Chapter 4

Sets

4.1 What is a Set?

A *set* is a well-defined collection of objects called *elements* or *members* of the set.

Here, *well-defined* means accurately and unambiguously stated or described. Any given object must either be an element of the set, or not be an element of the set. There is no concept of partial membership, and there is no possibility of an being a member more than once.

If x is a member of the set S, we write $x \in S$, and if x is a not member of the set S, we write $x \notin S$.

Sets are defined in terms of the objects they contain. We say sets A and B are *equal*, and write $A = B$ if they have exactly the same elements. That is, $A = B$ when $x \in A \Leftrightarrow x \in B$.

The *barber paradox* gives an example of a set that is not well-defined: *There is only one barber in a certain town. He is male. All of the men in the town are clean-shaven. The barber shaves all and only the men in the town who do not shave themselves. Who shaves the barber?* Now, if the barber shaves himself, then since the barber only the men who do not shave themselves, he does not shave himself. And if he does not shave himself, then since he shaves all of the men who don't shave themselves, he shaves himself. Hmmm. One explanation for this paradox is that the set, S, of men in the town who are shaved by the barber is not well-defined, as the barber must simultaneously be a member of the set, and not be a member of the set.

Sets can be described in several ways. One way to describe a set is to write a description of the set in words, as in "the set of all integers that can be written as the sum of two squares". There are three main ways of describing a set using mathematical notation.

1. *Explicit listing*: list the elements between brackets, as in $\{2, 3, 5, 7\}$.

   The elements of a set that's described by explicit listing are exactly the (different) objects in the list obtained when the outer brackets are erased. For example, the elements of $\{\text{car}, \pi, X\}$ are car, π, and X. The elements of $\{-1, \{3\}\}$ are -1 and $\{3\}$. (Sets can be members of other sets.)

2. *Implicit listing*: list enough its elements to establish a pattern and use an ellipsis "...". Proper use of the ellipsis requires that at least two elements be listed so that the pattern is established. (It could be that more elements must be listed before the pattern is apparent.) For example, $\{0, 2, 4, \ldots, 120\}$ is the set of non-negative even integers less than or equal to 120, while $\{\ldots - 3, -1, 1, 3, \ldots\}$ is the set of odd integers.

   The elements of a set that's described by implicit listing are those that follow the pattern, and respect any limits set. For example, the elements of $\{\{1\}, \{2\}, \ldots, \{6\}\}$ are $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}$, and $\{6\}$.

3. *Set-builder notation*: specify the set of the collection of all objects of a particular type that satisfy a given condition. Examples are: the set of all prime numbers less than 10 is $\{x : (x \text{ is prime}) \wedge (x < 10)\} = \{2, 3, 5, 7\}$; and the set of all positive even integers is $\{2k : k = 1, 2, \ldots\} = \{2, 4, 6, \ldots\}$.

   The elements of a set described using set-builder notation are those objects of the given type that make the stated condition true. For example, the elements of the set $\{a/b : a \text{ and } b \text{ are integers, and } a/b = 0.25\}$ are exactly the fractions whose numerical value is 0.25. There are infinitely many of these including $1/4, 3/12$ and $-5/(-20)$.

By the definition of equality of sets, it does not matter how a set is described; what matters is which elements it contains. Any particular object either belongs to the collection or it doesn't. All of $\{1, 2, 2, 3\}, \{1, 2, 3, 3\}$ $\{3, 2, 3, 1\}$ and $\{1, 2, 3\}$ all describe the same set because they all have the same three elements: 1, 2, and 3.

Some sets are well-known, and are denoted by special symbols.

- The set of *natural numbers* is $\mathbb{N} = \{1, 2, 3, \ldots\}$. Some people include 0 as an element of this set. It is always wise to check the definition that a particular author is using.
- The set of *integers* $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, The use of the symbol $\mathbb{Z}$ can be traced back to the German word *zählen*.
- The set of *rational numbers* is $\mathbb{Q} = \{a/b : a, b \in \mathbb{Z}, \text{ and } b \neq 0\}$. The symbol $\mathbb{Q}$ is used because these are *quotients* of integers.
- The set of *real numbers*, denoted by $\mathbb{R}$, has as elements all numbers that have a decimal expansion.
- The set of *complex numbers* is $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}, \text{ and } i^2 = -1\}$.

4.2 The Empty Set

It is certainly possible for a collection to have nothing in it. A good example would be the collection of years after 1967 in which the Toronto Maple Leafs have won the Stanley Cup.

The *empty set* is the set that has no elements, that is $\{\}$. It is commonly denoted by $\emptyset$.

The following sets are all equal to $\emptyset$: $\{x \in \mathbb{R} : x^2 + 1 = 0\}$, $\{n \in \mathbb{Z} : n^2 - 1 = 7\}$ and $\{a/b \in \mathbb{Q} : a/b = \sqrt{2}\}$.

The empty set is a perfectly legitimate object, and as such can occur as an element of a set. Notice that $\emptyset$ is different from $\{\emptyset\}$. The former set has no elements, while the latter set has one element, $\emptyset$. The set $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}, \emptyset\}\}$ has three elements: $\emptyset, \{\emptyset\}$, and $\{\{\emptyset\}, \emptyset\}\}$.

4.3 Subsets

We say that a set A is a *subset* of a set B if every element of A is an element of B (i.e., $x \in A \Rightarrow x \in B$). If A is a subset of B we write $A \subseteq B$, and otherwise we write $A \nsubseteq B$.

For example, $\mathbb{N} \subseteq \mathbb{Z}$, $\mathbb{Z} \subseteq \mathbb{Q}$, and $\mathbb{Q} \subseteq \mathbb{R}$. Also, $\{1, 3, 5\} \subseteq \{1, 3, 5\}$, and $\{2, 4\} \nsubseteq \{4, 5, 6\}$.

Notice that every set is a subset of itself (why?), that is $X \subseteq X$ for every set X.

