omega\),
\[
(N_K(x) + \mathbb{R}_+ y, N_K(y) + \mathbb{R}_+ x) = \hat{N}_\Omega(x, y).
\]

**Proof.** If \(m = 1\), then the possible cases are \(x = 0, y \in \text{int} K\) or \(x \in \text{int} K, y = 0\) or \(x = y = 0\). In these three cases, according to (18) and the formula of the regular normal cone given in Proposition 3.1 we have
\[
(N_K(x) + \mathbb{R}_+ y, N_K(y) + \mathbb{R}_+ x) = \hat{N}_\Omega(x, y).
\]

If \(m = 2\), according to the proof of Proposition 8.1 it only needs to show
\[
(N_K(x) + \mathbb{R}_+ y, N_K(y) + \mathbb{R}_+ x) \supset \hat{N}_\Omega(x, y)
\]
for \(x, y \in \text{bd} K \setminus \{0\}\) or \(x = 0, y \in \text{bd} K \setminus \{0\}\) or \(x \in \text{bd} K \setminus \{0\}, y = 0\).

- Let \((u, v) \in \text{bd} K \setminus \{0\}\). Take \((u, v) \in \hat{N}_\Omega(x, y)\). Then it is easy to see that \(u \perp x, v \perp y\) and hence \(u \in \mathbb{R}\hat{x}, v \in \mathbb{R}\hat{y}\). Since from (20) we have \((N_K(x) + \mathbb{R}_+ y, N_K(y) + \mathbb{R}_+ x) = (\mathbb{R}\hat{x}, \mathbb{R}\hat{y})\). It follows that
\[
(N_K(x) + \mathbb{R}_+ y, N_K(y) + \mathbb{R}_+ x) \supset \hat{N}_\Omega(x, y).
\]

- Let \(x = 0\) and \(y \in \text{bd} K \setminus \{0\}\). According to Proposition 3.1 and (21), \(\hat{N}_\Omega(x, y) = \{(u, v) | u \in \hat{y}^\circ, v \in \mathbb{R}_- \hat{y}\}\) and \((N_K(x) + \mathbb{R}_+ y, N_K(y) + \mathbb{R}_+ x) = (-K + \mathbb{R}_+ y, \mathbb{R}_- \hat{y})\). Hence it suffices to show that \(\hat{y}^\circ \subset -K + \mathbb{R}_+ y\). Let \(u \in \hat{y}^\circ\), i.e., \(u_1 y_1 - u_2 y_2 \leq 0\). Since \(y_1 = |y_2|\) due to the assumption that \(y \in \text{bd} K \setminus \{0\}\), consider the following two cases. If \(y_1 = y_2\), then \(u_1 \leq u_2\). Let \(t > 0\) be sufficiently large so that \(ty_1 - u_2 \geq 0\). Then \(ty_1 - u_1 \geq ty_1 - u_2 = |ty_1 - u_2| = |ty_2 - u_2|\). It means that \(ty - u \in K\), i.e., \(u \in -K + \mathbb{R}_+ y\). If \(y_1 = -y_2\), then \(u_1 + u_2 \leq 0\). Let \(t > 0\) be sufficiently large so that \(ty_1 + u_2 \geq 0\). Then \(ty_1 - u_1 \geq ty_1 + u_2 = |ty_1 + u_2| = |ty_2 - u_2|\), where the second equation holds due to the fact that \(y_1 = -y_2\). Hence \(ty - u \in K\), i.e., \(u \in -K + \mathbb{R}_+ y\).

In both cases, we have shown that \(\hat{y}^\circ \subset -K + \mathbb{R}_+ y\).
where the last inequality follows from the fact that $0 = \langle x, u \rangle$.

Proof. An M-stationary point must be an C-stationary point. We now verify this.

It is well known that the KKT conditions and the S-stationary conditions are equivalent for MPCC. However, according to Proposition 8.2 and Example 8.1, this equivalence holds only for the case where all $m_i \leq 2$ but may fail to hold for SOCMPCC as $m_i \geq 3$ for some $i \in \{1, \ldots, J\}$.

Since the S-stationary point is defined in terms of the regular normal cone and the M-stationary point is defined in terms of the limiting normal cone, it is obvious that an M-stationary point is an S-stationary point. However, unlike MPCC, it is not so easy to see that an M-stationary point must be an C-stationary point. We now verify this implication.

Theorem 8.1 An M-stationary point must be an C-stationary point.

Proof. It suffices to show that for every $(u, v) \in N_{\Omega}(x, y)$, one has $\langle u, v \rangle \geq 0$. The cases where $x = 0, y \in \text{int} K$ or $x \in \text{int} K, y = 0$ or $x = 0, y \in \text{bd} K \{ 0 \}$ or $x \in \text{bd} K \{ 0 \}, y = 0$ are clear. It suffices to prove for the cases where $x, y \in \text{bd} K \{ 0 \}$ and where $x = y = 0$. Let $x, y \in \text{bd} K \{ 0 \}$. Then by Proposition 3.5, $x_1 \hat{u} + y_1 v = \beta x$ for some $\beta \in \mathbb{R}$ and $u \perp x$. Since $y_1 = \|y_2\| \neq 0$, it follows that $v = \frac{\beta x - x_1 \hat{u}}{y_1}$. Hence

$$\langle u, v \rangle = \frac{1}{y_1} \langle u, \beta x - x_1 \hat{u} \rangle = -\frac{x_1}{y_1} \langle u, \hat{u} \rangle = -\frac{1}{y_1} (u_1^2 - \|u_2\|^2) \geq 0,$$

where the last inequality follows from the fact that $0 = \langle x, u \rangle = x_1 u_1 + x_2^Tu_2$ and $x_1 = \|x_2\| > 0$. Now consider the case where $x = y = 0$. In this case, it only needs to consider the case where there exists $\alpha \in [0, 1], \beta \in \mathbb{R}$ and $\xi \in \mathbb{R}^m$ with $\xi = (1, w), w \in \mathbb{R}^{m-1}, \|w\| = 1$ such that $\alpha u + (1 - \alpha) \hat{v} = \beta \xi, u \perp \xi, v \perp \xi$. If $\alpha = 0$, then $\hat{v} = \beta \xi$ which implies that $v = \beta \xi$ and hence $u \perp v = 0$. If $\alpha = 1$, then $u = \beta \xi$ and hence $u \perp v$. If $\alpha \in (0, 1)$, then $\alpha u + (1 - \alpha) \hat{v} = \beta \xi$ implies that $u = \frac{\beta \xi - (1 - \alpha) \hat{v}}{\alpha}$. Hence

$$\langle v, u \rangle = \frac{1}{\alpha} \langle v, \beta \xi - (1 - \alpha) \hat{v} \rangle = \frac{\alpha - 1}{\alpha} \langle v, \hat{v} \rangle = \frac{\alpha - 1}{\alpha} (v_1^2 - \|v_2\|^2) \geq 0,$$

where the last inequality follows from the fact that $0 = \langle v, \xi \rangle = v_1 - v_2^w$ and $\|w\| = 1$. ■

We can now summarize the relation between various stationary points as follows.

\[
K - \text{stationary point} \implies S - \text{stationary point} \implies M - \text{stationary point} \\
\implies C - \text{stationary point} \implies W - \text{stationary point}.
\]
9 New optimality conditions for MPCC via SOCMPCC

Consider the vector MPCC:

\[
\text{(MPCC)} \quad \min f(z) \\
\text{s.t. } h(z) = 0, \ g(z) \leq 0, \\
0 \leq G_i(z) \perp H_i(z) \geq 0, \ i = 1, \ldots, J,
\]

where \( G_i(z), H_i(z) : \mathbb{R}^n \to \mathbb{R}. \) We reformulate MPCC as the following SOCMPCC:

\[
\min f(z) \\
\text{s.t. } h(z) = 0, g(z) \leq 0 \\
K_i \ni \tilde{G}_i(z) \perp \tilde{H}_i(z) \in K_j, \ i = 1, \ldots, J.
\]

where \( \tilde{G}_i(x) := (G_i(x), 0, \ldots, 0) \in \mathbb{R}^{m_i} \) and \( \tilde{H}_i(x) := (G_i(x), 0, \ldots, 0) \in \mathbb{R}^{m_i} \) for \( i = 1, \ldots, J. \)

Let us discuss the relationship of the various stationary points between MPCC and its SOCMPCC reformulations.

