1 Logic Questions

1. Suppose that the statement $p \to \neg q$ is false. Find all combinations of truth values of r and s for which $(\neg q \to r) \wedge (\neg p \vee s)$ is true.

2. Find all combinations of truth values for p, q and r for which the statement $\neg p \leftrightarrow (q \wedge \neg(p \to r))$ is true.

3. Is $(p \to q) \to [(p \to q) \to q]$ a tautology? Why or why not?

4. (a) Show that $(p \to q) \leftrightarrow (q \to p)$ is neither a tautology nor a contradiction. What does that tell you about possible relationships between the truth values of a statement and its converse?

   (b) Suppose $\neg[(p \to q) \leftrightarrow (q \to p)]$ is false. Can $p \leftrightarrow q$ have both possible truth values? Explain.

5. Show that $[(p \vee q) \wedge (r \vee \neg q)] \to (p \vee r)]$ is a tautology by making a truth table, and then again by using an argument that considers the two cases "q is true" and "q is false".

6. Write each of the following statements, in English, in the form "if p, then q".

   (a) I go swimming on Mondays.

   (b) In order to be able to go motorcycling on Sunday, the weather must be good.

   (c) Eat your vegetables or you can't have dessert.

   (d) You can ride a bicycle only if you wear a helmet.

   (e) Polynomials are continuous functions.

   (f) A number n that is a multiple of 2 and also a multiple of 3 is a multiple of 6.

   (g) You can't have any pudding unless you eat your meat.

   (h) The cardinality of a set is either finite or infinite.

7. Write in English the converse, contrapositive and negation of each statement.

   (a) If I had $1,000,000, I'd buy you a fur coat.

   (b) If it is not raining and not windy, then I will go running or cycling.

   (c) A day that's sunny and not too windy is a good day for walking on the waterfront.

   (d) If 11 pigeons live in 10 pigeonholes, then there are two pigeons that live in the same pigeonhole.

   (e) If every domino covers a black square and a white square, then the number of black squares equals the number of white squares.

   (f) If the average of n numbers is 70, then one of the n numbers is less than or equal to 70.

8. Determine if each statement below is true or false, and explain your reasoning.

   (a) It is possible for an implication and its contrapositive to have different truth values.

   (b) The negation of "*Every golf shot is a hook or a slice*" is "*Some golf shots are hooks and slices*".

(c) If the statement q is true, then, for any statement p, the statement $p \to q$ is true.

(d) If $s_1 \to s_2$ is a contradiction, then so is its contrapositive.

(e) The negation of "*All enforcers skate slowly and pass badly*" is "*Some enforcers skate fast and pass well*".

(f) For integers m and n, arguing that if mn is odd then m and n are odd proves that if m or n is even then mn is even.

(g) There are truth values for p and q such that $p \to q$ and $q \to p$ are both false.

(h) $(\neg p \vee q) \wedge \neg(\neg p \vee q)$ is a contradiction.

(i) If the statement $\mathcal{P}$ is a contradiction, then, for any statement q, the statement $\mathcal{P} \to q$ is a tautology.

(j) If two statements are logically equivalent, then so are their negations.

9. Consider the following (correct) argument in which all variables represent integers.

   Suppose n and k are odd.
   Then $n = 2t + 1$ for some integer t, and $k = 2\ell + 1$ for some integer ℓ.
   Hence, $nk = (2t + 1)(2\ell + 1) = 4t\ell + 2t + 2\ell + 1$.
   Therefore, nk is odd.

   (a) Write the implication proved by the argument in plain English.

   (b) Write the contrapositive of the implication in plain English. Is it also proved by the argument?

   (c) Write the converse of your statement in (a). Is it also proved by the argument?

10. Consider the following. All variables represent integers.

    Proposition: If n^2 is a multiple of 8, then n is a multiple of 8.
    Proof: Let $n = 8m$. Then $n^2 = 64m^2 = 8(8m^2)$, which is a multiple of 8, as desired. $\square$.