A more subtle point is that $\emptyset$ is a subset of every set A. According to the definition, this is the same as the logical implication $x \in \emptyset \Rightarrow x \in A$ which, in turn, is the same as the implication $(x \in \emptyset) \rightarrow (x \in A)$ being a tautology. The implication has only the truth value "true" because its hypothesis, $x \in \emptyset$, is false for any x. A different way to say it is that every element in the collection of members of the empty set – there aren't any – is a member of A.

How many subsets does $\{a, b\}$ have? Let's count the options. Any particular subset either contains a or it does not. In both situations, there are two further options: the subset either contains b or it does not. Thus there are four possibilities $\{a, b\}, \{a\}, \{b\}, \{\}$.

The above reasoning can be extended to show that *a set with n elements has exactly 2^n subsets.*

Sometimes confusion arises in making the distinction between $\in$ and $\subseteq$. The first one makes the assertion that *a particular object belongs* to a set; the second one says that *every element of one set belongs* to another set.

In the following we show that the subset relation is *transitive*, that is, if A is a subset of B, and B is a subset of C, then A is a subset of C. (There is a more general meaning for the word "transitive". It will arise later in the course.) Before beginning the proof, it is useful to identify the statement to be proved, and the hypotheses that can be used in the argument. The statement to be proved is "A is a subset of C". That is, it needs to be argued that every element of A is an element of C. Equivalently, it needs to be argued that an arbitrary element of A is an element of C. The hypotheses that can be used in the argument are: "A is a subset of B", and "B is a subset of C". Constructing the proof involves using these to help argue that an arbitrary element of A must be an element of C.

Proposition 4.3.1 *Let A, B and C be sets. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.*

Proof. Take any $x \in A$. Since $A \subseteq B$, the element $x \in B$. Since $B \subseteq C$, the element $x \in C$. Therefore, if $x \in A$ then $x \in C$. That is, $A \subseteq C$. $\square$

Recall that if p and q are statements, then the logical equivalence $p \Leftrightarrow q$ is the same as the two logical implications $p \Rightarrow q$ and $q \Rightarrow p$. The logical equivalence is proved once the two logical implications are proved.

Proposition 4.3.2 *Let A and B be sets. Then $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.*

Proof. ($\Rightarrow$) Suppose $A = B$. Then every element of A is an element of B, and every element of B is an element of A. Thus, $A \subseteq B$ and $B \subseteq A$.

($\Leftarrow$) Suppose $A \subseteq B$ and $B \subseteq A$. Then every element of A is an element of B (because $A \subseteq B$), and every element of B is an element of A. This means A and B have exactly the same elements, so $A = B$. $\square$

4.4 Proper Subsets

The word "proper" occurs frequently in mathematics. Each time it has essentially the same meaning, roughly "*and not equal to the whole thing*".

A set A is a *proper subset* of a set B if $A \subseteq B$ and $A \neq B$. That is, A is a proper subset of B when every element of A belongs to B (so $A \subseteq B$) and there is an element in B which is not in A (so $A \neq B$).

Three common ways to denote that A is a proper subset of B are $A \subset B$, $A \subsetneq B$, and $A \subsetneqq B$. The last two of these are clear. The first one is, unfortunately, used by some authors to denote that A is a subset of B. While we we not do that, this is yet another reminder that it is always wise to check what the notation means instead of assuming.

We know, for example, that $\mathbb{Z} \subsetneqq \mathbb{Q}$ because $\mathbb{Z} \subseteq \mathbb{Q}$, and $1/2 \in \mathbb{Q}$ but $1/2 \notin \mathbb{Z}$.

From above, a set X with n elements has 2^n subsets. All but one of them is a proper subset.

Proposition 4.4.1 *Let A, B and C be sets. If $A \subseteq B$ and $B \subsetneqq C$, then $A \subsetneqq C$.*

Proof. Two things need to be shown: (i) $A \subseteq C$, and (ii) $A \neq C$. Since $B \subsetneqq C$ implies that $B \subseteq C$, statement (i) is true by Proposition 4.3.1.

To show statement (ii) we must find an element C which is not an element of A. Since $B \subsetneqq C$, there exists $x \in C$ such that $x \notin B$. Since every element of A is an element of B, x can not be an element of A. Therefore $A \neq C$.

Both statements have been shown, and the proof is now complete. $\square$

It is a good exercise to prove similar statements, for example If $A \subsetneqq B$ and $B \subseteq C$, then $A \subsetneqq C$. The argument is essentially the same as the one above.

4.5 The Power Set

The *power set* of a set A is the set whose elements are the subsets of A. The notation $\mathcal{P}(A)$ is commonly used to denote the power set of A.

Consider the set $A = \{a, b\}$. We know that A has four subsets, $\{a, b\}, \{a\}$, $\{b\}, \{\}$, so that $\mathcal{P}(A) = \big\{\{a, b\}, \{a\}, \{b\}, \{\}\big\}$.

The name "power set" comes from the fact that a set with n elements has exactly 2^n subsets. Thus, there are 2^n elements in the power set of a set with n elements.

The following facts are important to remember. For any set X:

- $\mathcal{P}(X)$ is a set.
- The elements of $\mathcal{P}(X)$ are sets (too).
- $A \in \mathcal{P}(X) \Leftrightarrow A \subseteq X$ (this is the definition of the power set).

- By the previous point, $\emptyset \in \mathcal{P}(X)$ and $X \in \mathcal{P}(X)$.

The following proposition is included because its proof forces us to think about power sets and their elements.

Proposition 4.5.1 *Let A and B be sets. Then $A \subseteq B$ if and only if $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.*

Proof. ($\Rightarrow$) Suppose $A \subseteq B$. We need to show that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Take any $X \in \mathcal{P}(A)$. Then $X \subseteq A$. Since $A \subseteq B$, we have by Proposition 4.3.1 that $X \subseteq B$. Therefore $X \in \mathcal{P}(B)$. Therefore $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

($\Leftarrow$) Suppose $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. We need to show that $A \subseteq B$.

Since $A \subseteq A$, $A \in \mathcal{P}(A)$. Since $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, $A \in \mathcal{P}(B)$. By definition of $\mathcal{P}(B)$, $A \subseteq B$. $\square$

4.6 Set Operations: The Laws of Set Theory

Let A and B be sets.