**Theorem 9.1** The following statements hold:

(a) If \( z^* \) is an S-stationary point of vector MPCC with \((\lambda^0, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^{J} \times \mathbb{R}^{\tau}\) then \( z^* \) is a S-stationary point of SOCMPCC with \((\lambda^0, \lambda^h, \tilde{\lambda}^G, \tilde{\lambda}^H) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^{\tau} \times \mathbb{R}^{\tau}\)

where \( \tilde{\lambda}^G_i = (\lambda^G_i, 0, \ldots, 0) \in \mathbb{R}^{m_i} \) and \( \tilde{\lambda}^H_i = (\lambda^H_i, 0, \ldots, 0) \in \mathbb{R}^{m_i} \) for \( j = 1, \ldots, J. \)

Conversely, if \( z^* \) is a S-stationary point of SOCMPCC with \((\lambda^0, \lambda^h, \tilde{\lambda}^G, \tilde{\lambda}^H) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^{\tau} \times \mathbb{R}^{\tau}\), then \( z^* \) is a S-stationary point of vector MPCC with \((\lambda^0, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^{J} \times \mathbb{R}^{J}\) where \( \lambda_i^G = (\tilde{\lambda}_i^G)_1 \) and \( \lambda_i^H = (\tilde{\lambda}_i^H)_1 \) for \( i = 1, \ldots, J. \)

(b) If \( z^* \) is an M-,C-stationary point of vector MPCC with \((\lambda^0, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^{J} \times \mathbb{R}^{J}\), then \( z^* \) is an M-,C-stationary point of SOCMPCC with \((\lambda^0, \lambda^h, \tilde{\lambda}^G, \tilde{\lambda}^H) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^{\tau} \times \mathbb{R}^{\tau}\)

where \( \tilde{\lambda}^G_i = (\lambda^G_i, 0, \ldots, 0) \in \mathbb{R}^{m_i} \) and \( \tilde{\lambda}^H_i = (\lambda^H_i, 0, \ldots, 0) \in \mathbb{R}^{m_i} \) for \( i = 1, \ldots, J. \)

**Proof.** Part (a). Recall that a point \( z^* \) is said to be an S-stationary point of the MPCC if there exists \((\lambda^0, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^{J} \times \mathbb{R}^{J}\) such that

\[
\begin{align*}
\nabla f(z^*) + \nabla g(z^*) \lambda^0 + \nabla h(z^*) \lambda^h + \sum_{i=1}^J \nabla G_i(z^*) \lambda^G_i + \sum_{i=1}^J \nabla H_i(z^*) \lambda^H_i &= 0, \\
\lambda^0 &\geq 0, \quad g(z^*)^T \lambda^0 = 0, \\
\lambda^H_i &= 0 \quad \text{if} \quad i \in I_G(z^*) \cap I_H^+(z^*), \\
\lambda^G_i &= 0 \quad \text{if} \quad i \in I_G^+(z^*) \cap I_H(z^*), \\
\lambda_i^G &\leq 0, \quad \lambda_i^H \leq 0, \quad \text{if} \quad i \in I_G(z^*) \cap I_H(z^*). \quad (22)
\end{align*}
\]

Note that

\[
I_G(z^*) = I_G(z^*), \quad I_G^{+}(z^*) = I_G^{+}(z^*), \quad I_H(z^*) = I_H(z^*), \quad I_H^{+}(z^*) = I_H^{+}(z^*).
\]

Let

\[
\tilde{\lambda}^G = (\tilde{\lambda}^G_1, \ldots, \tilde{\lambda}^G_J) \quad \text{with} \quad \tilde{\lambda}^G_i = (\lambda^G_i, 0, \ldots, 0) \in \mathbb{R}^{m_i},
\]

\[
\tilde{\lambda}^H = (\tilde{\lambda}^H_1, \ldots, \tilde{\lambda}^H_J) \quad \text{with} \quad \tilde{\lambda}^H_i = (\lambda^H_i, 0, \ldots, 0) \in \mathbb{R}^{m_i}.
\]
From $\lambda_i^G \leq 0, \lambda_i^H \leq 0$, we have $\tilde{\lambda}_i^G \in -K_i, \tilde{\lambda}_i^H \in -K_i$. Thus (22) implies that
\[
\begin{cases}
\nabla f(z^*) + \nabla g(z^*)\lambda^g + \nabla h(z^*)\lambda^h + \sum_{i=1}^J \nabla G_i(z^*)\tilde{\lambda}_i^G + \sum_{i=1}^J \nabla \widetilde{H}_i(z^*)\tilde{\lambda}_i^H = 0, \\
\lambda^g \geq 0, \ g(z^*)^T\lambda^g = 0, \\
\tilde{\lambda}_i^H = 0 \quad \text{if} \quad i \in I_G(z^*) \cap I_H^+(z^*), \\
\tilde{\lambda}_i^G = 0 \quad \text{if} \quad i \in I_G^+(z^*) \cap I_H(z^*), \\
\tilde{\lambda}_i^G \in -K_i, \quad \tilde{\lambda}_i^H \in -K_i, \quad \text{if} \quad i \in I_G(z^*) \cap I_H(z^*). 
\end{cases}
\tag{23}
\]
It is obvious that $B_G(z^*)$ and $B_H(z^*)$ are empty. Hence it follows from (23) that $z^*$ is an S-stationary point of the vector MPCC.

Conversely, assume that $z^*$ is an S-stationary point of the SOCMPCC reformulation, i.e., there exists $(\lambda^g, \lambda^h, \tilde{\lambda}_i^G, \tilde{\lambda}_i^H) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^r \times \mathbb{R}^t$ such that (23) holds, where $\tilde{\lambda}_i^G = (\tilde{\lambda}_i^G, \ldots, \tilde{\lambda}_i^G), \tilde{\lambda}_i^G \in \mathbb{R}^{m_i}$ and $\tilde{\lambda}_i^H = (\tilde{\lambda}_i^H, \ldots, \tilde{\lambda}_i^H), \tilde{\lambda}_i^H \in \mathbb{R}^{m_i}$ for $i = 1, \ldots, J$. Notice that
\[
\nabla G_i(z^*)\tilde{\lambda}_i^G = (\lambda_i^G)\nabla G_i(z^*) \quad \text{and} \quad \nabla \widetilde{H}_i(z^*)\tilde{\lambda}_i^H = (\lambda_i^H)\nabla H_i(z^*).
\]
In addition, $\tilde{\lambda}_i^G \in -K_i, \tilde{\lambda}_i^H \in -K_i$ implies $(\tilde{\lambda}_i^G) \leq 0, (\tilde{\lambda}_i^H) \leq 0$. Hence $z^*$ is an S-stationary point of MPCC with $(\lambda^g, \lambda^h, \tilde{\lambda}_i^G, \tilde{\lambda}_i^H) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^r \times \mathbb{R}^t$ satisfying (22) where $\lambda_i^G = (\tilde{\lambda}_i^G)$ and $\lambda_i^H = (\tilde{\lambda}_i^H)$ for $i = 1, \ldots, J$.

