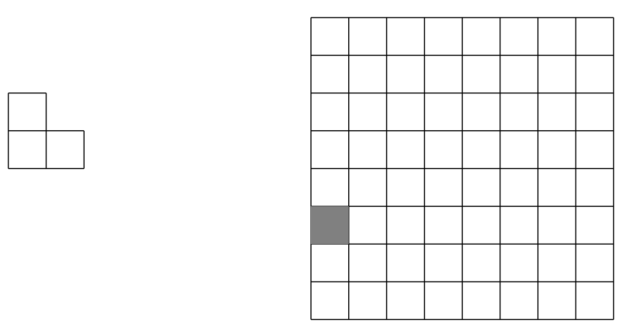
Chapter 3

Induction and Recursion

3.1 Induction: An informal introduction

This section is intended as a somewhat informal introduction to *The Principle of Mathematical Induction* (PMI): a theorem that establishes the validity of the proof method which goes by the same name. There is a particular format for writing the proofs which makes it clear that PMI is being used. We will not explicitly use this format when introducing the method, but will do so for the large number of different examples given later.

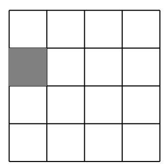
Suppose you are given a large supply of L-shaped tiles as shown on the left of the figure below. The question you are asked to answer is whether these tiles can be used to exactly cover the squares of an $2^n \times 2^n$ *punctured grid* – a $2^n \times 2^n$ grid that has had one square cut out – say the 8×8 example shown in the right of the figure.



In order for this to be possible at all, the number of squares in the punctured grid has to be a multiple of three. It is. The number of squares is

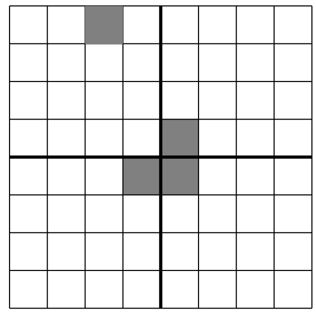
$$2^n 2^n - 1 = 2^{2n} - 1 = 4^n - 1 \equiv 1^n - 1 \equiv 0 \pmod 3.$$

But that does not mean we can tile the punctured grid. In order to get some traction on what to do, let's try some small examples. The tiling is easy to find if $n = 1$ because 2×2 punctured grid is exactly covered by one tile. Let's try $n = 2$, so that our punctured grid is 4×4. By rotating, we can assume the missing square is in the upper left quadrant, say as illustrated below.



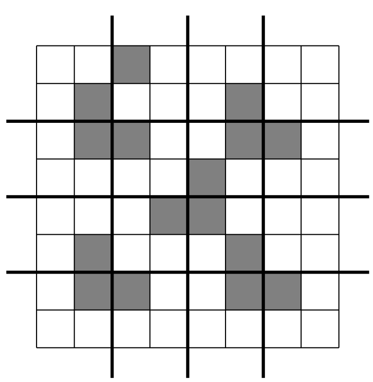
Imagine the punctured grid partitioned into four 2×2 grids, one of which has a square missing, as shown on the left of the figure below. As shown on the right of the figure, we can astutely place one tile to transform our problem into four 2×2 problems, each of which we know how to solve.

It is clear that this method works no matter which square in the upper left quadrant has been removed. Hence, if we can cover any 2×2 punctured grid, then we can cover and 4×4 punctured grid. Now we can see what to do to cover the 8×8 punctured grid: partition it into four 4×4 grids, one of which has a square removed, then astutely place a tile to transform the problem into four 4×4 problems we know how to solve because of our previous work.



There is nothing special about the numbers 4 and 8 in the previous examples. Once we know how to cover all possible punctured grids of size 2×2, 4×4, and 8×8, we can use the same method on any 16×16 punctured grid. And we can keep going. Once we know how to cover all punctured grids of size $2 \times 2,\ 4 \times 4,\ \ldots, 2^k \times 2^k$, we can use the same method to reduce the problem of covering a $2^{k+1} \times 2^{k+1}$ grid to four smaller problems we know how to solve because of previous work. Therefore, for any $n \geq 1$, the squares of a $2^n \times 2^n$ punctured grid can be exactly covered by L-shaped tiles.

The previous example illustrates the strong form of the *Principle of Mathematical Induction* (PMI). One meaning of the word *induction* is "the act of bringing forward". Above, we brought forward our knowledge of how to solve smaller instances of the problem to solve all instances of the next possible size. Notice also that the solution can be obtained recursively. For example, to cover an 8×8 punctured grid, we cover four 4×4 punctured grids, and each of these is covered via covering four 2×2 punctured grids. This is illustrated in the figure below. Completing the tiling of each 2×2 punctured grid gives the tiling of the 8×8 punctured grid.



3.2 PMI: The Principle of Mathematical Induction

The Principle of Mathematical Induction (PMI) is a theorem that gives a method for establishing the truth of statements quantified over all integers greater than or equal to some given integer. An example of such a statement is "*For any* $n \geq 1$*, a* $2^n \times 2^n$ *punctured grid can be exactly covered by L-shaped tiles*". Another is "*Every integer greater than or equal to six can be written as a sum of twos and threes*". In computer science, statements like these regularly arise in the analysis of algorithms. But not only that, proofs by induction also tend to imply recursive algorithms for solving the problem at hand. Further, PMI is a main tool in proving the correctness of recursive algorithms. Witness the L-shaped tiles example in the previous paragraph. *Whenever you need to prove a statement that is quantified over all integers greater than of equal to some given integer, then one tool you should consider trying to use is PMI.* (As usual, it may or not be successful to complete the task at hand.)

It turns out that there are two forms of PMI – a so-called strong form and a so-called weak form – but they are of identical expressive power. In other words, any statement that can be proved by one of them can be proved by the other. However, it is often true that a proof using one form (usually the strong form) involves a lot less writing than a proof using the other form. The choice of which to use is really a matter of mathematical aesthetics, and sheer laziness (wanting to write less, or wanting the writing to be easier). We will begin our discussion of PMI with the strong form of induction, and come to the weak form later.

Theorem 3.2.1 (Strong Form of PMI) *Let* $S(n)$ *be a statement whose truth depends on the integer* n*. If the following two conditions hold:*

1. *the statement* $S(n)$ *is true when* n *is any of the integers* $n_0, n_0+1, \ldots, t$*, for some* $t \geq n_0$*;*

2. *the truth of the statement* $S(n)$ *for all of the integers* $n_0, n_0+1, \ldots, k$*, where* $k \geq t$*, logically implies the truth of* $S(n)$ *when* $n = k+1$*;*

then, the statement $S(n)$ *is true for all integers* $n \geq n_0$*.*

We discuss three important matters before outlining the proof of this theorem. The first is the proof method implied by the theorem. Next, we discuss why it implies such a method and, finally, we discuss the Well-Ordering Principle, on which the proof of PMI depends.