- The *union* of A and B is the set $A \cup B = \{x : (x \in A) \vee (x \in B)\}$.
- The *intersection* of A and B is the set $A \cap B = \{x : (x \in A) \wedge (x \in B)\}$.

Notice that the set union symbol looks vaguely like the symbol for the logical connective "or", and the set intersection symbol looks vaguely like the symbol for the logical connective 'and"/ Indeed, union is defined using "or", and intersection is defined using "and".

The definition of union and intersection allows us to use the laws of logic to prove statements about sets. As an example, we prove on of the associative laws. The proof amounts to using set builder notation and demonstrating that the sets on each side of the equals sign are described by logically equivalent conditions.

Proposition 4.6.1 *Let A, B and C be sets. Then $(A \cup B) \cup C = A \cup (B \cup C)$.*

Proof.

$$\begin{aligned}(A \cup B) \cup C &= \{x : (x \in A \cup B) \vee (x \in C)\}\\ &= \{x : ((x \in A) \vee (x \in B)) \vee (x \in C)\}\\ &= \{x : (x \in A) \vee ((x \in B) \vee (x \in C))\}\\ &= \{x : (x \in A) \vee (x \in B \cup C)\}\\ &= A \cup (B \cup C)\end{aligned}$$

□

For each Law of Logic, there is a corresponding "Law of Set Theory". For example, for sets A, B and C:

- $(A \cap B) \cap C = A \cap (B \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

In each case, the proof can be carried out similarly to the above.

It would be nice to have a set operation that corresponds to negation, but the set $\{x : x \notin A\}$ does not make any sense unless we first specify the collection of all objects that are permitted to be an element of any set under discussion. This is the *universe* (in which we are working). The universe is itself a set, and is typically denoted by $\mathcal{U}$.

We can now define set operations corresponding all logical connectives (since every logical connective can be expressed using only "and", "or" and "not"), and also obtain a Law of Set Theory corresponding to each Law of Logic.

Let A and B be sets (which are subsets of a universe $\mathcal{U}$).

- The *set difference of* A *and* B is the set $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$. This is the subset of A obtained by deleting from A all of the elements that are also in B. For this reason, the notation $A - B$ is also commonly used.
- The *complement of* A is the set $A^c = \{x : x \notin A\} = \mathcal{U} \setminus A$. In analogy with a notation commonly used for negation in logic (but not by us), the complement of A is also sometimes denoted by $\overline{A}$.

For example, $\{a, b, c\} \setminus \{b, d\} = \{a, c\}$. The set $\{1, 2\}^c$ depends on the universe of discourse. If $\mathcal{U} = \mathbb{N}$, then $\{1, 2\}^c = \{3, 4, \ldots\}$, and if $\mathcal{U} = \{1, 2\}$, then $\{1, 2\}^c = \emptyset$. It is worth noticing that $\{b, d\} \setminus \{a, b, c\} = \{d\}$ which shows that, in general, $A \setminus B \neq B \setminus A$.

By definition, $A \setminus B = A \cap B^c$.

As in the situation for logical connectives, *there is no precedence among set operations, except that complements are done first*. The moral of the story is that one should always use brackets for clarity.

Having defined the complement, it is now possible to have DeMorgan's Laws for set theory. These can be proved using the same method is Proposition 4.6.1. They can also be proved by showing that $LHS \subseteq RHS$ and $RHS \subseteq LHS$. For the purposes of illustration, we choose the latter method.

Proposition 4.6.2 (DeMorgan's Laws) *Let A and B be sets. Then*

- $(A \cup B)^c = A^c \cap B^c$*; and*
- $(A \cap B)^c = A^c \cup B^c$.

Proof. We prove only the first statement. The proof of the second statement can be done in a similar way.

($LHS \subseteq RHS$) Let $x \in (A \cup B)^c$. Then $x \notin A \cup B$. Thus, $x \notin A$ and $x \notin B$. That is, $x \in A^c$ and $x \in B^c$. Therefore $x \in A^c \cap B^c$, and so $(A \cup B)^c \subseteq A^c \cap B^c$.

($RHS \subseteq LHS$). Let $x \in A^c \cap B^c$. Then $x \in A^c$ and $x \in B^c$. Thus, $x \notin A$ and $x \notin B$. Therefore, $x \notin A \cup B$, that is, $x \in (A \cup B)^c$. Hence, $A^c \cap B^c \subseteq (A \cup B)^c$. $\square$

A careful look at the argument reveals that the second part of the proof, ($RHS \subseteq LHS$), is really the same steps as in the first part in the reverse order. That's because each step is actually an equivalence rather than (just) an implication. These are the same equivalences that would be used if the statement were proved using the Laws of Logic. The same thing happens frequently proofs about set equality. Once half of a proof is constructed, it pays to think about whether the other half is already in hand.

There are ways in which the universe plays a similar role in set theory as a tautology does in logic. Similarly, the empty set can be seen to play a similar role in set theory as a contradiction does in logic. The following proposition is the set theory version of the logical equivalences:

- $p \vee \neg p \Leftrightarrow \mathbf{1}$;
- $p \wedge \neg p \Leftrightarrow \mathbf{0}$;
- $\mathbf{1} \wedge p \Leftrightarrow p$; and
- $\mathbf{0} \vee p \Leftrightarrow p$.

Proposition 4.6.3 *Let A be a set. Then*

- $A \cup A^c = \mathcal{U}$*;*
- $A \cap A^c = \emptyset$*;*
- $\mathcal{U} \cap A = A$*;*
- $\emptyset \cup A = A$*.*

Proof. To see the first statement, recall that every element x is either in A or in A^c, so that $A \cup A^c = \mathcal{U}$. To see the second statement, note that, by definition, no element x can be in both A and A^c, so that $A \cap A^c = \emptyset$. The last two statements follow immediately from the definitions. $\square$

For statements p and q, the logical connective *exclusive or* of p and q is defined to be the statement $p \veebar q$ which is true when p is true, or q is true, but not when both are true. It can be checked using a truth table that $p \veebar q \Leftrightarrow \neg(p \leftrightarrow q)$, and (using logical equivalences) that $p \veebar q \Leftrightarrow (p \wedge \neg q) \vee (\neg p \wedge q) \Leftrightarrow (p \vee q) \wedge \neg(p \wedge q)$.