Part (b). Recall that a point $z^*$ is said to be an M-stationary point of the vector MPCC if there exists $(\lambda, \mu, u, v) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^r \times \mathbb{R}^t$ such that
\[
\begin{cases}
\nabla f(z^*) + \nabla g(z^*)\lambda^g + \nabla h(z^*)\lambda^h + \sum_{i=1}^J \nabla G_i(z^*)\tilde{\lambda}_i^G + \sum_{i=1}^J \nabla H_i(z^*)^T\lambda_i^H = 0, \\
\lambda^g \geq 0, \ g(z^*)^T\lambda^g = 0, \\
\lambda_i^H = 0 \quad \text{if} \quad i \in I_G(z^*) \cap I_H^+(z^*), \\
\tilde{\lambda}_i^G = 0 \quad \text{if} \quad i \in I_G^+(z^*) \cap I_H(z^*), \\
\lambda_i^G < 0, \lambda_i^H < 0, \text{ or } \lambda_i^G\lambda_i^H = 0 \quad \text{if} \quad i \in I_G(z^*) \cap I_H(z^*).
\end{cases}
\]
Let
\[
\begin{align*}
\tilde{\lambda}_i^G &= (\tilde{\lambda}_i^G, \ldots, \tilde{\lambda}_i^G) \quad \text{with} \quad \tilde{\lambda}_i^G = (\lambda_i^G, 0, \ldots, 0) \in \mathbb{R}^{m_i}, \\
\tilde{\lambda}_i^H &= (\tilde{\lambda}_i^H, \ldots, \tilde{\lambda}_i^H) \quad \text{with} \quad \tilde{\lambda}_i^H = (\lambda_i^H, 0, \ldots, 0) \in \mathbb{R}^{m_i}.
\end{align*}
\]
Then
\[
\begin{cases}
\nabla f(z^*) + \nabla g(z^*)\lambda^g + \nabla h(z^*)\lambda^h + \sum_{i=1}^J \nabla G_i(z^*)\tilde{\lambda}_i^G + \sum_{i=1}^J \nabla \widetilde{H}_i(z^*)\tilde{\lambda}_i^H = 0, \\
\lambda^g \geq 0, \ g(z^*)^T\lambda^g = 0, \\
\tilde{\lambda}_i^H = 0 \quad \text{if} \quad i \in I_G(z^*) \cap I_H^+(z^*), \\
\tilde{\lambda}_i^G = 0 \quad \text{if} \quad i \in I_G^+(z^*) \cap I_H(z^*), \\
\lambda_i^G, \lambda_i^H \in -K_i, \quad \text{or} \quad \tilde{\lambda}_i^G = 0, \tilde{\lambda}_i^H \in \mathbb{R}^{m_i}, \quad \text{or} \quad \tilde{\lambda}_i^G = 0, \tilde{\lambda}_i^H \in \mathbb{R}^{m_i} \quad \text{if} \quad i \in I_G(z^*) \cap I_H(z^*). 
\end{cases}
\]
Hence $z^*$ is an M-stationary point for the corresponding SOCMPCC. The proof for the C-stationary condition is similar and is omitted.

**Example 9.1** Consider an example of MPCC given in [6].
\[
\begin{align*}
\min & \quad z_1 - \frac{25}{8}z_2 - z_3 - \frac{1}{2}z_4 \\
\text{s.t.} & \quad z_4^2 \leq 0, \\
& \quad 0 \leq G_i(z) \perp H_i(z) \geq 0, \quad i = 1, 2
\end{align*}
\]
where $G_1(x) = 6z_1 - z_3 - z_4$, $G_2(x) = z_1$, $H_1(x) = -6z_2 - z_3$, and $H_2(x) = -z_2$.

It is easy to see that $z^* = (0, 0, 0, 0)$ is the unique optimal solution. The only nonempty index set is $I_G(z^*) \cap I_H(z^*) = \{1, 2\}$. Consider the $W$-stationary system for MPCC:

\[
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
\frac{25}{8} \\
-1 \\
-\frac{1}{2}
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} \lambda^g + \begin{bmatrix}
6 \\
0 \\
-1
\end{bmatrix} \lambda^G_1 + \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} \lambda^G_2 + \begin{bmatrix}
-6 \\
0 \\
-1
\end{bmatrix} \lambda^H_1 + \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} \lambda^H_2,
\]

where $\lambda^g \geq 0$. The solutions are $\lambda^g \geq 0, \lambda^G_1 = \frac{1}{2}, \lambda^G_2 = 2, \lambda^H_1 = -\frac{1}{2}, \lambda^H_2 = -\frac{1}{8}$ and hence $z^* = (0, 0, 0, 0)$ is a $W$-stationary point. But since $\lambda^G_1 \lambda^H_2 < 0$, $z^*$ is not a $C$-stationary point and hence not an M-stationary point. Now we reformulate the problem as an $SOCMPCC$:

\[
\min \quad z_1 - \frac{25}{8}z_2 - z_3 - \frac{1}{2}z_4
\]

\[
s.t. \quad z_4^2 \leq 0
\]

\[
\mathcal{K} \ni \tilde{G}_i(z) \perp \tilde{H}_i(z) \in \mathcal{K}, \ i = 1, 2,
\]

where $\mathcal{K}$ is the 2-dimensional second-order cone, $\tilde{G}_1(x) = (6z_1 - z_3 - z_4, 0)$, $\tilde{G}_2(x) = (z_1, 0)$, $\tilde{H}_1(x) = (-6z_2 - z_3, 0)$, and $\tilde{H}_2(x) = (-z_2, 0)$. The only nonempty index set is $I_{\tilde{G}}(z^*) \cap I_{\tilde{H}}(z^*) = \{1, 2\}$. We now increase the dimensions of the multipliers from 1 to 2 with the first components kept the same. Let

\[
\tilde{\lambda}^G_1 := \left(\frac{1}{\frac{1}{2}}\right), \quad \tilde{\lambda}^G_2 := \left(\frac{2}{-2}\right), \quad \tilde{\lambda}^H := \left(\frac{-\frac{1}{2}}{-\frac{1}{2}}\right), \quad \tilde{\lambda}^H_2 := \left(\frac{-\frac{1}{8}}{-\frac{1}{8}}\right).
\]

Then $\tilde{\lambda}^G_1, \tilde{\lambda}^H \in -\mathcal{K}$. Let $\xi = (1, 1)$. Then $\tilde{\lambda}^G_2 \perp \xi, \tilde{\lambda}^H_2 \perp \xi$. Hence $z^*$ is an M-stationary point for the corresponding $SOCMPCC$.

From this example, it is inspiring to see that by increasing the dimension of the second-order cone, we can obtain new and weaker M- or C-stationary conditions which can be used to identify candidates for optimality when the M- or C-stationary conditions of the original MPCC do not hold.

10 Appendix

Proof of Proposition 3.1. Recall that by the relationship between the regular normal cone and the regular coderivative of the projection operator in (5),

\[
(u, v) \in \tilde{N}_\Omega(x, y) \iff -v \in \tilde{D}^*\Pi_{\mathcal{K}}(x - y)(-u - v).
\]

Case 1 $x = 0, y \in \text{int}\mathcal{K}$. In this case, $x - y = -y \in -\text{int}\mathcal{K}$. It follows from Proposition 2.4(ii) that $\mathcal{J}\Pi_{\mathcal{K}}(x - y) = \emptyset$. Hence

\[
(u, v) \in \tilde{N}_\Omega(x, y) \iff -v \in \tilde{D}^*\Pi_{\mathcal{K}}(x - y)(-u - v) = \{0\} \iff u \in \mathbb{R}^m, v = 0.
\]

Case 2 $x \in \text{int}\mathcal{K}, y = 0$. This case is symmetric to Case 1 and we omit the proof.