The strong form of PMI is commonly referred to as *strong induction* or sometimes just *induction*. We will discuss the qualifier "strong" after introducing "weak induction" The theorem says we can prove that a $S(n)$ is true for all $n \geq n_0$ by doing two things:

1. Directly check that $S(n)$ is true for the first few possible values of n, say $n = n_0, n = n_0 + 1, \ldots, n = t$, where $t \geq n_0$. (It turns out that the size of t depends on what you're trying to prove.) This is called the *Basis* because it is the foundation that the rest of the argument rests on.

2. Prove that if $S(n)$ is true for all possible values of n from n_0 up to k, where $k \geq t$, then it is also true when $n = k + 1$. This is called the *Induction* because we use (bring forward) the truth of $S(n)$ for smaller values of n to prove that $S(n)$ is true for the next possible value of n. Usually the induction is separated into two parts. In the *Induction Hypothesis* one assumes the truth of $S(n)$ for all vales $n = n_0,\ n = n_0 + 1,\ \ldots,\ n = k$ for *some* k which is at least as large as the biggest value checked in the Basis. In the *Induction Step* one uses this information to show that $S(n)$ is also true when $n = k + 1$.

Now, why do these two points imply the conclusion we want? The first point says the statement $S(n)$ is true for all values of n from n_0 up to t. Using this, the second point says that the statement $S(n)$ is also true when $n = t+1$. So, now, we have that $S(n)$ is true or all values of n from n_0 up to $t + 1$. But using this and the second point (again), we get that $S(n)$ is true or all values of n from n_0 up to $t + 2$. This procedure can be repeated over and over. For any particular integer $x \geq n_0$, after enough applications of the second assertion we have that the statement $S(n)$ is true when $n = x$. But x is an arbitrary integer which is greater than or equal to n_0. Hence, after we know the first assertion is true, enough applications of the second assertion would allow us to conclude that $S(n)$ is true for any given integer $x \geq n_0$. Thus it follows that $S(n)$ is true for all integers $n \geq n_0$.

By the above discussion, a proof using PMI has four components:

1. A *Basis* in which it is checked that the statement $S(n)$ is true for all values of n from n_0 up to some $t \geq n_0$ (whose size depends on what is needed in the Induction Step, described below).
2. An *Induction Hypothesis*, in which it is assumed that the statement $S(n)$ is true for all values of n from n_0 up to k, where $k \geq t$.
3. An *Induction Step*, in which the Induction Hypothesis (an assumption) is used to argue that the statement $S(n)$ is true when $n = k + 1$.
4. A *Conclusion*, in which it is asserted that PMI implies that the statement $S(n)$ is true for all $n \geq n_0$.

It is customary to carry out these four steps in clearly labelled sections. Many examples follow in later subsections.

In what follows we give many examples of the use of PMI in different situations. We also discuss recursive definitions, and eventually the weak form of PMI. And then we conclude with more examples.

3.3 Non-algebraic Examples

In this section we use induction to prove results that involve neither formulas nor inequalities. Instead, we prove theorems about numbers, puzzles and games. Careful analysis of each argument gives a method for finding the object whose existence is implies by the proof (for example, a solution to a puzzle).

3.3.1 Postage stamp problems

Problems of this type involve showing that all integers greater than or equal to some given integer can be written as a sum of a's and b's, for some relatively prime integers a and b. In other words, any postage of at least some number of cents can be made using a-cent and b-cent stamps. The restriction that $gcd(a, b) = 1$ is essential: and sum of a's and b's is a multiple of $gcd(a, b)$. Hence all integers greater than a certain threshold can be made only when $gcd(a, b) = 1$.

It is a theorem of number theory that every integer greater than or equal to $(a-1)(b-1)$ can be so written. Instead of (being good mathematicians and) proving the general theorem, we will prove a particular instance as a way to practice our skills with PMI. Postage stamp arguments are always the same. First check that the given statement is true for the first few cases starting at the given integer n_0, and then use the assumption that it is true for a large enough collection of integers all greater than or equal to n_0 to show that it is true for the first integer not in the collection.

Example. *Prove that any positive integer greater than or equal to 8 can be written as a sum of 3's and 5's.*

The statement $S(n)$ we want to prove is "*any integer* $n \geq 8$ *can be written as a sum of 3's and 5's*".

Basis. Since 8 = 5+3, 9 = 3+3+3, and 10 = 5+5, each of 8, 9, and 10 can be written as a sum of 3s and 5s.

Induction Hypothesis. Assume each of $8, 9, 10, \ldots, k$ can be written as a sum of 3's and 5's. Because of the numer of cases checked in the Basis we can assume $k \geq 10$.

Induction Step We want to show that $k+1$ can be written as a sum of 3's and 5's. Since $k \geq 10$, $k+1-3 \geq 8$, so by the Induction Hypothesis, $k+1-3$ can be written as a sum of 3's and 5's But then adding 3 to this sum gives $k+1$ as a sum of 3's and 5's, which is what we wanted.

Conclusion. Therefore, by the strong form of PMI, any integer $n \geq 8$ can be written as a sum of 3's and 5's. $\square$

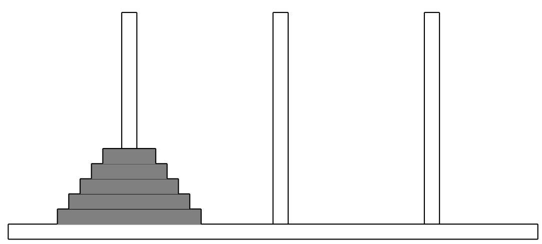
Here are some comments on the proof:

- To illustrate the Induction Step, suppose $k+1 = 11$. Then $k+1-3 = 11-3 = 8 = 5+3$, so $k+1 = 8+3 = 5+3+3$, a sum of 3's and 5's.
- Notice that 7 can not be written as a sum of 3's and 5's, so the integer 8 is the smallest possible value of n_0 such that our statement is true for all $n \geq n_0$.
- How did we know to use three cases in the Basis? It is because that's how many are needed to make the Induction Step work out. In the Induction Step, you first subtract 3, then you apply the Induction Hypothesis to the smaller number, and then you add 3 to get the con-

clusion you want. When you subtract 3, you need to make sure that you're reducing to a situation covered by the Induction Hypothesis. The very first time you use the Induction Hypothesis is to go from knowing the truth of the statement when $n = 8, 9, \ldots t$ to knowing it when $n = 8, 9, \ldots, t + 1$. The method is to subtract 3, to $(t + 1) - 3$ must be greater than or equal to 8. So the truth is that *you may not know how many cases to cover in the Basis step until after completing the Induction Hypothesis.*

3.3.2 The Towers of Hanoi

The *Towers of Hanoi* is a puzzle that begins with $n \geq 1$ rings, each with a different diameter, stacked in decreasing order of size on one of three towers. An example with five rings is shown below. The objective is to move the rings one at a time so that they are eventually stacked in the same order on one of the other towers. At no point in time may a larger ring rest on top of a smaller one.