    Why does the given argument not prove the proposition? Either give a correct proof, or give an example to show that the proposition is false.

11. Consider the statement "if the goods are unsatisfactory, then your money will be refunded". This was an advertising slogan of the T. Eaton Company. Is the given statement logically equivalent to "goods satisfactory or money refunded"? What about to "if your money is not refunded, then the goods are satisfactory"? And what about to "if the goods are satisfactory, then your money will not be refunded".

12. A sign posted outside of Tokyo says "*In order to attack the city, you must be green and related to Godzilla. If you are not green and not related to Godzilla, then you can not attack the city*".

    (a) Render the two statements on the sign in symbols, making any quantifiers explicit. Start with: *Let $a(x)$ be the assertion "x can attack the city"*, and carry on from there. Assume that the collection of allowed replacements for x is the collection of all monsters.

    (b) Argue that the two statements on the sign are not logically equivalent, contrary to what the author probably intended. Which is more restrictive on who can attack Tokyo?

(c) Correct the second statement so that it is logically equivalent to the first one.

13. Show that the two statements $(p \wedge q) \rightarrow r$ and $(p \rightarrow r) \wedge (q \rightarrow r)$ are not logically equivalent.

14. Assume that a and b are integers. Consider the statements:
    $\mathcal{A}$ If c is a prime number such c divides ab,
    then c divides a or c divides b.
    $\mathcal{B}$ If c is a prime number such c divides ab, and c does not divide b,
    then c divides a.

    (a) Write the statements $\mathcal{A}$ and $\mathcal{B}$ in symbolic form and then show that they are logically equivalent.

    (b) Write the contrapositive of each statement in English.

15. Determine if each statement below is true or false, and explain your reasoning.

    (a) When the statement "*There is no largest integer.*" is written is symbols, both of the quantifiers $\forall$ and $\exists$ appear.

    (b) For the universe of real numbers, $\forall x, \exists y, xy = 1$ is false.

    (c) For the universe of integers, $\exists x, (x^2 < 0) \rightarrow (x > 10)$ is true.

16. Let $(0, 1)$ denote the open interval consisting of the real numbers x such that $0 < x < 1$. Consider the statement $\mathcal{A} : \exists x \in (0, 1), \forall y \in (0, 1),\ y \leq x$.

    (a) Write $\mathcal{A}$ in English without using symbols except $x,\ y,\ (0, 1)$.

    (b) Write down the negation of statement $\mathcal{A}$ in symbols without using either of $\neg$ and $\not\leq$.

    (c) Show that $\mathcal{A}$ is false.

17. Write each statement in plain English. Do not use any symbols except the letters that denote elements of the universe.

    (a) $\forall x, \forall y, (x \neq -y) \rightarrow (x + y) \neq 0$, where the universe is the real numbers.

    (b) $\exists s, \forall t, p(s) \wedge [(t \neq s) \rightarrow \neg p(t)]$, where the universe of s and t is the collection of all students who completed Math 122 last fall, and $p(s)$ is the assertion "s got 100% on the final exam".

18. Suppose the universe of m and n is $\{-1, 0, 1\}$. Then, for example,

$$\exists n, n^2 + n > 0 \Leftrightarrow ((-1)^2 + (-1) > 0) \vee (0^2 + 0 > 0) \vee (1^2 + 1 > 0).$$

    For each of the following statements,

    (i) write a compound statement involving neither quantifiers nor variables that is logically equivalent to the given quantified statement,

    (ii) determine whether the statement is TRUE or FALSE, and

    (iii) write the negation of the quantified statement in symbols, with quantifiers, and without using negation ($\neg$) or any negated mathematical symbols like $\neq$ or $\not<$.

    (a) $\forall n, n^3 - n = 0$

(b) $\exists n, \forall m, n + m < 1$.