Case 3 $x = 0, y \in \text{bd}\mathcal{K}\setminus\{0\}$. Then $x - y \in -\text{bd}\mathcal{K}\setminus\{0\}$. By Proposition 2.4(v),

\[
\tilde{D}^*\Pi_{\mathcal{K}}(x - y)(-u - v) = \{z^* | z^* \in \mathbb{R}_+c_2(x - y), (-u - v - z^*, c_2(x - y)) \geq 0\}.
\]
Since \( y \in \text{bd}\mathcal{K}\setminus\{0\} \), \( y_1 = \|y_2\| > 0 \) and hence \( c_2(x - y) = \frac{1}{2}(1, -\bar{y}_2) = \frac{1}{2\bar{y}_1}\bar{y}_1. \) So

\[
(u, v) \in \hat{N}_\Omega(x, y) \iff -v \in \hat{D}_+ \Pi_\mathcal{K}(x - y)(-u - v) \\
\iff -v \in \mathbb{R}_+ c_2(x - y) = \mathbb{R}_+ \hat{y}, \ (u, \hat{y}) \geq 0 \\
\iff v \in \mathbb{R}_- \hat{y}, \ u \in \hat{y}^\circ.
\]

**Case 4** \( x \in \text{bd}\mathcal{K}\setminus\{0\}, y = 0. \) This case can be proven similarly as in Case 3.

**Case 5** \( x, y \in \text{bd}\mathcal{K}\setminus\{0\} \) with \( x^T y = 0. \) In this case, by virtue of Proposition 2.2, we have \( x - y = ((1 - k)x_1, (1 + k)x_2) \) with \( k = y_1/x_1 > 0. \) Note that \( x - y \in (-\mathcal{K} \cup \hat{\mathcal{K}})^c. \) So according to Proposition 2.4(iii),

\[
\mathcal{J}_\mathcal{K}(x - y) = \frac{1}{1 + k} I + \frac{1}{2} \left[ -\frac{1-k}{1+k} \frac{x^T}{x_2} \frac{x}{x_2} - \frac{k}{1+k} \frac{x_2^T}{x_2} \right].
\]

Hence

\[
(u, v) \in \hat{N}_\Omega(x, y) \iff -v \in \hat{D}_+ \Pi_\mathcal{K}(x - y)(-u - v) \\
\iff \left( \frac{1}{1 + k} I + \frac{1}{2} \left[ -\frac{1-k}{1+k} \frac{x^T}{x_2} \frac{x}{x_2} - \frac{k}{1+k} \frac{x_2^T}{x_2} \right] \right) (u_1 + v_1) = (v_1) \\
\iff \left\{ \begin{array}{l} u_1 + \frac{x_2}{x_2} (u_2 + v_2) = v_1 \\
\left[ (1 + k)(u_1 + v_1) - (1 - k)\frac{x_2}{x_2} (u_2 + v_2) \right] x_2 = 2k v_2 - 2u_2. \end{array} \right.
\]

In what follows, we first show that the following inclusion holds

\[
\tilde{N}_\Omega(x, y) \subset \{(u, v) \mid v \perp y, \ u \perp x, \ x_1 \bar{u} + y_1 v \in \mathbb{R} x\}
\]

and then show the converse inclusion holds. Let \( (u, v) \in \tilde{N}_\Omega(x, y). \) Take \( x' \in \text{bd}\mathcal{K}\setminus\{0\} \) and \( y' := kx' \in \text{bd}\mathcal{K}\setminus\{0\}. \) Then \( (x', y') = 0, \) i.e., \( (x', y') \in \Omega. \) Hence

\[
\frac{(u, v), (x', y') - (x, y)}{\| (x', y') - (x, y) \|} = \frac{\langle (u, x') - x, (v, y') - y \rangle}{\| (x' - x, y' - y) \|} = \frac{\langle (u, x') - x, k\hat{v}, -k \hat{x} \rangle}{\| (x' - x, kx' - k\hat{x}) \|} = \frac{\langle (u + k\hat{v}, x' - x) \rangle}{\sqrt{1 + k^2} \| x' - x \|} = \frac{\langle u + k\hat{v}, x' - x \rangle}{\sqrt{1 + k^2} \| x' - x \|},
\]

where we have used the fact that \( \langle a, \hat{b} \rangle = \langle \hat{a}, b \rangle \) and \( \| (a, \hat{b}) \| = \| (a, b) \| \) for arbitrary vectors \( a, b \in \mathbb{R}^m. \) Since \( (u, v) \in \tilde{N}_\Omega(x, y), \) it follows from (25) that

\[
\limsup_{x' \to x} \frac{(u + k\hat{v}, x' - x)}{\sqrt{1 + k^2} \| x' - x \|} \leq 0,
\]

which implies that

\[
 u + k\hat{v} \in \text{bd}\mathcal{K}\setminus\{0\}(x).
\]

Since \( x_2 \neq 0, \) \( \text{bd}\mathcal{K} = \{x | x_1 - \| x_2 \| = 0 \} \) is a smooth manifold near \( x. \) So \( \tilde{N}_{\text{bd}\mathcal{K}\setminus\{0\}}(x) = \{\mathbb{R} \hat{x}\} \) (see also [17, Example 6.8]). Thus

\[
 u + k\hat{v} \in \mathbb{R} \hat{x}.
\]
On the other hand, if in particular we choose \((x', y') := (x, k'y)\) with \(k' \to 1\). Then 
\((x', y') \in \Omega\) and
\[
\frac{\langle (u, v), (x', y') - (x, y) \rangle}{\| (x', y') - (x, y) \|} = \frac{\langle u, x' - x \rangle + \langle v, y' - y \rangle}{\| (x' - x, y' - y) \|} = \frac{(k' - 1) \langle v, y \rangle}{|k' - 1| \| y \|}. \tag{26}
\]

Since \((u, v) \in \hat{N}_\Omega(x, y)\), it follows from the definition of regular normal cone and (26) that
\[
\limsup_{k' \to 1} \frac{(k' - 1) \langle v, y \rangle}{|k' - 1| \| y \|} = \limsup_{(x', y') \to (x, y)} \frac{\langle (u, v), (x', y') - (x, y) \rangle}{\| (x', y') - (x, y) \|} \leq 0,
\]
which implies that \(v \perp y\). Similarly, we obtain \(u \perp x\).