The Principle of Mathematical Induction can be used to prove that a solution exists. In fact, following the proof leads to a recursive method of solving the puzzle.

Example. *Prove that, for any integer $n \geq 1$, it is possible to solve the Towers of Hanoi problem with n rings.*

Basis. It is clear how to solve the puzzle if $n = 1$. There is only one ring – just move it!

Induction Hypothesis. Suppose we know how to solve the puzzle when there is 1 ring, when there are 2 rings, and so on up to when there are k rings, for some $k \geq 1$.

Induction Step. Consider the puzzle with $k+1$ rings. It can be solved in three phases. First, legally move the k smallest rings to one of the other towers. (That is, move them according to the rules of the puzzle.) We know how to do this by the Induction Hypothesis. Leaving the large ring in place will not cause the constraint that a larger ring may not rest atop a smaller one to be violated. Second, move the largest ring to the empty tower. Finally, legally move the k smallest rings so that they are on top of the largest one. Again, this is possible because we know how to solve the puzzle with k rings.

Conclusion. Therefore, by PMI, for any $n \geq 1$ there is a solution to the Towers of Hanoi puzzle when there n rings. □

It is possible go a bit farther and use the Principle of Mathematical Induction to find the number of moves needed to (most efficiently) solve the puzzle with n rings. (It is always possible to use more moves by making stupid moves.) Some experimenting with 1, 2, and 3 rings leads to the conjecture that if there are n rings, then the number of moves needed is $2^n - 1$. Let's outline the argument that this statement is true. When there is one ring, exactly one move is needed, so that statement is true when $n = 1$. Assume that solving the puzzle with 1 ring takes $2^1 - 1$ moves, that solving it with 2 rings takes $2^2 - 1$ moves, and so on, until solving it with k rings takes $2^k - 1$ moves, for some $k \geq 1$. Consider an instance of the puzzle with $k + 1$ rings. Before the largest ring can be moved, the k other rings must be stacked in decreasing order of size on one of the other towers. By the Induction Hypothesis, this takes $2^k - 1$ moves. Moving the largest ring takes one move, and then properly stacking the remaining k rings on top of it takes $2^k - 1$ more moves (by the Induction Hypothesis). Thus the number of moves needed is $2(2^k - 1) + 1 = 2^{k+1} - 1$, as desired. Therefore, by PMI, for any $n \geq 1$, solving the puzzle with n rings takes $2^n - 1$ moves.

A good exercise is to take the argument above and write it out so the four components are explicitly listed.

Legend has it that the end of the world would come before a person could complete the solution to the puzzle with 64 rings. By the above, it would take $2^{64} - 1$ moves. There are $60 \times 60 \times 24 \times 365 = 31536000 \approx 2^{24.9}$ seconds in a year, ignoring leap years. Hence, if a person could move one ring per second, then solving the puzzle would take about 2^{39} years.

3.3.3 Analysis of 2-pile Nim

The game 2-pile of Nim starts with two piles of objects, say coins. There are two players who alternate turns. On her turn, each player removes a positive number of coins from one of the two piles, and leaves the other pile alone. The first player unable to take a coin loses. (Equivalently, the player who takes the last coin wins.)

For example, suppose that when the game starts, one pile has 3 coins and the other pile has 2 coins. The first player removes a coin from the 3-coin pile, so that both piles now have 2 coins. The second player then removes a coin from a pile. Now the first player removes a coin from the opposite pile so that both piles now have one coin each. At this point it is clear that the second player has lost. His only move is to take the single coin from one of the two piles, whereupon the first player takes the other coin and wins.

The question to be answered is: if the piles have size r and s, where $r \geq s \geq 0$ when the game starts, which player wins and what is the winning strategy?

Example. *Prove that, for each $n \geq 0$, if the two piles together contain a total of n coins, then the first player to play has a winning strategy if and only if the two piles are not of equal size.*

Basis. When $n = 0$ there is only one possible configuration: both piles are empty. In this case the first player loses because she is the first player unable to make a legal move (there are no coins to take). Hence the statement to be proved is true when $n = 0$

Induction Hypothesis. Suppose that, for some $k \geq 0$, whenever total number of coins in the two piles is one of $0, 1, \ldots, k$, then the first player has a winning strategy if and only if the two piles are not of equal size.

Induction Step. Suppose the two piles together contain a total of $k+1$ coins. There are two cases to consider.

First, if the two piles are not the same size, then the first player removes some coins from the larger pile so that the two piles become size. Since a positive number of coins have been removed, the two piles together contain k or fewer coins. In this new, reduced game, the second player (in the original game) is first to play (that is, the players have swapped roles). By the Induction Hypothesis, he does not have a winning strategy. Hence his opponent does. That is, the first player to play the original game has a winning strategy.

To prove the opposite implication, we argue that its contrapositive is true. Suppose the two piles are the same size. Any move that the first player makes results in a situation where the two piles together contain k or fewer coins, and are not of equal size. By the Induction Hypothesis, the first player to play this reduced game has a winning strategy. That is, the second player to play in the original game has a winning strategy.

Combining the results from both cases, the first player (in the original game) has a winning strategy if and only if the two piles are not of equal size, which is what we wanted to show).

Conclusion. Therefore, by PMI, for each $n \geq 0$, if the two piles together contain a total of n coins, then the first player to play has a winning strategy if and only if the two piles are not of equal size. $\square$

The winning strategy can be determined by following the argument in the proof. When the two piles are not of equal size, remove some coins from the larger pile so that they become the same size. Keep doing it, eventually both piles have size zero. When the two piles are of equal size, such a move is possible only for the opponent, who then has a winning strategy.