The set operation corresponding to exclusive or is "symmetric difference". For sets A and B, the *symmetric difference* of A and B is the set $A \oplus B = (A \setminus B) \cup (B \setminus A)$. Using the definitions of set difference and union gives $A \oplus B = \{x : (x \in A \wedge x \notin B) \vee (x \notin A \wedge x \in B)\}$. (The two steps that show this are embedded in the proof of the proposition below.)

For example, $\{1, 2, 3\} \oplus \{2, 4, 6\} = \{1, 3, 4, 6\}$, and $\{1, 2\} \oplus \emptyset = \{1, 2\}$.

The proposition below corresponds to the logical equivalence $p \veebar q \Leftrightarrow (p \vee q) \wedge \neg(p \wedge q)$. The proof of the set equality looks a lot like the proof of the logical equivalence. We now have enough Laws of Set Theory to write the proof using them.

Proposition 4.6.4 *Let A and B be sets. Then $A \oplus B = (A \cup B) \setminus (A \cap B)$*

Proof.

$$\begin{aligned}
A \oplus B &= (A \setminus B) \cup (B \setminus A) \\
&= (A \cap B^c) \cup (B \cap A^c) \\
&= ((A \cap B^c) \cup B) \cap ((A \cap B^c) \cup A^c) \\
&= [(A \cup B) \cap (B^c \cup B)] \cap [(A \cup A^c) \cap (B^c \cup A^c)] \\
&= [(A \cup B) \cap \mathcal{U}] \cap [\mathcal{U} \cap (B^c \cup A^c)] \\
&= (A \cup B) \cap (B^c \cup A^c) \\
&= (A \cup B) \cap (A \cap B)^c \\
&= (A \cup B) \setminus (A \cap B)
\end{aligned}$$

□

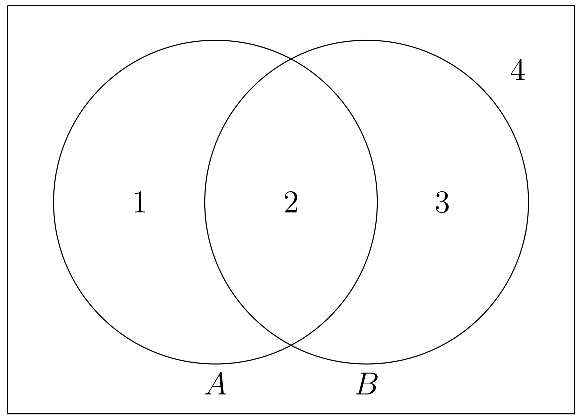
4.7 Venn Diagrams

Informally, a Venn diagram is a picture that shows all possible memberships between elements of the universe and a collection of sets.

Let A and B be sets. For any element of the universe, there are four mutually-exclusive possibilities, where *mutually exclusive* means only one possibility holds at a time.

1. it belongs to both A and B, that is, to $A \cap B$;
2. it belongs to A and not to B, that is, to $A \setminus B$;
3. it belongs to B and not to A, that is, to $B \setminus A$; or
4. it belongs to neither A nor B, that is, to $(A \cup B)^c$.

These four possibilities correspond to the four regions in the diagram below, if we imagine each element of the universe being somehow located in the diagram depending on which of the possibilities holds.



Each of the sets defined in the previous section can be associated with a collection of regions in the diagram:

Set	Represented by Regions
A	$1, 2$
B	$2, 3$
$A \cup B$	$1, 2, 3$
$A \cap B$	2
$\mathcal{U}$	$1, 2, 3, 4$
A^c	$2, 3$
B^c	$1, 4$
$A \setminus B$	1
$B \setminus A$	3
$A \oplus B$	$1, 3$

The regions that represent a set correspond exactly to its elements in the situation where $\mathcal{U} = \{1, 2, 3, 4\}$, $A = \{1, 2\}$ and $B = \{2, 3\}$.

It is apparent from the table above that, for example, $A \setminus B \neq B \setminus A$, because the set on the left hand side is represented by region 1, and the set on the right hand side is represented by region 3. And the diagram can be used to get an example of a universe $\mathcal{U}$ and sets A and B such that $A \setminus B \neq B \setminus A$. Directly from what was just done, if we take $\mathcal{U} = \{1, 2, 3, 4\}$, $A = \{1, 2\}$ and $B = \{2, 3\}$, then $A \setminus B = \{1\}$ and $B \setminus A = \{3\}$.

We thus have an important principle: *If two sets are represented by different collection of regions in a Venn diagram, then an example showing the sets are not equal can be obtained directly from the diagram.*

On the other hand, Venn diagrams can provide intuition about equality between sets. As a first example, let's investigate whether $A \cup B$ is equal to $(A \setminus B) \cup B$. Using the diagram from before, we have:

Set	Represented by Regions
A	$1, 2$
B	$2, 3$
$A \cup B$	$1, 2, 3$
$A \setminus B$	1
$(A \setminus B) \cup B$	$1, 2, 3$

Since both sets are represented by the same collection of regions, we expect that they are equal. There are several different ways to construct a proof.

- Construct a truth table to show that the statement $x \in A \cup B \leftrightarrow x \in (A \setminus B) \cup B$ is a tautology. To do that, one has to express the memberships on each side in terms of compound statements, as in $[x \in A \vee x \in B] \leftrightarrow [(x \in A \wedge \neg(x \in B)) \vee x \in B]$.

- Use the definition of the two sets and show they are described by logically equivalent conditions.