From the above arguments, we have
\[
\hat{N}_\Omega(x, y) \subset \{ (u, v) \mid u \perp x, \ v \perp y, \ u + kv \in \mathbb{R} x \} = \{ (u, v) \mid u \perp x, \ v \perp y, \ x_1 \hat{\nu} + y_1 v \in \mathbb{R} x \}.
\]

Now we show that the converse inclusion holds. Let \((u, v)\) lie in the right hand side of the above inclusion. Then there exists \(\beta \in \mathbb{R}\) such that
\[
\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + k \begin{pmatrix} v_1 \\ -v_2 \end{pmatrix} = \beta \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}, \tag{27}
\]
\[
\begin{align*}
u_1 x_1 + u_2^T x_2 &= 0 \quad \text{and} \quad v_1 y_1 + v_2^T y_2 &= 0,
\end{align*}
\]
which implies that \(u_1 + u_2^T \bar{x}_2 = 0\) and \(v_1 + v_2^T \bar{y}_2 = 0\) since \(x_1 = \| x_2 \| > 0\) and \(y_1 = \| y_2 \| > 0\). Since \(\bar{x}_2 = -\bar{y}_2\), it follows that \(v_1 - v_2^T \bar{x}_2 = 0\). Hence
\[
u_1 + \bar{x}_2^T (u_2 + v_2) = u_1 + \bar{x}_2^T u_2 + \bar{x}_2^T v_2 = \bar{x}_2^T v_2 = v_1,
\]
and
\[
\begin{align*}
\left[ (1 + k)(u_1 + v_1) - (1 - k) \bar{x}_2^T (u_2 + v_2) \right] \bar{x}_2 &= \left[ (1 + k)(u_1 + v_1) - (1 - k)(-u_1 + v_1) \right] \bar{x}_2 \\
&= \left[ 2u_1 + 2kv_1 \right] \bar{x}_2 \\
&= 2\beta x_1 \bar{x}_2 \\
&= 2\beta x_2 \\
&= 2kv_2 - 2u_2,
\end{align*}
\]
where the third and fifth equalities follow from (27). Thus \((u, v)\) satisfies (24), i.e., \((u, v) \in \hat{N}_\Omega(x, y)\).

**Case 6** \(x = y = 0\). By Proposition 2.4(vi),
\[
\hat{D}^* \Pi_K(x - y)(-u - v) = \{ z^* \mid z^* \in K, \ -u - v - z^* \in K \}.
\]

Hence
\[
(u, v) \in \hat{N}_\Omega(x, y) \iff -v \in \hat{D}^* \Pi_K(x - y)(-u - v) \iff u \in -K, v \in -K.
\]
Proof of Proposition 3.2. We only prove for \( \Pi_K \) since the proof for \( \Pi_{K^c} \) exactly similar. Consider the following six cases.

Case 1 \( x \in \text{int} K \). In this case \( \Pi_K(x) = x \), \( \Pi_K(x + h) = x + h \) for \( h \) sufficiently close to 0, and \( \Pi_K(x; h) = h \) by Proposition 2.3(i). So

\[
\Pi_K(x + h) - \Pi_K(x) - \Pi_K'(x; h) = x + h - x - h = 0 = O(\|h\|^2).
\]

Case 2 \( x \in -\text{int} K \). This case is symmetric to Case 1 and we omit the proof.

Case 3 \( x \in (-K \cup K)^c \). Then for \( h \) sufficiently close to 0, we have \( x + h \in (-K \cup K)^c \) and so \( \lambda_1(x) = x_1 - \|x_2\| < 0, \lambda_1(x + h) = (x_1 + h_1) - \|x_2 + h_2\| < 0 \). By (3) and Proposition 2.4(iii),

\[
\Pi_K(x) = \frac{1}{2} (x_1 + \|x_2\|) \left( \frac{1}{\|x_2\|} \right), \quad \Pi_K(x + h) = \frac{1}{2} (x_1 + h_1 + \|x_2 + h_2\|) \left( \frac{1}{\|x_2 + h_2\|} \right)
\]

and

\[
\Pi_K'(x; h) = \frac{1}{2} \left[ \frac{1}{\bar{x}_2} I + \frac{x_1}{\|x_2\|} (I - \bar{x}_2 \bar{x}_2^T) \right] \left( \begin{array}{c} h_1 \\ h_2 \end{array} \right).
\]

Therefore

\[
2 \left[ \Pi_K(x + h) - \Pi_K(x) - \Pi_K'(x; h) \right]
\]

\[
= (x_1 + h_1 + \|x_2 + h_2\|) \left( \frac{1}{\|x_2 + h_2\|} \right) - (x_1 + \|x_2\|) \left( \frac{1}{\|x_2\|} \right)
\]

\[- \left[ \frac{1}{\bar{x}_2} I + \frac{x_1}{\|x_2\|} (I - \bar{x}_2 \bar{x}_2^T) \right] \left( \begin{array}{c} h_1 \\ h_2 \end{array} \right). \quad (28)
\]

Since \( x_2 \neq 0 \) in this case, the norm \( \| \cdot \| \) is second-order continuously differentiable at \( x_2 \), which in turn implies that

\[
\|x_2 + h_2\| = \|x_2\| + \bar{x}_2^T h_2 + O(\|h_2\|^2).
\]

Thus the first component of the right hand side of (28) is equal to

\[
x_1 + h_1 + \|x_2 + h_2\| - (x_1 + \|x_2\|) - (h_1 + \bar{x}_2^T h_2)
\]

\[
= \|x_2 + h_2\| - \|x_2\| - \bar{x}_2^T h_2 = O(\|h_2\|^2) = O(\|h\|^2).
\]

The second component of the right hand side of (28) is equal to

\[
(x_1 + h_1 + \|x_2 + h_2\|) \frac{x_2 + h_2}{\|x_2 + h_2\|} - (x_1 + \|x_2\|) \frac{x_2}{\|x_2\|} - h_1 \frac{x_2}{\|x_2\|} - h_2 - \frac{x_1}{\|x_2\|} (I - \bar{x}_2 \bar{x}_2^T) h_2
\]

\[
= x_1 \left[ \frac{x_2 + h_2}{\|x_2 + h_2\|} - \frac{x_2}{\|x_2\|} \right] - h_1 \frac{x_2}{\|x_2\|} - h_2 - \frac{x_1}{\|x_2\|} (I - \bar{x}_2 \bar{x}_2^T) h_2
\]

\[
= x_1 \left[ \frac{x_2 + h_2}{\|x_2 + h_2\|} - \frac{x_2}{\|x_2\|} \right] - h_1 \frac{x_2}{\|x_2\|} - h_2 - \frac{x_1}{\|x_2\|} (I - \bar{x}_2 \bar{x}_2^T) h_2
\]

\[
= O(\|h\|^2),
\]
where the second equality holds by the Lipschitz continuity of $x_2/\|x_2\|$ and the last equality follows from the second-order continuous differentiability of $x_2/\|x_2\|$, i.e.,

$$\frac{x_2 + h_2}{\|x_2 + h_2\|} \rightarrow \frac{x_2}{\|x_2\|} = \frac{I - \bar{x}_2 \bar{x}_2^T}{\|x_2\|} h_2 = O(\|h_2\|^2).$$

**Case 4** $x \in \mathbf{b} \mathcal{K} \setminus \{0\}$. In this case $\lambda_1(x) = 0$ and $\lambda_2(x) > 0$. Then by (3) and Proposition 2.3(ii), for $h$ sufficiently close to 0,

$$\Pi_K(x) = \frac{1}{2} \left( x_1 + \|x_2\| \right) \left( \frac{1}{x_2} \right),$$

$$\Pi_K(x+h) = \frac{1}{2} \left( x_1 + h_1 - \|x_2 + h_2\| \right) + \left( -\frac{1}{\|x_2 + h_2\|} \right) + \frac{1}{2} \left( x_1 + h_1 + \|x_2 + h_2\| \right) \left( \frac{1}{\|x_2 + h_2\|} \right),$$

and

$$\Pi'_K(x; h) = h - \frac{1}{2} (h_1 - \bar{x}_2^T h_2)_- \left( \frac{1}{-\bar{x}_2} \right).$$