3.4 Silly analogies for PMI

In carrying out a proof by PMI, it is important to carry out all four of the steps. The only two that require any real work are checking that the Basis holds, and arguing that the logical implication needed for the Induction Step holds. The other two steps are important, however, especially for communication; *it is definitely worth making an effort to clearly state the Induction Hypothesis.*

Some people draw an analogy between PMI and climbing as high as you want on a really tall ladder, starting from rung n_0. Assertion (2) is saying that if you have climbed up the steps $n_0, n_0+1, \ldots, k$, where $k \geq t$, then you can climb up to step $k+1$. By itself, this does not matter much. You have to be able to get on the ladder and complete the steps $n+0, n_0+1, \ldots, t$, otherwise you can't use assertion (2) repeatedly to conclude that you can climb as high as you want on the ladder. Thus assertion (1) is of crucial importance in the argument. It might be that, depending on the situation, the rung t to which you have to climb before being assured of the ability to continue differs.

Other people draw an analogy between PMI and toppling dominoes. Suppose you have an infinite row of dominoes that are arranged close together, but that dominoes $n_0, n_0 + 1, \ldots, t$ are exceptionally heavy. Suppose you happen to be able to prove that if you can make dominoes $n_0, n_0 + 1, \ldots, k$ fall over, where $k \geq t$, then the next domino in the row is guaranteed to fall over. Pushing over domino n_0 alone won't help if domino $n_0 + 1$ is so heavy that it won't fall over when struck by domino n_0. And pushing over the first few of $n_0, n_0 + 1, \ldots, t$ won't help if the the next domino is also very heavy. The only thing to do is make sure you individually push over each of the dominoes $n_0, n_0 + 1, \ldots, t$. After you do that, you can conclude from your argument that all of the dominos will fall over.

A classical example of needing both of the assertions the fallacious argument that *in any group of* $n \geq 1$ *people, all people in the group have the same hair colour*. Certainly it is true that in any group of 1 people, all people in the group have the same hair colour. Suppose that it is true that in any group of 1 up to t people, all people in the group have the same hair colour, for some $t \geq 1$. Now consider a group of $t + 1$ people. We want to use the Induction Hypothesis to argue that all people in this group have the same hair colour. Consider any member of the group. Call her Anna. By the Induction Hypothesis, all t members of the group who are not Anna have the same hair colour. Now consider any other member of the group. Call him Bill. By the Induction Hypothesis, all t members of the group who are not Bill have the same hair colour. But now Anna and Bill each have the same hair colour as all the remaining members of the group, and so all $t + 1$ members of the group have the same hair colour. Therefore, by PMI, in any group of $n \geq 1$ people, all people in the group have the same hair colour.

Now, the statement "proved" in the previous paragraph is certainly not true, and so there must something wrong with the argument. In the Basis we checked only up to $t = 1$. (Had we checked up to $t = 2$, and been the least bit alert, there would have been trouble.) So this means the first application of the Induction Step is supposed to take us from the truth of the statement for all values of n from 1 up to 1, to the truth of the statement for all values of n from 1 up to 2. But the argument does not work as there are no group members besides Anna and Bill. In saying that they each have the same hair colour as all members the rest of the group we are assuming that there is at least one more person in the group. There isn't. Thus the argument given to establish the Induction Step is wrong, as it does not work when $k = 1$. A different way to view the problem is that we have an Induction Step that is valid so long as $k \geq 2$, but no Basis that supports the truth of the Induction Hypothesis in that case.

3.5 The weak form of induction

Look back at the induction examples we have done so far. In some of them, for example the Towers of Hanoi, completing the Induction Step required only that we assume the result to be true for k (and not all values between n_0 and k). In others, for example the statements about factorization of integers, completing the Induction Step required only that we assume the result to be true for several values between n_0 and k.

Mathematicians care about aesthetics, and so we do not like to assume more than we need. If completing the Induction Step requires only that the result be true for k, we don't want to assume any more than that. It is also true that some proofs become much easier to write using the weak form of induction because it is much easier to state the Induction Hypothesis.

Theorem 3.5.1 (Weak Form of PMI) *Let $S(n)$ be a statement whose truth depends on the integer n. If the following two conditions hold:*

1. *the statement $S(n)$ is true when $n = n_0$;*
2. *the truth of the statement $S(n)$ when $n = k$, where $k \geq n_0$, logically implies the truth of $S(n)$ when $n = k + 1$;*

then, the statement $S(n)$ is true for all integers $n \geq n_0$.

As before, a proof using the result given by this theorem has four parts: a Basis, an Induction Hypothesis, and Induction Step, and a Conclusion. There are two differences between the strong form of PMI and the weak form of PMI. One of them is that the weak form has only one case in the Basis, whereas the strong form may involve many cases in the Basis. The main difference, however, is that in the weak form the Induction Hypothesis only that the statement to be proved holds when $n = k$ (and not all values from n_0 up to k). The terms "strong form" and "weak form" arise from this comparison between the Induction Hypotheses. The strong form has a stronger Induction Hypothesis in the sense that more seems to be being assumed.

The reason the conclusion holds is the same as before. We know that the statement is true for n_0. The induction (assertion (2)) then allows us to conclude that the statement is true for $n_0 + 1$. Using this, the induction (assertion (2)) then allows us to conclude that the statement is true for n_0+2. And so on, until finally we can reach any integer $x \geq n_0$. Thus, as before, the only reasonable conclusion is that the statement is true for all integers $n \geq n_0$. Note, also, that by the time we have applies assertion (2) enough times to know the statement is true when $n = k$, we have actually proved that it is true for all integers between n_0 and k (identical to the assumption in the strong form of induction).

The proof of the weak form of PMI is virtually identical to the proof given for the strong form. It is a good exercise to write it out and see the underlying logic for yourself.

In what follows, we will give some more example of using PMI, and will freely use the weak form when it is possible to do so. How do you know which form to use? Sometimes you don't until after completing the Induction Step and seeing the smaller values for which you need the truth of the statement. *It is always safe to use the strong form of PMI.* but your proofs might look a lot prettier (and you might look more aware of what's being assumed) with the weak form.

3.6 Examples involving sums and inequalities

3.6.1 Summations

The key point in using PMI to prove summation identities occurs in the Induction Step: *remember the meaning of the ellipsis "...", and substitute* *the assumed value from the Induction Hypothesis for the first k terms in the sum (and don't forget to keep the $(k+1)$-st term, then do algebra to get what you want.*

Example. *Prove that, for any natural number $n \geq 1$,*

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Basis. When $n = 1$, we have LHS$= 1$ and RHS$= 1(1+1)/2 = 1$. Thus the statement is true when $n = 1$.

Induction Hypothesis. Assume that $1+2+3+\cdots+k = k(k+1)/2$ for some $k \geq 1$.