- Write a proof in words, showing $LHS \subseteq RHS$ and $RHS \subseteq LHS$. The written proofs tend to follow the flow of logic used in constructing the set of regions that represent a set, except in the reverse order. In this example:

  ($LHS \subseteq RHS$) *Take any $x \in A \cup B$. Then $x \in A$ or $x \in B$. We consider two cases, depending on whether $x \in B$. If $x \in B$, then $B \cup (A \setminus B) = (A \setminus B) \cup B$. If $x \notin B$, then x must be in A since it is in $A \cup B$. Thus $x \in A \setminus B$, and hence $x \in (A \setminus B) \cup B$. In either case, $x \in (A \setminus B) \cup B$. Therefore $A \cup B \subseteq (A \setminus B) \cup B$.*

  ($RHS \subseteq LHS$) *Take any $x \in (A \setminus B) \cup B$. Then either $x \in A \setminus B$ or $x \in B$. If $x \in A \setminus B$, then $x \in A$ so $x \in A \cup B$. If $x \in B$, then $x \in B \cup A = A \cup B$. In either case, $x \in A \cup B$. Therefore $(A \setminus B) \cup B \subseteq A \cup B$.* $\square$

Venn diagrams can also give insight into other types of relationships between. An example is the statement $A \subseteq B \Leftrightarrow A \cup B = B$. It is clear that if $A \subseteq B$ then $A \cup B = B$. What follows is not a proof, but will prove to be quite easy to turn into a proof. The condition $A \subseteq B$ says that every element of A is in B so, referring to the Venn diagram, no elements of A would be located in region 1. When region 1 contains no points of A, the set $A \cup B$ is (actually) represented by regions 2 and 3, so $A \cup B = B$. For the other logical implication, in the Venn diagram above, $A \cup B$ is represented by regions 1, 2, and 3, while B is represented by regions 2 and 3. The condition $A \cup B = B$, says that there are no elements of A that would be located in region 1 of the diagram. When this happens, A is (actually) represented by region 2 and, since B is represented by regions 2 and 3, this means $A \subseteq B$.

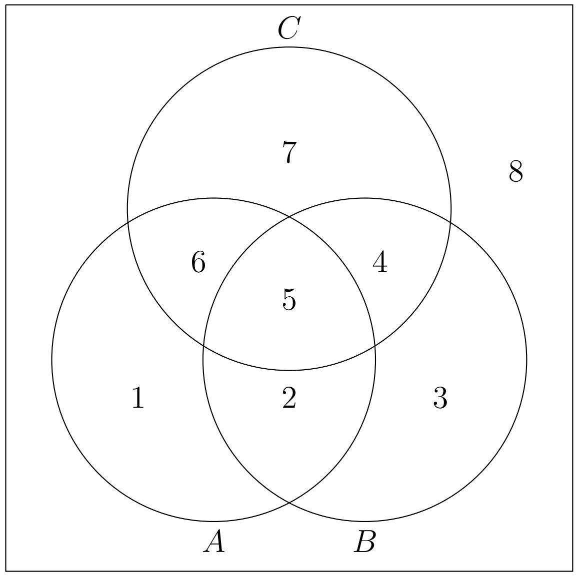
We now transform the observations in the preceding paragraph into a proof. There are two things so show:

($A \subseteq B \Rightarrow A \cup B = B$) The goal is to prove that $A \cup B = B$. By definition of union, $B \subseteq A \cup B$. It remains to argue that $A \cup B \subseteq B$. Take any $x \in A \cup B$. Then $x \in A$ or $x \in B$. If $x \in B$ there is nothing to show. If $x \in A$, then since $A \subseteq B$, $x \in B$. This completed the proof that $A \cup B = B$.

($A \cup B = B \Rightarrow A \subseteq B$) The goal is to prove that $A \subseteq B$. Take any $x \in A$. Then, by definition of union, $x \in A \cup B$. Since $A \cup B = B$, $x \in B$. Therefore $A \subseteq B$. $\square$

Because the definition of union involves the logical connective "or", it is important to remember that *proofs of set relationships where one set involves the operation of union often use the method of proof by cases.*

Let A, B and C be sets. For any element of the universe, there are eight mutually-exclusive possibilities: it belongs to none of them (one possibility), it belongs to exactly one of them (three possibilities), it belongs to exactly two of them (three possibilities), or it belongs to all of them (one possibility). These are represented by the eight regions in the Venn diagram below.



Let's use the diagram above to investigate whether $A \cup (B \cap C)$ equals $(A \cup B) \cap C$.

Set	Represented by Regions
A	$1, 2, 5, 6$
B	$2, 3, 4, 5$
C	$4, 5, 6, 7$
$B \cap C$	$4, 5$
$A \cup (B \cap C)$	$1, 2, 4, 5, 6$
$A \cup B$	$1, 2, 3, 4, 5, 6$
$(A \cup B) \cap C$	$4, 5, 6$

As before, the regions correspond to the sets that would arise if we performed the set operations using $\mathcal{U} = \{1, 2, \ldots, 8\}, A = \{1, 2, 5, 6\}, B = \{2, 3, 4, 5\}$ and $C = \{4, 5, 6, 7\}$. Hence, when the sets in question are represented by different regions, these sets provide a counterexample. Doing so for the the example above, $A \cup (B \cap C) = \{1, 2, 4, 5, 6\}$ and $(A \cup B) \cap C = \{4, 5, 6\}$. Therefore the two expressions determine different sets in general.

The Venn diagram suggests $(A \cup B) \cap C \subseteq A \cup (B \cap C)$. Proving it would be a good exercise.

In general, a *Venn Diagram* is a collection of n simple closed curves

(curves that don't intersect themselves) that partition the plane into 2^n connected regions (regions that are in one piece). These regions illustrate all possible memberships an element of the universe might have with respect to these sets. It is known that Venn diagrams for n sets exist for every non-negative integer n.

4.8 Counting sets and subsets

A set is called *finite* if it is empty, or has exactly n elements for some positive integer n. In this case we write $|X| = n$, and say that X has *size* or *cardinality* n.

A set that isn't finite is called *infinite*. We will study infinite sets in a later chapter.

We reasoned before that *if X is a set such that $|X| = n$, then X has exactly 2^n subsets*. Imagine constructing a subset of X. For each element of X there are two options – either it belongs to the subset or it doesn't – end each choice leads to a different subset.