Then the first component of $2\left[ \Pi_K(x + h) - \Pi_K(x) - \Pi'_K(x; h) \right]$ is

\[
\begin{align*}
&\left( x_1 + h_1 - \|x_2 + h_2\| \right)_+ + (x_1 + h_1 + \|x_2 + h_2\|) - (x_1 + \|x_2\|) - \left( 2 h_1 - (h_1 - \bar{x}_2^T h_2)_- \right) \\
&= \left( x_1 + h_1 - \|x_2 + h_2\| \right)_+ - \left( x_1 + h_1 - \|x_2 + h_2\| \right)_- + (x_1 + h_1 + \|x_2 + h_2\|) - (x_1 + \|x_2\|) \\
&\quad - \left( 2 h_1 - (h_1 - \bar{x}_2^T h_2)_- \right) \\
&= \left( x_1 + h_1 - \|x_2 + h_2\| \right)_+ + \left( h_1 - \bar{x}_2^T h_2 \right)_- \\
&= \left( h_1 - \bar{x}_2^T h_2 + O(\|h_2\|^2) \right)_+ + \left( h_1 - \bar{x}_2^T h_2 \right)_- \\
&= O(\|h_2\|^2) = O(\|h\|^2),
\end{align*}
\]

(29)

where the fourth equality holds since

$$\left( h_1 - \bar{x}_2^T h_2 + O(\|h_2\|^2) \right)_- = \left( h_1 - \bar{x}_2^T h_2 \right)_- + O(\|h_2\|^2)$$

by virtue of Lipschitz continuity of the function $(\cdot)_-$. According to (29) we have that

$$\left( x_1 + h_1 - \|x_2 + h_2\| \right)_+ = (x_1 + h_1 + \|x_2 + h_2\|) - (x_1 + \|x_2\|) - \left( 2 h_1 - (h_1 - \bar{x}_2^T h_2)_- \right) + O(\|h\|^2).$$

(30)

The second component of $2\left[ \Pi_K(x + h) - \Pi_K(x) - \Pi'_K(x; h) \right]$ is

\[
\begin{align*}
&\left[ - \left( x_1 + h_1 - \|x_2 + h_2\| \right)_+ + (x_1 + h_1 + \|x_2 + h_2\|) \right] \frac{x_2 + h_2}{\|x_2 + h_2\|} - (x_1 + \|x_2\|) \frac{x_2}{\|x_2\|} - (2 h_2 + (h_1 - \bar{x}_2^T h_2)_- \bar{x}_2) \\
&= \left[ 2 \|x_2 + h_2\| + (h_1 - \bar{x}_2^T h_2)_- + O(\|h\|^2) \right] \frac{x_2 + h_2}{\|x_2 + h_2\|} - (x_1 + \|x_2\|) \frac{x_2}{\|x_2\|} - (2 h_2 + (h_1 - \bar{x}_2^T h_2)_- \bar{x}_2) \\
&= \left( h_1 - \bar{x}_2^T h_2 \right)_- \left[ \frac{x_2 + h_2}{\|x_2 + h_2\|} - \frac{x_2}{\|x_2\|} \right] + O(\|h\|^2) \\
&= O(\|h\|^2),
\end{align*}
\]
where the second equality follows from (30) and the last equality follows from the fact that $h_1 - \bar{x}_2^T h_2 = O(||h||)$ and the Lipschitz continuity of $x_2/||x_2||$ since $x_2 \neq 0$ in this case.

**Case 5** \(x \in -\text{bd}\mathcal{K}\setminus\{0\}\). In this case \(\Pi_{\mathcal{K}}(x) = 0\) and for \(h\) that is very close to zero, \(x+h \not\in \mathcal{K}\) and hence \(\lambda_1(x+h) < 0\) and by (3) and Proposition 2.3(iii),

\[
\Pi_{\mathcal{K}}(x+h) = \frac{1}{2}\left( x_1 + h_1 + ||x_2 + h_2|| \right) + \left( \frac{1}{||x_2 + h_2||} \right)
\]

and

\[
\Pi'_{\mathcal{K}}(x; h) = \frac{1}{2}(h_1 + \bar{x}_2^T h_2)_+ \left( \frac{1}{\bar{x}_2} \right).
\]

The first component of \(2[\Pi_{\mathcal{K}}(x+h) - \Pi_{\mathcal{K}}(x) - \Pi'_{\mathcal{K}}(x; h)]\) is

\[
\left( x_1 + h_1 + ||x_2 + h_2|| \right)_+ - \left( h_1 + \bar{x}_2^T h_2 \right)_+ + \left( h_1 + \bar{x}_2^T h_2 + O(||h_2||^2) \right)_+ = O(||h_2||^2).
\]

The second component of \(2[\Pi_{\mathcal{K}}(x+h) - \Pi_{\mathcal{K}}(x) - \Pi'_{\mathcal{K}}(x; h)]\) is

\[
\left( x_1 + h_1 + ||x_2 + h_2|| \right)_+ \left( \frac{x_2 + h_2}{||x_2 + h_2||} \right) - \left( h_1 + \bar{x}_2^T h_2 \right)_+ \left( \frac{x_2}{||x_2||} \right)
\]

\[
= \left( h_1 + \bar{x}_2^T h_2 \right)_+ \left( \frac{x_2 + h_2}{||x_2 + h_2||} - \frac{x_2}{||x_2||} \right) + O(||h_2||^2)
\]

\[
= O(||h||^2),
\]

where the last equality follows from \(h_1 + \bar{x}_2^T h_2 = O(||h||)\) and the Lipschitz continuity of \(x_2/||x_2||\) since \(x_2 \neq 0\) in this case.

**Case 6** \(x = 0\). Then \(\Pi_{\mathcal{K}}(x) = 0\), \(\Pi_{\mathcal{K}}(x+h) = \Pi_{\mathcal{K}}(h)\) and \(\Pi'_{\mathcal{K}}(x; h) = \Pi_{\mathcal{K}}(h)\) by Proposition 2.3(iv). Thus

\[
\Pi_{\mathcal{K}}(x+h) - \Pi_{\mathcal{K}}(x) - \Pi'_{\mathcal{K}}(x; h) = 0 = O(||h||^2).
\]

\[\blacksquare\]

**Proof of Proposition 3.4.** Let \((x, y) \in \Omega\). Consider the following cases.

**Case 1** \(x \in \text{int}\mathcal{K}, y = 0\), or \(x = 0, y \in \text{int}\mathcal{K}, y = 0 \in \text{bd}\mathcal{K}\setminus\{0\}\). In this case \(\Pi_{\mathcal{K}}\) is continuously differentiable at \(x - y\). By Proposition 3.3, \((u, v) \in N^\pi_{\Omega}(x, y)\) if and only if (8) holds. Since \(\Pi_{\mathcal{K}}(x - y)\) is continuously differentiable at \(x - y\), (8) takes the form

\[
(\nabla \Pi_{\mathcal{K}}(x-y)(u+v) - v, h) \leq 0, \quad \forall h \in \mathbb{R}^m,
\]

or equivalently,

\[
\nabla \Pi_{\mathcal{K}}(x-y)(u+v) - v = 0.
\]

By Proposition 2.6, the above equation holds if and only if \((u, v) \in \tilde{N}_{\Omega}(x, y)\) and hence \(N^\pi_{\Omega}(x, y) = \tilde{N}_{\Omega}(x, y)\).