Induction Step. We want to prove that

$$1 + 2 + 3 + \cdots + (k+1) = (k+1)((k+1)+1)/2 = (k+1)(k+2)/2.$$

Consider the LHS:

$1 + 2 + \cdots + (k+1)$	
$= 1 + 2 + \cdots + k + (k+1)$	(meaning of the elipsis)
$= k(k+1)/2 + 2(k+1)/2$	(by IH, and getting a common denominator)
$= (k+1)(k+2)/2$	as desired.

Conclusion. Therefore, by induction, $1+2+3+\cdots+n = n(n+1)/2$ for all $n \geq 1$. $\square$

3.6.2 Summation identities to memorize

There are a number of summations that arise frequently. You should both memorize them, and know how to prove each one. Induction always works, though there can be other proofs as well.

- For any natural number $n \geq 1$,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

- For any natural number $n \geq 1$,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

- For any natural number $n \geq 1$,

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

  It is a fluke that the RHS is the square of the first identity above. The pattern does not continue.

- (Sum of a geometric series.) For any natural number $n \geq 1$ and any real number r,

$$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}.$$

3.6.3 Inequalities

The following sort of argument arises all the time in proving inequalities where one "side" is a polynomial. Suppose $n \geq 6$, and consider $n^3 + 4n^2 + 5n + 3$. Since $n \geq 6$, we can change the right hand 3 to n, that is $n^3 + 4n^2 + 5n + 3 \leq n^3 + 4n^2 + 5n + n = n^3 + 4n^2 + 6n$ In the same way, replacing $6n$ by n^2 only makes the expression larger (because $n \geq 6$). Thus, $n^3 + 4n^2 + 6n \leq n^3 + 4n^2 + n^2 = n^3 + 5n^2$. And, doing the same again to replace $5n^2$ by n^3 (because $n \geq 6$) gives $n^3 + 5n^2 \leq n^3 + n^3$. Putting this all together, we have just shown that if $n \geq 6$, then $n^3 + 4n^2 + 5n + 3 \leq 2n^3$. We use this method in the example below.

Example. *Prove that for all $n \geq 5$, $2^n > n^2$.*

Basis. When $n = 5$, $2^n = 2^5 = 32$ and $n^2 = 5^2 = 25$. As $32 > 25$ the statement is true for $n = 5$.

Induction Hypothesis. Suppose $2^k > k^2$, for some $k \geq 5$.

Induction Step. We want to show $2^{k+1} > (k+1)^2$. Consider $(k+1)^2 = k^2 + 2k + 1 < k^2 + 2k + k$ (as $k \geq 5 > 1$) $= k^2 + 3k < k^2 + k(k)$ (as $k \geq 5 > 3$) $= 2k^2 < 2(2^k)$ (by the induction hypothesis) $= 2^{k+1}$, which is what we wanted.

Conclusion. Therefore, by the Principle of Mathematical Induction, for all $n \geq 5$, $2^n > n^2$. □

There is a fairly established hierarchy of the growth rates of functions, and it is used all the time when comparing the performance of algorithms on inputs of given size (for example, algorithms that operate on n items usually use a number of steps proportional to n^2, or to $n \log_2(n)$; when n is large, this difference matters in terms of how long it takes for the task to be completed.). What that means is that for all large enough values (and maybe not for small ones), functions at a higher level in the hierarchy are greater that those at a lower level. Constants are at the bottom of the hierarchy, then logs. And then polynomials. The higher the degree, the faster the growth. Exponential functions always eventually become greater than any polynomial, and factorials always eventually become larger that exponentials. Finally, functions like n^n eventually become larger than factorials. Inequalities between functions at the various levels of this hierarchy can be proved with induction.

Example. *Prove that $n! > 3^n$ for all $n \geq 7$.*

Basis. When $n = 7$ we have $n! = 7! = 5040$ and $3^n = 3^7 = 2187$. Hence the statement to be proved is true when $n = 7$.

Induction Hypothesis. Assume $k! > 3^k$ for some $k \geq 7$.

Induction Step. We want to show that $(k+1)! > 3^{k+1}$. Consider the RHS. We have $3^{k+1} = 3 \cdot 3^k < 3 \cdot k!$ by the Induction Hypothesis. Now, since $k + 1 \geq 8 > 3$, we have $3 \cdot k! < (k+1)k! = (k+1)!$, as wanted.

Conclusion. Therefore, by PMI, $n! > 3^n$ for all $n \geq 7$. □

3.7 Recursive definitions

The word “recursive” originates from the Latin word *recurs*, which means “returned”, and which arises from a verb that means “go back”. Informally, we will call a process “recursive” if it refers back to itself. In mathematics, a process is recursive if successive results depend on previous ones; a function is recursive if the value of the function at some elements of the domain depends on its value at other elements of the domain. In order to avoid an infinite regression of self-references, some basic outcomes (results, values) must be explicitly known without any self-reference.

A *recursive definition of a sequence* consists of two parts:

1. one or more *base cases* that explicitly state one or more terms of the sequence, and
2. a *recursion* (that is, a function) that gives other terms of the sequence in terms of those already known.

Some examples follow.

- The sequence $1, 2, 4, 8, \ldots, 2^n, \ldots$ is recursively defined by $a_0 = 1$, and $a_{n+1} = 2a_n$, for all $n \geq 0$.
- The sequence $-5, -2, 1, 4, \ldots, 3n - 5, \ldots$ is recursively defined by $a_0 = -5$ and $a_{n+1} = a_n + 3$ for all $n \geq 0$.
- The sequence $a_1, a_2, \ldots$ where $a_n = 1 + 2 + \cdots + n$ is recursively defined by $a_1 = 1$, and $a_{n+1} = a_n + (n + 1)$.
- The *Fibonacci Sequence* $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$ is recursively defined by $f_1 = 1, f_2 = 1$, and $f_{n+1} = f_n + f_{n-1}$.

After describing the first few terms explicitly, the key to writing recursive definitions is to imagine that all terms up to the n-th are of the correct form, and then to describe how to get the $(n+1)$-st term from those already defined. Go back over the examples above with this in mind.

We now generalize the example in the first bullet point. A *geometric progression* (or geometric sequence) is a sequence $a, ar, ar^2, ar^3, \ldots$, where $a, r \in \mathbb{R}$. (Remember that $a = ar^0$, so the sequence can also be written as $ar^0, ar, ar^2, ar^3, \ldots$.) Geometric progressions (with common ratio r) have the property that the ratio of each term to the one immediately before it is (the same number) r. These sequences can be recursively defined by $g_0 = a$, and $g_{n+1} = rg_n$ for all $n \geq 0$.