Let's use induction to prove the statement "For any integer $n \geq 0$, if $|X| = n$, then X has exactly 2^n subsets". For the basis, notice that the empty set has 0 elements, and $2^0 = 1$ subsets, so the statement is true when $n = 0$. For some integer $k \geq 0$, assume that if $|X| = k$, then X has exactly 2^k subsets. Consider a set S such that $|S| = k + 1$. We want to show that S has exactly 2^{k+1} subsets. Since $k + 1 \geq 1$, the set $S \neq \emptyset$. Let a be any element of S. A subset of S either contains a or it doesn't. Any subset of S that does not contain a is a subset of $S \setminus \{a\}$, which is a set with exactly k elements. Since any subset of $S \setminus \{a\}$ is a subset of S that does not contain a, by the induction hypothesis, there are exactly 2^k subsets of S that do not contain a. On the other hand, if T is a subset of S that contains a, then $T \setminus \{a\} \subseteq S \setminus \{a\}$. Since the union of $\{a\}$ and any subset of $S \setminus \{a\}$ is a subset of S that contains a, by the induction hypothesis, there are exactly 2^k subsets of S that do contain a. Since these two cases (not containing a, and containing a) can not arise simultaneously, he number of subsets of S equals $2^k + 2^k = 2^{k+1}$, as wanted. The statement is now proved.

It follows from what we have just proved that the number of proper subsets of a set X such that $|X = n$ equals $2^n - 1$: only the set X itself is not a proper subset of X.

Let $X = \{x_1, x_2, \ldots, x_n\}$. Then $|X| = n$. Let's figure out the number of subsets of X that contain x_1. Any subset of X that contains x_1 is the union of $\{x_1\}$ and a subset of $X \setminus \{x_1\}$, so that there 2^{n-1} such subsets. Another point of view is that we can construct a subset of X that contains x_1 in two steps: first, put x_1 into the subset (this choice is forced), and then pick one of the 2^{n-1} subsets of $X \setminus \{x_1\}$ and put all of its elements into the subset. The outcome of the first step does not affect the number of options we have at the second step, so the number of different outcomes of the construction is $1 \times 2^{n-1}$. Since each outcome leads to a different subset of X that contains x_1, the number of X that contain x_1 equals 2^{n-1}.

Similar reasoning as above shows that:

- the number of subsets of X that do not contain x_2 equals 2^{n-1};
- the number of subsets of X that contain x_3 and x_4, but not x_5, equals 2^{n-3}.

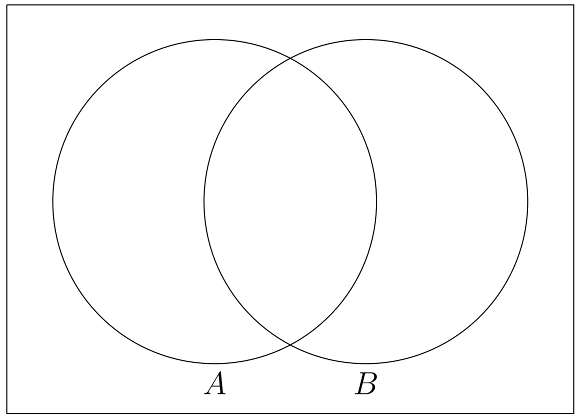
It is a bit more tricky to count the number of subsets of X that contain x_1 or x_2. It isn't the number that contain x_1 plus the number that contains x_2 because subsets that contain both x_1 and x_2 are included twice. We could consider 3 cases: (i) subsets that contain x_1 and not x_2; (ii) subsets that contain x_2 and not x_1; and (iii) subsets that contain x_1 and x_2. This leads to the answer $2^{n-2} + 2^{n-2} + 2^{n-2} = 3 \cdot 2^{n-2}$. An alternative method uses the Principle of Inclusion and Exclusion, which is discussed below.

Let A and B be finite sets. Referring to the Venn diagram below, let's calculate $|A \cup B|$. The number $|A| + |B|$ counts each element in $A \setminus B$ exactly once, each element in $B \setminus A$ exactly once, and each element in $A \cap B$ exactly twice. *Therefore,* $|A| + |B| - |A \cap B|$ *counts each element of the union exactly once.*

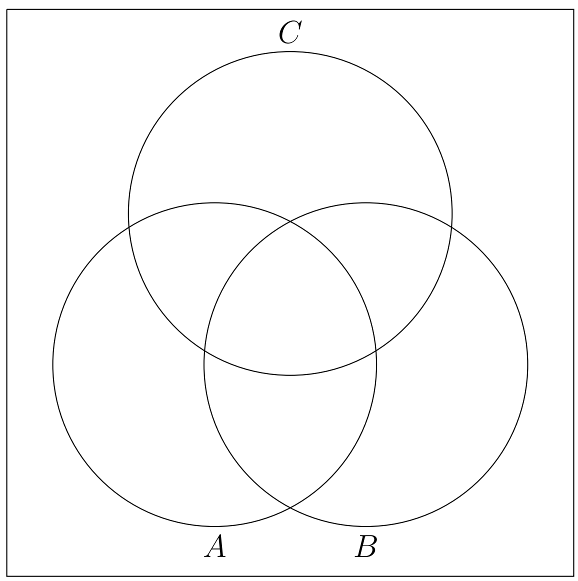
The size of each single set is *included* and then the size of the intersection is *excluded.*

For sets A, B and C, a similar argument gives that

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$



The size of each single set is *included*, the size of each intersection of two of the sets is *excluded*, and then the size of the intersection of all three sets is *included*.



The argument can be extended beyond three sets. The resulting theorem is called the *Principle of Inclusion and Exclusion*. It says that the cardinality of the union of n finite sets can be computed by including the size of each single set, excluding the size of all possible intersections of two sets, excluding the size of all possible intersections of three sets, excluding the size of all possible intersection of four sets, and so on.