**Case 2** \(x = 0\) and \(y \in \text{bd}\mathcal{K}\setminus\{0\}\). In this case \(x - y = -y \in -\text{bd}\mathcal{K}\setminus\{0\}\). Hence by Proposition 2.3(iii) and the fact that \(c_2(-y) = c_1(y)\), \(\Pi'_{\mathcal{K}}(x-y; h) = 2(c_1(y)^T h)_+ c_1(y)\). So
(8) takes the form
\[
\langle u + v, 2(c_1(y)^T h) + c_1(y) \rangle - \langle v, h \rangle \leq 0 \quad \forall h \in \mathbb{R}^m
\]
\[\Longleftrightarrow \]
\[
\begin{cases}
\langle -v, h \rangle \leq 0 & \text{if } c_1(y)^T h \leq 0 \\
\langle u + v, 2c_1(y)^T h c_1(y) \rangle - \langle v, h \rangle \leq 0 & \text{if } c_1(y)^T h \geq 0
\end{cases}
\]
\[\Longleftrightarrow \]
\[
\begin{cases}
\langle -v, h \rangle \leq 0 & \text{if } c_1(y)^T h \leq 0 \\
2(u + v)^T c_1(y)c_1(y) - \langle v, h \rangle \leq 0 & \text{if } c_1(y)^T h \geq 0
\end{cases}
\]
\[\Longleftrightarrow \exists \alpha, \beta \geq 0 \text{ such that } -v = \alpha c_1(y) \quad \text{and} \\
st_{1}(u + v)^T c_1(y)c_1(y) - v = -\beta c_1(y)
\]
\[\Longleftrightarrow \exists \alpha, \beta \geq 0 \text{ such that } -v = \alpha c_1(y) \quad \text{and} \\
2u^T c_1(y)c_1(y) = -\beta c_1(y)
\]
\[\Longleftrightarrow \exists \alpha, \beta \geq 0 \text{ such that } v \in \mathbb{R}^{-}c_1(y) \quad \text{and} \quad \langle u, c_1(y) \rangle \leq 0.
\]

Since \(y_1 = ||y_2|| > 0\), we have \(c_1(y) = \frac{1}{2y_1^2} \hat{y}\) and hence \((u, v) \in N^\pi_{\Omega}(x, y)\) if and only if \(u \in \hat{y}_o, v \in \mathbb{R}^{-}\hat{y}\). The equivalence of the two normal cones follows from the exact formula of \(N^\pi_{\Omega}(x, y)\) in Proposition 3.1.

**Case 3** \(x \in \text{bd}K \setminus \{0\}\) and \(y = 0\). In this case \(x - y = x\) and \(c_1(x - y) = c_1(x)\). Hence by Proposition 2.3(ii), \(\Pi^\prime_{\Omega}(x - y; h) = h - 2(c_1(x)^T h) - c_1(x)\). So (8) takes the form
\[
\langle u + v, h - 2(c_1(x)^T h) - c_1(x) \rangle - \langle v, h \rangle \leq 0, \quad \forall h \in \mathbb{R}^m
\]
\[\Longleftrightarrow \]
\[
\begin{cases}
\langle u + v, h \rangle - \langle v, h \rangle \leq 0 & \text{if } c_1(x)^T h \geq 0 \\
\langle u + v, h - 2c_1(x)^T h c_1(x) \rangle - \langle v, h \rangle \leq 0 & \text{if } c_1(x)^T h \leq 0
\end{cases}
\]
\[\Longleftrightarrow \]
\[
\begin{cases}
\langle u, h \rangle \leq 0 & \text{if } c_1(x)^T h \geq 0 \\
\langle u - 2(u + v)^T c_1(x)c_1(x), h \rangle \leq 0 & \text{if } c_1(x)^T h \leq 0
\end{cases}
\]
\[\Longleftrightarrow \exists \alpha, \beta \geq 0 \text{ such that } u = -\alpha c_1(x) \quad \text{and} \\
st_{1}(u - 2(u + v)^T c_1(x)c_1(x)) = \beta c_1(x)
\]
\[\Longleftrightarrow \exists \alpha, \beta \geq 0 \text{ such that } u = -\alpha c_1(x) \quad \text{and} \\
2v^T c_1(x)c_1(x) = \beta c_1(x)
\]
\[\Longleftrightarrow \exists \alpha, \beta \geq 0 \text{ such that } v \in \mathbb{R}_{-}c_1(x) \quad \text{and} \quad \langle u, c_1(x) \rangle \leq 0.
\]

Since \(x_1 = ||x_2|| > 0\), we have \(c_1(x) = \frac{1}{2x_1^2} \hat{x}\) and hence \((u, v) \in N^\pi_{\Omega}(x, y)\) if and only if \(u \in \mathbb{R}_{-} \hat{x}, v \in \mathbb{R}_{-} \hat{x}\). The equivalence of the two normal cones follows from the exact formula of \(N^\pi_{\Omega}(x, y)\) in Proposition 3.1.

**Case 4** \(x = 0\) and \(y = 0\). In this case (8) takes the form
\[
\langle u + v, \Pi_K(h) \rangle - \langle v, h \rangle \leq 0, \quad \forall h \in \mathbb{R}^m
\]
\[\Longleftrightarrow \]
\[
\langle u, \Pi_K(h) \rangle - \langle v, \Pi_K(h) \rangle \leq 0, \quad \forall h \in \mathbb{R}^m
\]
\[\Longleftrightarrow \]
\[
u \in K^o = -K \quad \text{and} \quad v \in -K.
\]

The equivalence of the two normal cones follows from the exact formula of \(N^\pi_{\Omega}(x, y)\) in Proposition 3.1.

**Proof of Proposition 3.5.** Consider the following cases.

**Case 1** \(x = 0, y \in \text{int}K, \text{ or } x \in \text{int}K, y = 0 \text{ or } x, y \in \text{bd}K \setminus \{0\}\). In these cases, it is easy to prove since all points in \(\Omega\) near \((x, y)\) belong to the same type and hence the regular normal cone and the limiting normal coincide.
Case 2 \( x = 0 \) and \( y \in \text{bd}\mathcal{K}\{0\} \). Let \( z := x - y \). Then \( z \in -\text{bd}\mathcal{K}\{0\} \) and hence according to Proposition 2.4(v),
\[
D^*\Pi_K(z)(-u - v) = \left\{ O, \frac{1}{2} \begin{bmatrix} \frac{1}{2} z_2^T \bar{z}_2 \end{bmatrix} (-u - v) \cup \left\{ z^* | z^* \in \mathbb{R}_+c_2(z), \langle -u - v - z^*, c_2(z) \rangle \geq 0 \right\} \right\}
\]
Since \( O \in D^*\Pi_K(z)(-u - v) \), it follows from (6) that \((u, v)\) with \( u \in \mathbb{R}^m \) and \( v = 0 \) belongs to \( N_\Omega(x, y) \). Take \( \frac{1}{2} \begin{bmatrix} 1 \bar{z}_2 \bar{z}_2^T \end{bmatrix} (-u - v) \in D^*\Pi_K(z)(-u - v) \). Since \( \bar{z}_2 = -\bar{y}_2 \), the following equivalences hold.
\[
\begin{align*}
-v &= \frac{1}{2} \begin{bmatrix} 1 & -\bar{y}_2 \\ -\bar{y}_2 & \bar{y}_2^T \end{bmatrix} (-u - v) \\
\iff & \left\{ \begin{array}{l}
u_1 + v_1 - (u_2 + v_2)^T \bar{y}_2 = 2v_1 \\
-v_1 + v_1 \bar{y}_2 + (u_2 + v_2)^T \bar{y}_2 \bar{y}_2 = 2v_2
\end{array} \right.
\end{align*}
\]
It follows from (6) that \( \{(u, v) | u \perp \hat{y}, v \in \mathbb{R} \hat{y} \} \subset N_\Omega(x, y) \).