We now generalize the example in the second bullet point. An *arithmetic progression* (or arithmetic sequence) is a sequence $a, a + d, a + 2d, a + 3d, \ldots$, where $a, d \in \mathbb{R}$. Arithmetic progressions are sequences such that the difference between any term and the one after it is (*the common difference*) d.

These sequences can be recursively defined by $b_0 = a$, and $b_{n+1} = b_n + d$ for all $n \geq 0$.

The Fibonacci sequence has many wild and wonderful properties. Every third Fibonacci number is even, every fourth is a multiple of three, every fifth is a multiple of 5, every sixth is a multiple of 8. In general, every n-th Fibonacci number is a multiple of f_n. All of these facts can be proved using PMI. Another remarkable fact which can also be proved by these methods is that

$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1-\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

This is even more stunning when you stop to think that f_n is an integer! Just for the sake of interest, let's look at the right hand side a bit more closely. The quantity $\frac{1-\sqrt{5}}{2}$ is less than one, so $\left(\frac{1-\sqrt{5}}{2}\right)^n$ converges to zero (quickly) as n grows. Because of this, it turns out that f_n is the nearest integer to $\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$, i.e., the integer that arises from rounding.

Other things can also be defined recursively in the same way. As with the example of sequences, a *recursive definition* consists of two parts:

1. one or more *bases cases* that explicitly describe some of the basic items, and

2. a *recursion* that gives other items in terms of those already known.

One example is the recursive definition of n-*factorial*, that is, the quantity $n! = 1 \times 2 \times \cdots \times n$, where n is a non-negative integer (remember that an empty product equals zero): $0! = 1$ *and* $n! = n \times (n-1)!,\ n \geq 1.$

Suppose you are given a machine that will add two numbers like, say, your calculator. How can you use it to compute the sum of n numbers? Surely you add the first two numbers, then add the third to the total, and then the fourth, and so on. It may not be apparent that doing so implicitly uses a recursive definition of summation. Given numbers $x_1, x_2, \ldots, x_n$, let $S_i = x_1 + x_2 + \cdots + x_i$. Then $S_2 = x_1 + x_2$, and $S_k = S_{k-1} + x_k$ for $k \geq 3$.

Other (associative) operations like multiplication, set union, set intersection, conjunction of logical propositions, and disjunction of logical propositions can be recursively defined in a similar way.

3.8 Examples involving recursively defined sequences

In this section we present a few examples of induction proofs involving recursively defined sequences. Typically these involve showing that some sort of formula holds. An important point to remember is that *the Basis of the induction often has the same number of cases as the Basis of the recursive definition*.

Proofs involving recursively defined sequences in which the recursion has more than one term are almost always by strong induction. The reason is that, in the Induction Step, you will want to apply the recursion and then make a substitution for each of the terms that arise from doing so. Typically this involves substituting for more than just the k-th term of the sequence.

3.8.1 Formulas for the terms of recursively defined sequences

This subsection illustrates a method to use when proving the correctness of a formula for the terms of a recursively defined sequence.

Example. *Let a_n be the sequence recursively defined by $a_0 = 1$, $a_1 = 2$, and for $n \geq 2$, $a_n = 3a_{n-1} - 2a_{n-2}$. Show that $a_n = 2^n$ for all $n \geq 0$.*

Basis. When $n = 0$ we have $a_0 = 1 = 2^0$ and when $n = 1$ we have $a_1 = 2 = 2^1$. Hence the statement is true when $n = 0$ and $n = 1$.

Induction Hypothesis. Assume $a_0 = 2^0, a_1 = 2^1, \ldots, a_k = 2^k$, for some $k \geq 1$.

Induction Step. We went to show that $a_{k+1} = 2^{k+1}$. Consider a_{k+1}. Since $k + 1 \geq 2$ we have

$$a_{k+1} = 3a_k - 2a_{k-1} = 3 \times 2^k - 2 \times 2^{k-1}$$

by the Induction Hypothesis. The RHS of this expression equals $3 \times 2^k - 2^k = 2^k(3 - 1) = 2^{k+1}$, as needed.

Conclusion. Therefore, by PMI, $a_n = 2^n$ for all $n \geq 0$. $\square$

3.8.2 Bounds for the n-th Fibonacci number

Recall the Fibonacci numbers $f_1, f_2, \ldots$ are defined by $f_1 = f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 3$.

Example. *Use the strong form of mathematical induction to prove that $f_n \leq 2^{n-1}$ for any natural number $n \geq 1$.*

Basis. We have $f_1 = 1 \leq 2^0$ and $f_2 = 1 \leq 2^1$. Thus the statement is true when $n = 1$ and when $n = 2$.

Induction Hypothesis. Assume that $f_1 \leq 2^0,\ f_2 \leq 2^1, \ldots, f_k \leq 2^{k-1}$ for some $k \geq 2$.

Induction Step. We want to prove that $f_{k+1} \leq 2^{(k+1)-1} = 2^k$.
Consider f_{k+1}. Since $k+1 \geq 3$ we have

$$\begin{array}{rcll} f_{k+1} & = & f_k + f_{k-1} & \text{(by definition of } f_{k+1}\text{)} \\ & \leq & 2^{k-1} + 2^{(k-1)-1} & \text{(by IH)} \\ & = & 2^{k-2}(2+1) & \text{(algebra)} \\ & \leq & 2^{k-2}2^2 & \text{(because } 3 \leq 4 = 2^2\text{)} \\ & = & 2^k & \text{as wanted.} \end{array}$$

Conclusion. Therefore, by PMI, $f_n \leq 2^{n-1}$ for all natural numbers $n \geq 1$. □

It is worth emphasizing the importance of having two cases in the Basis. In the Induction Step we want to take f_{k+1} and replace it by $f_k + f_{k-1}$. The recursive part of the definition can only be applied when $k+1$ is at least 3.

By using a bit more algebra, a better upper bound is possible.

Example. *Prove that for all integers $n \geq 1$, $f_n \leq (7/4)^{n-1}$.*

Basis When $n = 1$ we have $f_n = f_1 = 1$ and $(7/4)^{n-1} = (7/4)^9 = 1$, so the statement is true when $n = 1$. When $n = 2$ we have $f_n = f_2 = 1$ and $(7/4)^{n-1} = (7/4)^1 = (7/4)$, so the statement is true when $n = 2$.

Induction Hypothesis Assume that $f_i \leq (7/4)^{i-1}$ for $i = 1, 2, \ldots, k$, for some $k \geq 2$.