Let's go back to the example of computing the number of subsets of $S = \{x_1, x_2, \ldots, x_n\}$ that contain x_1 or x_2. Let A be the collection of subsets of S that contain x_1, and B be the collection of subsets of S that contain x_2. The subsets we want to count are exactly the elements of $A \cup B$. By the Principle of Inclusion and Exclusion, $|A \cup B| = |A| + |B| - |A \cap B| = 2^{n-1} + 2^{n-1} - 2^{n-2} = 3 \cdot 2^{n-2}$, which agrees with our previous calculation.

Suppose that in a group of 200 students, there are 150 taking Math 122, 100 taking Math 101, and 50 taking both of these classes. We can use this information to answer questions like:

1. How many of these students are taking neither Math 122 nor Math 101?

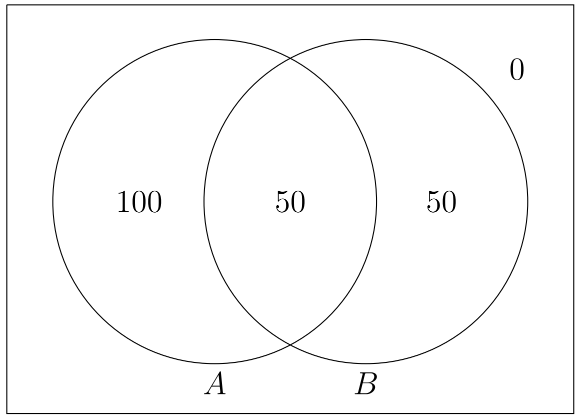
2. How many of these students are taking exactly one of Math 122 and Math 101?

To do this, first let A be the set of students taking Math 122, and B be the set of students taking Math 101. The information given is that $\mathcal{U} = 200$, $|A| = 150$, $|B| = 100$, and $|A \cap B| = 50$. We can work backwards and fill in the number of elements in the 4 regions of the Venn Diagram. That is, we fill the diagram in starting with the region corresponding to intersection of all sets, and working "outwards" to the region corresponding to the elements not in any of the sets. It is given that $|A \cap B| = 50$. Since $|A| = 150$, and $|A \cap B| = 50$, it follows that $|A \setminus B| = 150 - 150 = 100$. Similarly, $|B \setminus A| = 100 - 50 = 50$. Therefore $|A \cup B| = 50 + 100 + 50$, the sum of the numbers in the 3 regions of the Venn Diagram that comprise $A \cup B$. Finally $|\mathcal{U} \setminus (A \cup B)| = 200 - (50 + 100 + 50) = 0$.

The answer to the questions is therefore:

1. This is $|(A \cup B)^c| = |\mathcal{U} \setminus (A \cup B)| = 0$.

2. This is $|(A \setminus B) \cup (B \setminus A)| = 100 + 50 = 150$. Notice that the sets associated with corresponding regions of the Venn Diagram regions are disjoint (their intersection is empty), so that the number of elements in their union is just the sum of the elements in the sets.



The same can be done for three (or more) sets. Suppose that, of 250 programmers, 75 can program in Ada, 47 can program in Basic, and 60 can program in C++. There are 30 who can program in both Ada and Basic, 22 who can program in both Basic and C++, 7 who can program in both C++ and Ada, and 5 who can program in all three languages.

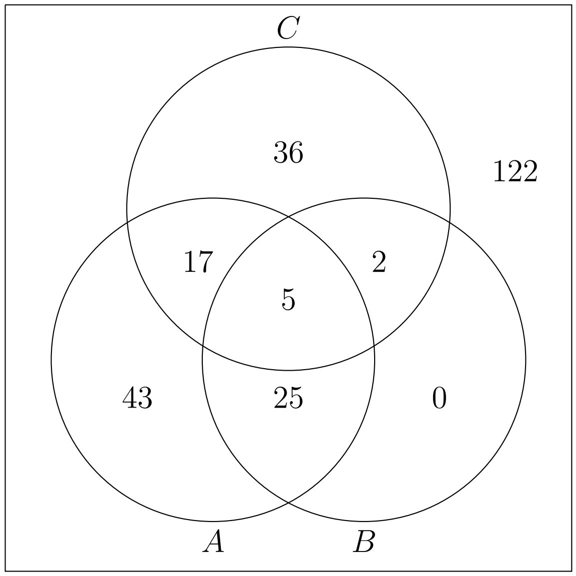
1. How many can program in at most one of them?

2. How many can program in Ada and exactly one of the other two languages?

Let A, B and C be the set of programmers who can program in Ada, Basic and C++, respectively. Filling in the regions of a Venn Diagram as above leads to the picture below. The answers to the questions can then be read directly from the picture.

1. We want $|(A \cup B \cup C)^c| + |A \setminus (B \cup C)| + |B \setminus (A \cup C)| + |C \setminus (A \cup B)| = 122 + 43 + 0 + 36 = 201$

2. We want $|(A \cap B) \setminus C| + |(A \cap C) \setminus B| = 25 + 2 = 27.$



4.9 Set Theory Questions

1. Let $A = \{1, 2, \{1, 2\}\}$. Answer each question true or false, and briefly explain your reasoning.

   (a) $\{2\} \in A$
   (b) $\{1, 2\} \subsetneq A$
   (c) $\{2, \{1, 2\}\} \subseteq A$
   (d) $\emptyset \in A$
   (e) $A \cap \mathcal{P}(A) = \emptyset$

2. Answer each question true or false, and briefly explain your reasoning.

   (a) If A, B, C are sets, then $(A \cup B) \cup C = (C \cup B) \cup A$.
   (b) If $A \cap B$ is not empty, then $A \setminus B$ is a proper subset of A.
   (c) If $x \in A$, then $\{x\} \in \mathcal{P}(A)$.
   (d) $\{\emptyset\}$ has two different subsets.

3. Let A and B be sets. Prove that any two of the following statements are (logically) equivalent.

(a) $A \subseteq B$

(b) $A \cup B = B$

(c) $A \cap B = A$

(d) $A \setminus B = \emptyset$

(e) $A \oplus B \subseteq B$

(f) $B^c \subseteq A^c$

Note: by a result from the Logic questions, it suffices to establish a cycle of 6 implications, for example (a) $\Rightarrow$ (b) $\Rightarrow \cdots \Rightarrow$ (f) $\Rightarrow$ (a). On the other hand, it is good practice to prove directly that any pair of statements are equivalent.