19. Write each statement in plain English.

    (a) $\neg[\exists x, (p(x) \wedge \neg(q(x)))]$, where the universe of x is all Canadian citizens, $p(x)$ is the statement "x is eligible to vote in a municipal election" and $q(x)$ is the statement "x is 18 years old or older."

    (b) $\forall x, [(x \neq \text{ Quebec}) \to v(x)] \wedge \neg v($Quebec, where the universe of x is the collection of all major Canadian cities, and $v(x)$ is the assertion "Gary has visited x."

20. (a) Let n be in integer. Explain what is wrong with the following argument which "shows" that *if n is a multiple of 2 and a multiple of 3, then n is a multiple of 6.*

    Suppose n is a multiple of 6. Then $n = 6k$ for some integer k.
    Since $6 = 2 \times 3$, we have that $n = 2 \times (3k)$, so it is a multiple of 2,
    and $n = 3 \times (2k)$, so it is a multiple of 3. ■

    (b) Give a correct proof of the assertion.

21. (a) Suppose that m and n are integers. It is claimed that the argument below proves that *if mn is odd, then m and n are both odd.* Does it? Explain your reasoning.

    Suppose that the integers m and n are both even. Then there exists an integer k such that $m = 2k$, and there exists an integer ℓ such that $n = 2\ell$. Thus,

    $$mn = (2k)(2\ell) = 2(2k\ell).$$

    Since $2k\ell$ is an integer, mn is even.

22. Suppose the set of allowed replacements for the variables is the integers. Let $p(n)$ be "*n is even*" and $q(n)$ be "*n is odd*". Determine the truth value of each statement and provide a brief explanation of your reasoning.

    (a) $\forall n, p(n) \vee q(n)$

    (b) $[\exists n, p(n)] \wedge [\exists n, q(n)]$

    (c) $\exists n, p(n) \to q(n)$

    (d) $[\forall n, p(n)] \wedge [\forall n, q(n)]$

    (e) $\forall n, \exists m, n + m = 0$

    (f) $\exists n, \forall m, n + m = 0$

23. Suppose that the set of allowed replacements for the variable p is $\{Gary, Christi\}$ and the set of allowed replacements for the variable c is $\{Whitehorse, Ottawa, Halifax\}$. Let $v(p, c)$ be the statement "p has visited c". Write each statement in symbolic form without quantifiers.

    (a) Christi has visited every city.

    (b) There is a city Gary has not visited.

    (c) For every person there is a city which they have visited.

24. (a) Use any method to show that $\neg(p \to q) \leftrightarrow (p \wedge \neg q)$ is a tautology.

    (b) Use known logical equivalences to show that $\neg(p \leftrightarrow q) \Leftrightarrow (p \vee q) \wedge (\neg p \vee \neg q)$.

25. Use known logical equivalences to show that $(\neg a \to b) \wedge (\neg b \vee (\neg a \vee \neg b))$ is logically equivalent to $\neg(a \leftrightarrow b)$.

26. Use known logical equivalences to show that $\neg(p \leftrightarrow q)$ is logically equivalent to $(p \vee q) \wedge (p \to \neg q)$.

27. Use known logical equivalences to do each of the following.

    (a) Show $p \to (q \vee r) \Leftrightarrow (p \wedge \neg q) \to r$.

    (b) Show $\neg(p \vee q) \vee (\neg p \wedge q) \vee \neg(\neg p \vee \neg q) \Leftrightarrow \neg(p \wedge \neg q)$.

    (c) Find an expression logically equivalent to $\neg(p \leftrightarrow q)$ that involves only $\neg$ and $\vee$.

28. Let s be the statement whose truth table is given below.

p	q	r	s
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	0

    (a) Express the statement s in terms of p, q and r in such a way that only negation ($\neg$) and the logical connectives $\vee$ and $\wedge$ are used.

    (b) Find an equivalent formulation of s that uses only $\neg$ and $\vee$.