Induction Step We want to show that $f_{k+1} \leq (7/4)^k$. Consider f_{k+1} Since $k+1 \geq 3$ we can use the recursive definition to write $f_{k+1} = f_k + f_{k-1}$. By the Induction Hypothesis, $f_k + f_{k-1} < (7/4)^{k-1} + (7/4)^{k-2} = (7/4)^{k-2}(7/4+1) =$

$(7/4)^{k-2}(11/4) = (7/4)^{k-2}(44/16) \leq (7/4)^{k-2}(49/16) = (7/4)^{k-2}(7/4)^2 = (7/4)^k$, which is what we wanted.

Conclusion. Therefore, by the strong form of PMI, for all $n \geq 0$, $f_n \leq (7/4)^{n-1}$. $\square$

3.8.3 Iterating one term recurrences and proving the formula obtained to be correct

For our work in this subsection it is important to know the value of the sum of a geometric progression of finite length, that is, of $a + ar + ar^2 + \cdots + ar^n$, for any integer n. Since $a + ar + ar^2 + \cdots + ar^n = a(1 + r + \cdots + r^n)$, it is enough to know the value of the bracketed sum. Suppose $S = 1 + r + \cdots + r^n$. Then $rS = r + r^2 + \cdots + r^{n+1}$, so that $rS - S = r^{n+1} - 1$. All other terms cancel. Therefore, factoring the left hand side, $S(r-1) = r^{n+1} - 1$ so that $S = \frac{r^{n+1}-1}{r-1}$.

Let $a_1, a_2, \ldots$ be the sequence recursively defined by $a_1 = 2$ and $a_n = 7a_{n-1} + 2$ for $n \geq 2$.

By direct computation,

$$\begin{aligned}
a_1 &= 2 \\
a_2 &= 7a_1 + 2 = 16 \\
a_3 &= 7a_2 + 2 = 114 \\
a_4 &= 7a_3 + 2 = 800
\end{aligned}$$

Computing the exact values in this way does no help find a formula for the n-therm of the sequence unless you happen to have amazing powers of observation. The best way to obtain formula is to write out the derivation for the first few cases, but don't perform any additions or multiplications (except for collecting exponents with the same base), and then try to recognize what you have as something you know. *If there is a pattern, it is typically fairly apparent after working out enough cases that the calculation is routine and* *boring – typically that means working out about 4 cases.*

$$\begin{aligned}
a_1 &= 2\\
a_2 &= 7a_1+2 = 7\cdot 2+2\\
a_3 &= 7a_2+2 = 7(7\cdot 2+2)+2 = 7^2\cdot 2+7\cdot 2+2\\
a_4 &= 7a_3+2 = 7(7^2\cdot 2+7\cdot 2+2)+2 = 7^3\cdot 2+7^2\cdot 2+7\cdot 2+2\\
a_5 &= 7a_4+2 = 7(7^3\cdot 2+7^2\cdot 2+7\cdot 2+2)+2\\
&= 7^4\cdot 2+7^3\cdot 2+7^2\cdot 2+7\cdot 2+2
\end{aligned}$$

At this point it seems reasonable to conjecture that

$$a_n = 2(7^{n-1}+7^{n-2}+\cdots+1) = 2\frac{7^n-1}{7-1} = \frac{7^n-1}{3}$$

for all $n \geq 1$.

We can prove the conjectured formula is correct using PMI.

Basis. When $n=1$ we have $a_1 = 2 = \frac{7^1-1}{3}$, as desired. Thus the statement is true when $n=0$.

Induction Hypothesis Assume that $a_i = \frac{7^i-1}{3}$ for $i = 0, 1, \ldots, k$, for some $k \geq 0$.

Induction Step We want to show that $a_{k+1} = \frac{7^{(k+1)}-1}{3}$. Since $k+1 \geq 2$ we can use the recursion to write $a_{k+1} = 7a_k + 2 = 7(\frac{7^k-1}{3}+2$, by the Induction Hypothesis. Hence $a_{k+1} = \frac{7^{k+1}-7}{3} + \frac{6}{3} = \frac{7^{k+1}-1}{3}$, as desired.

Conclusion. Therefore, by PMI, $a_n = \frac{7^n-1}{3}$ for all $n \geq 1$. $\square$

Example. *Let $a_0, a_1, \ldots$ be the sequence recursively defined by $a_0 = 0$ and $a_n = a_{n-1} + 3n^2$ for $n \geq 1$. Find, with proof, a formula for a_n for all $n \geq 0$.*

Finding the formula. Write out the first few values, but be very judicious about doing multiplications or collecting terms. Keep going until the calculation becomes boring.

$$\begin{aligned}
a_0 &= 0 \\
a_1 &= a_0 + 3 \cdot 1^2 = 0 + 3 \cdot 1^2 \\
a_2 &= a_1 + 3 \cdot 2^2 = 0 + 3 \cdot 1^2 + 3 \cdot 2^2 \\
a_3 &= a_2 + 3 \cdot 3^2 = 0 + 3 \cdot 1^2 + 3 \cdot 2^2 + 3 \cdot 3^2 \\
a_4 &= a_3 + 3 \cdot 4^2 = 0 + 3 \cdot 1^2 + 3 \cdot 2^2 + 3 \cdot 3^2 + 3 \cdot 4^2
\end{aligned}$$

At this point is seems reasonable to conjecture that $a_n = 3(1^2+2^2+\cdots+n^2)$ for all $n \geq 0$. The bracketed expression is a known sum, so our conjecture really is that $a_n = 3n(n+1)(2n+1)/6 = n(n+1)(2n+1)/2$ for all $n \geq 0$. We now prove the conjecture by induction.

Basis When $n = 0$ we have $a_n = a_0 = 0$ and $n(n+1)(2n+1)/2 = 0(1)(1)/2 = 0$. Thus the statement is true when $n = 0$.

Induction Hypothesis. Assume that $a_k = k(k+1)(2k+1)/2$ for some $k \geq 0$.

Induction Step. We want to show that $a_{k+1} = (k+1)((k+1)+1)(2(k+1)+1)/2 = (k+1)(k+2)(2k+3)/2$. Look at a_{k+1}. Since $k+1 \geq 1$, we can use the recursion to write

$$\begin{aligned}
a_{k+1} &= a_k + 3(k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{2} + 3(k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{2} + \frac{6(k+1)^2}{2}
\end{aligned}$$

where, in the last two steps, we used the Induction Hypothesis, then got a common denominator. Now,

$$\begin{aligned}
\frac{k(k+1)(2k+1)}{2} + \frac{6(k+1)^2}{2} &= \frac{(k+1)[k(2k+1)+6(k+1)]}{2} \\
&= \frac{(k+1)[2k^2+7k+6)]}{2} \\
&= \frac{(k+1)(k+2)(2k+3)}{2}
\end{aligned}$$

which is what we wanted.