4. Let A and B be sets. Prove that $A \cup B = A \cap B \Leftrightarrow A = B$.

5. Let $A = \big\{\emptyset, \{x\}, B, \{1, \{x\}\}\big\}$, and $B = \{1, x\}$. Answer each question true or false, and briefly explain your reasoning.

   (a) $x \in A$.

   (b) $\{\emptyset\} \subseteq A$.

   (c) $B \subseteq A$.

   (d) $1 \in B \cap A$.

6. Determine the number of sets X such that $\{1, 2, 3\} \subseteq X \subsetneqq \{1, 2, 3, 4, 5, 6\}$. Explain your reasoning.

7. Prove that if $A \subsetneqq B$ and $B \subseteq C$, then $A \subsetneqq C$.

8. Prove or disprove each of the following statements about sets.

   (a) If $A \cap B \subseteq C$, then $\big((A \subseteq C) \wedge (B \subseteq C)\big)$.

   (b) $A \setminus B = (B \setminus A)^c$.

9. Prove that for all sets A, B and C, if $A \subseteq B$ and $B \cap C = \emptyset$, then $A \cap C = \emptyset$. Hint: Proof by contradiction.

10. Prove that for all sets A and B, $(A \backslash B) \cup (A \cap B) = A$.

11. Give a counterexample to each statement.

    (a) $(A \setminus B) \cap C = (A \cap C) \setminus B^c$, for all sets A, B, and C.

    (b) $(A \setminus B) \cup C^c = (A \cup B) \setminus C$, for all sets A, B, and C.

12. Let A, B and C be sets. Prove that $A \setminus (B \setminus C) = (A \setminus B) \cup (A \setminus C^c)$ without using set-builder notation and showing that the two sides are determined by logically equivalent expressions. Hint: an easy way uses the Laws of Set Theory.

13. Prove the same statement as in the previous question by showing LHS $\subseteq$ RHS and RHS $\subseteq$ LHS.

14. Prove or disprove: For all sets A, B and C, $(A \backslash B) \cup (B \backslash C) = A \backslash C$.

15. Give a recursive definition of the union $\bigcup_{i=1}^{n} A_i$ of the $n \geq 2$ sets, $A_1, A_2, \ldots, A_n$. Do the same for their intersection, $\bigcap_{i=1}^{n} A_i$

16. Let A, B, C be sets. Prove that $(A \cap B \cap C)^c = (A^c \cup B^c \cup C^c)$ by:

    (a) using the Associative Law to inset brackets and then DeMorgan's Law;

    (b) Showing LHS $\subseteq$ RHS and RHS $\subseteq$ LHS;

    (c) using set-builder notation and showing the LHS and RHS are defined by logically equivalent expressions

17. Repeat question 16 for the equality $(A \cup B \cup C)^c = (A^c \cap B^c \cap C^c)$.

18. Use induction to prove that, for any $n \geq 2$,

$$\left(\bigcup_{i=1}^{n} A_i\right)^c = \bigcap_{i=1}^{n} A_i^c.$$

    The base case is one of DeMorgan's Laws, which we have already proved to be true.

19. Use induction to prove that, for any $n \geq 2$,
$$\left(\bigcap_{i=1}^{n} A_i\right)^c = \bigcup_{i=1}^{n} A_i^c.$$
The base case is one of DeMorgan's Laws, which we have already proved to be true.

20. Let A and B be sets. Prove that the following statements are all (logically) equivalent.
    (a) $A = B$
    (b) $A \subseteq B$ and $B \subseteq A$
    (c) $A \setminus B = B \setminus A$
    (d) $A \oplus B = \emptyset$
    (e) $A \cap B = A \cup B$
    (f) $A^c = B^c$

21. Prove that for all sets A, B and C, $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$ by using set-builder notation and showing the LHS and RHS are defined by logically equivalent expressions.

22. Repeat the previous question but use the Laws of Set Theory instead of set-builder notation.

23. Prove that for all sets A, B and C, if $B \cap C \subseteq A$, then
$$(C \setminus A) \cap (B \setminus A) = \emptyset.$$
by using set-builder notation and using the fact that the hypothesis corresponds to the logical implication that for any x, $(x \in B) \wedge (x \in C) \Rightarrow (x \in A)$.

24. Repeat the previous question but use the Laws of Set Theory instead of set-builder notation.

25. Prove that for all sets A and B, if $B \subseteq A^c$, then $A \cap B = \emptyset$.

26. Let A, B, C be sets. Prove that if $A \cap B = \emptyset$, then $A \cap B \cap C = \emptyset$. Is the converse true? Explain.

27. Let $X = \{a, b, c, \ldots, z\}$. Determine the number of subsets $T \subseteq X$ that:
    (a) contain z;
    (b) do not contain a, e, i, o, u;
    (c) are such that $\{w, x, y\} \subset T$;
    (d) contain a and b but not c;
    (e) contain m or do not contain n;
    (f) contain at least one of p, q, r;
    (g) are such that $\{f, g, h\} \not\subseteq T$.
28. Two sets X and Y are called *disjoint* if $X \cap Y = \emptyset$.
    (a) Prove that if X and Y are disjoint finite sets, then $|X \cup Y| = |X| + |Y|$.
    (b) Prove that if A, B, C are pairwise disjoint finite sets (i.e., finite sets such that any two of them are disjoint), then $|A \cup B \cup C| = |A| + |B| + |C|$.
29. Suppose that in a group of 50 motorcyclists, 30 own a Triumph and 32 own a Honda. If 15 motorcyclists in the group own neither type of motorcycle, how many own a motorcycle of each type?
30. In a group of 35 ex-athletes, 17 play golf, 20 go cycling, and 12 do yoga. Exactly 8 play golf and go cycling, 8 play golf and do yoga, 7 go cycling and do yoga, and 4 do all three activities. How many of the ex-athletes do none of these activities?