For \( \{z^* | z^* \in \mathbb{R}_+c_2(z), \langle -u - v - z^*, c_2(z) \rangle \geq 0 \} \subset D^*\Pi_K(z)(-u - v) \) we have
\[
\begin{align*}
\left\{ \begin{array}{l}
v \in \mathbb{R}_+c_2(z) \\
\langle -u, c_2(z) \rangle \geq 0
\end{array} \right. & \iff \left\{ \begin{array}{l}
-v \in \mathbb{R}_+ \hat{y} \\
\langle -u, \hat{y} \rangle \geq 0
\end{array} \right. \iff \left\{ \begin{array}{l}
v \in \mathbb{R}_- \hat{y} \\
\langle u, \hat{y} \rangle \leq 0
\end{array} \right.
\end{align*}
\]
where the second equivalence comes from the fact that \( c_2(z) = \frac{1}{2} (1, -\bar{y}_2) = \frac{1}{2y_1} \hat{y} \) with \( y_1 > 0 \) since \( y \in \text{bd}\mathcal{K}\{0\} \). It follows from (6) that \( \{(u, v) | \langle u, \hat{y} \rangle \leq 0, v \in \mathbb{R}_- \hat{y} \} \subset N_\Omega(x, y) \).

Combining the above possibilities, we have
\[
N_\Omega(x, y) = \{ (u, v) | u \in \mathbb{R}^m, v = 0 \text{ or } u \perp \hat{y}, v \in \mathbb{R} \hat{y} \text{ or } \langle u, \hat{y} \rangle \leq 0, v \in \mathbb{R}_- \hat{y} \}.
\]
Case 3 \( x \in \text{bd}\mathcal{K}\{0\} \) and \( y = 0 \). The proof of this case is similar to Case 2.
Case 4 \((x, y) = (0, 0)\). By Proposition 2.4(vi), we have
\[
D^*\Pi_K(0)(-u - v) = \partial_B \Pi_K(0)(-u - v) \cup \{z^* | z^* \in \mathcal{K}, -u - v - z^* \in \mathcal{K} \} \cup \bigcup_{\xi \in \mathcal{C}} \{z^* | -u - v - z^* \in \mathcal{R}_+ \xi, \langle z^*, \xi \rangle \geq 0 \} \cup \bigcup_{\xi \in \mathcal{C}} \{z^* | z^* \in \mathcal{R}_+ \xi, \langle -u - v - z^*, \xi \rangle \geq 0 \}.
\]
Since \( O \in \partial_B \Pi_K(0) \), it follows from (6) that \((u, v)\) with \( v = 0 \) and \( u \in \mathbb{R}^m \) belongs to \( N_\Omega(x, y) \).
Since \( I \in \partial B \Pi_{K}(0) \), \((u,v)\) with \( u = 0 \) and \( v \in \mathbb{R}^{m} \) belongs to \( N_{\Omega}(x,y) \).

Since \( \alpha I + \frac{1}{2} \begin{bmatrix} 1 - 2\alpha & w^{T} \\ w & (1 - 2\alpha)w^{T} \end{bmatrix} \in \partial B \Pi_{K}(0) \) for any \( \alpha \in [0,1] \) and \( \|w\| = 1 \), by virtue of (6),

\[
\left( \alpha I + \frac{1}{2} \begin{bmatrix} 1 - 2\alpha & w^{T} \\ w & (1 - 2\alpha)w^{T} \end{bmatrix} \right) (-u - v) = -v \implies (u,v) \in N_{\Omega}(x,y),
\]

which can be rewritten equivalently as

\[
\begin{cases}
 u_1 + w^{T}(u_2 + v_2) = v_1 \\
 \alpha u_2 + \alpha u_1 w = (1 - \alpha)v_2 - (1 - \alpha)\nu_1 w
\end{cases} \implies (u,v) \in N_{\Omega}(x,y). \tag{31}
\]

We now claim that the solution set of the system of two equations in (31) is

\[
\{ (u,v) \mid u \perp \left( \begin{array}{c} 1 \\ w \end{array} \right), \ v \perp \left( \begin{array}{c} 1 \\ -w \end{array} \right), \ \alpha u + (1 - \alpha)\nu \in \mathbb{R} \left( \begin{array}{c} 1 \\ -w \end{array} \right) \}. \tag{32}
\]

Multiplying \( w \) to the second equation in the system of two equations in (31) yields

\[
v_2^{T}w - v_1 + \alpha v_1 = \alpha (u_2 + v_2)^{T}w + \alpha u_1 = \alpha (v_1 - u_1) + \alpha u_1 = \alpha v_1,
\]

where the second equality holds by the first equality in (31). This means that \( v_2^{T}w - v_1 = 0 \), i.e., \( v \perp (1, -w) \). Applying this to the first equation in (31) yields \( u_1 + w^{T}u_2 = 0 \), i.e., \( u \perp (1, w) \). Using (31) again yields

\[
(1 - \alpha)v_2 - \alpha u_2 = \left[ \alpha u_1 + (1 - \alpha)v_1 \right] w.
\]

Let \( \eta := \alpha u_1 + (1 - \alpha)v_1 \), then \( \alpha u_2 - (1 - \alpha)v_2 = -\eta w \). Hence \( \alpha u + (1 - \alpha)\nu = \eta(1, -w) \in \mathbb{R}(1, -w) \). Conversely, take \((u,v)\) satisfying (32), i.e., there exists \( \eta \in \mathbb{R} \) such that

\[
\alpha \left( \begin{array}{c} u_1 \\ u_2 \end{array} \right) + (1 - \alpha) \left( \begin{array}{c} v_1 \\ -v_2 \end{array} \right) = \eta \left( \begin{array}{c} 1 \\ -w \end{array} \right)
\]

and

\[
u \perp \left( \begin{array}{c} 1 \\ w \end{array} \right), \ v \perp \left( \begin{array}{c} 1 \\ -w \end{array} \right).
\]

Then

\[
u_1 + (u_2 + v_2)^{T}w = v_2^{T}w + u_1 + u_2^{T}w = v_2^{T}w = v_1
\]

and

\[
\alpha u_2 + \alpha u_1 w = -\eta w + (1 - \alpha)v_2 + \left[ \eta - (1 - \alpha)v_1 \right] w
\]

\[
= (1 - \alpha)v_2 - (1 - \alpha)v_1 w,
\]

i.e., \((u,v)\) satisfies the system of equations in (31). It follows that any element \((u,v)\) in the set (32) belongs to the limiting normal cone \( N_{\Omega}(x,y) \).

Since \( \{ z^{*} \mid z^{*} \in K, -u - v - z^{*} \in K \} \subset D^{*}\Pi_{K}(0)(-u - v) \), by (6) any \((u,v)\) such that \( v \in -K \) and \( u \in -K \) lies in \( N_{\Omega}(x,y) \). Similarly, from \( \{ z^{*} \mid -u - v - z^{*} \in \mathbb{R}_{+}\xi, \ (z^{*}, \xi) \geq 0 \} \) we derive that any \((u,v)\) such that \( u \in \mathbb{R}_{-}\xi \) and \( v \in \xi^{o} \) lies in \( N_{\Omega}(x,y) \) and from \( \{ z^{*} \mid z^{*} \in \mathbb{R}_{+}\xi, (-u - v - z^{*}, \xi) \geq 0 \} \) we derive that any \((u,v)\) such that \( v \in \mathbb{R}_{-}\xi \) and \( u \in \xi^{o} \) lies in \( N_{\Omega}(x,y) \). Combining all possibilities yields the formula of \( N_{\Omega}(x,y) \) at \((0,0)\) \( \blacksquare \).
References