Conclusion. Therefore, by PMI, $a_n = n(n+1)(2n+1)/2$ for all $n \geq 0$. $\square$

3.9 Examples involving divisibility

Example. *Use induction to show that* $12 \mid 25^n - 13^n$ *for all* $n \geq 0$.

Basis. When $n = 0$ we have $25^n - 13^n = 25^0 - 13^0 = 1 - 1 = 0$. Since $12|0$, the statement holds when $n = 0$.

Induction Hypothesis. Assume $12 \mid 25^k - 13^k$ for some $k \geq 0$.

Induction Step. We need to show that $12 \mid 25^{k+1} - 13^{k+1}$. Consider the RHS: $25^{k+1} - 13^{k+1} = 12 \cdot 25^k + 13 \cdot 25^k - 13 \cdot 13^k = 12 \cdot 25^k + 13(25^k - 13^k)$. Now, the first term on the RHS is clearly divisible by 12, and the second term on the RHS is divisible by 12 by the Induction Hypothesis. Therefore, $12 \mid 12 \cdot 25^k + 13(25^k - 13^k) = 25^{k+1} - 13^{k+1}$, as desired.

Conclusion. Therefore, by PMI, $12 \mid 25^n - 13^n$ for all $n \geq 0$. $\square$

There is an easier proof of results like this using congruences, which we wil study later.

Example. *Show that every third Fibonacci number is even.*

Let's first translate the problem. We want to show that $2 \mid f_{3n}$ for all $n \geq 1$.

Basis. We have $f_{3 \cdot 1} = f_3 = 2$. Since $2|2$, the statement is true when $n = 1$.

Induction Hypothesis. Suppose $2 \mid f_{3k}$ for some $k \geq 1$.

Induction Step. We want to show that $2 \mid f3(k+1) = f_{3k+3}$. Consider f_{3k+3}. Since $k \geq 3$, $3k + 3 \geq 6$ so we can use the recursion to write

$$f_{3k+3} = f_{3k+2} + f_{3k+1} = (f_{3k+1} + f_{3k}) + f_{3k+1} = f_{3k} + 2f_{3k+1}.$$

Now, the last term on the RHS is even because it is a multiple of 2, and the first term on the RHS is even by the Induction Hypothesis. Therefore, $2 \mid f_{3k} + 2f_{3k+1} = f_{3k+3}$, as desired.

Conclusion. Therefore, by PMI, $2 \mid f_{3n}$ for all $n \geq 1$. $\square$

3.10 Induction Problems

1. Prove that any integer greater than or equal to 35 can be written as a sum of 5's and 6's.

2. Prove by induction that the number of binary sequences of length n is 2^n, for any $n \geq 1$.

3. Prove by induction that, for any $n \geq 1$, the number of binary sequences of length n with an even number of ones equals the number of binary sequences of length n with an odd number of ones.

4. The binary sequences of length 1 can be listed so that consecutive sequences in the list, including the first and last, differ in exactly one place. One such list is $L_1 = 0, 1$. The binary sequences of length 2 can also be listed so that consecutive sequences in the list, including the first and last, differ in exactly one place. One such list is $L_2 = 00, 01, 11, 10$. The list L_2 is constructed from L_1 in several steps. First, let $0 \cdot L_1$ be the list constructed from L_1 by adding a 0 to the left end of every sequence in L_1, so that $0 \cdot L_1 = 00, 01$. The list $1 \cdot L_1$ is constructed similarly. Then L_2 consists of the sequence $0 \cdot L_1$ followed by the sequence $1 \cdot L_1$ in reverse order (say $L_2 = 0 \cdot L_1, \mathit{reverse}(1 \cdot L_1)$).

   (a) Show, by producing the list, that the binary sequences of length 3 can be listed so that consecutive sequences in the list, including the first and last, differ in exactly one place.

   (b) Prove that, for any $n \geq 1$, the binary sequences of length n can be listed so that consecutive sequences in the list, including the first and last, differ in exactly one place.

5. Prove by induction that if $n \geq 1$ distinct dice are rolled, then the number of outcomes where the sum of the faces is an even integer equals the number of outcomes where the sum of the faces is an odd integer.

6. Consider the following two player game. A pile of coins is places on a table. There are two players, Alice and Bob, who alternate moves. Alice moves first. A legal move consists of removing one or two coins from the pile. The player who takes the last coin wins. Prove that

Alice has a winning strategy if the number of coins in the pile is not a multiple of 3, and Bob has a winning strategy of the number of coins in the pile is a multiple of 3.

7. Consider the sequence $a_0, a_1, a_2, \ldots$ of integers defined by $a_0 = 10$ and $a_n = 2a_{n-1}$, $n \geq 1$. Prove that $a_n = 2^n 10$ for all $n \geq 0$.

8. Find, with proof, the least integer n_0 such that $n! > 3 \cdot 2^n$ for all $n \geq n_0$.

9. Find, with proof, the least integer n_0 such that $5^n > (n+1)^3$ for all $n \geq n_0$.

10. Guess and prove a formula for $1 - 2 + 3 - 4 + \cdots + (-1)^{n-1} n$ (i.e., one that works for any $n \geq 1$).

11. Prove that for all $n \geq 1$, $1(2) + 2(3) + 3(4) + \cdots + n(n+1) = n(n+1)(n+2)/3$.

12. Prove that for all $n \geq 1$, $1^3 + 2^3 + 3^3 + \cdots + n^3 = n^2(n+1)^2/4$.

13. Prove that for all $n \geq 1$, $\frac{1}{1(2)} + \frac{1}{2(3)} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$.

14. Prove that for all $n \geq 1$, $1(1!) + 2(2!) + \cdots + n(n!) = (n+1)! - 1$.

15. Prove by induction that for any integer $n \geq 1$, $n^3 + (n+1)^3 + (n+2)^3$ is a multiple of 9.

16. Prove that $6 \mid 15^n - 9^n$, for any $n \geq 0$.

17. Let f_n denote the n-th Fibonacci number. Prove that for all $n \geq 6$, $f_n \geq (3/2)^{n-1}$.

18. Prove that every fifth Fibonnacci number is a multiple of 5.

19. Let $a_0, a_1, \ldots$ be the sequence recursively defined by $a_0 = 3$ and $a_n = 2a_{\lfloor n/3 \rfloor} + 3$ for $n \geq 1$. Find a formula for a_{3^n} and prove it is correct by induction.

20. Let $a_0, a_1, \ldots$ be the sequence recursively defined by $a_0 = 2$ and $a_n = a_{n-1} + 2(n-1)$ for $n \geq 1$. Find a formula for a_n and prove it is correct by induction.