    (c) Find an equivalent formulation of s that uses only $\neg$ and $\wedge$.

29. Define the logical connective "nand" (not and) by $p \barwedge q \Leftrightarrow \neg(p \wedge q)$.

    (a) Find a representation of each of the following statements using only the logical connective nand.

        i. $\neg p$
        ii. $p \wedge q$
        iii. $p \vee q$
        iv. $p \to q$
        v. $p \leftrightarrow q$

    (b) Explain why every statement has a representation using only the logical connective nand.

30. Repeat question 29 using the logical connective "nor" (not or) defined by $p \barvee q \Leftrightarrow \neg(p \vee q)$.

31. Referring to questions 29 and 30, prove that $\neg(p \barvee q) \Leftrightarrow \neg p \barwedge \neg q$. Guess and prove a similar logical equivalence for $\neg(p \barwedge q)$.

32. (a) Argue that "logically implies" has the property (called *transitivity*) that if a, b and c are statements such that $a \Rightarrow b$ and $b \Rightarrow c$, then $a \Rightarrow c$.

    (b) Suppose a, b, c and d are statements such that $a \Rightarrow b$, $b \Rightarrow c$, $c \Rightarrow d$, and $d \Rightarrow a$. Argue that any two of these statements are logically equivalent.

33. Determine whether each statement is true or false, and briefly explain your reasoning.

    (a) If an argument is valid then it is possible the conclusion to be false when all premises are true.

    (b) If the premises can't all be true, then the argument is valid.

    (c) If $p \Leftrightarrow q$ and $q \Leftrightarrow r$, then $p \Leftrightarrow r$.

34. Show that the argument

$$\begin{array}{rl} & p \leftrightarrow q \\ & q \to r \\ & r \vee \neg s \\ & \neg s \to q \\ \hline \therefore & s \end{array}$$

    is invalid by providing a counterexample.

35. Use basic inference rules to establish the validity of the argument

$$\begin{array}{rl} & p \to \neg q \\ & q \vee r \\ & p \vee u \\ & \neg r \\ \hline \therefore & u \end{array}$$

36. If the argument below is valid, then use any method to prove it. Otherwise, give a counterexample to show that the argument is invalid.

$$\begin{array}{rl} & \neg r \to p \\ & q \to \neg p \\ & \neg(r \vee t) \\ \hline \therefore & q \end{array}$$

37. Use any method to show the following argument is valid.

$$\begin{array}{rl} & p \\ & \neg q \leftrightarrow \neg p \\ \hline \therefore & q \end{array}$$

38. Show that the following argument is not valid.

$$\begin{array}{rl} & p \vee r \\ & p \vee q \\ \hline \therefore & q \vee r \end{array}$$

39. Use logical equivalences and the rules of inference to determine whether the following argument is valid.

$$
\begin{array}{rl}
 & \neg(\neg p \vee q) \\
 & \neg z \to \neg s \\
 & (p \wedge \neg q) \to s \\
 & \neg z \vee r \\
\hline
\therefore & r
\end{array}
$$

40. Write the argument below in symbolic form. If the argument is valid, prove it. If the argument is not valid, give a counterexample:

$$
\begin{array}{rl}
 & \text{If I watch football, then I don't do mathematics} \\
 & \text{If I do mathematics, then I watch hockey} \\
\hline
\therefore & \text{If I don't watch hockey, then I watch football}
\end{array}
$$

41. Write the argument below in symbolic form. If the argument is valid, prove it. If the argument is not valid, give a counterexample:

$$
\begin{array}{rl}
 & \text{If you are pregnant or have a heart condition, then you can not use the hot tub.} \\
 & \text{You do not have a heart condition.} \\
 & \text{You can use the hot tub.} \\
\hline
\therefore & \text{You are not pregnant.}
\end{array}
$$

42. (a) Show that $p \to (q \to r)$ is logically equivalent to $(p \wedge q) \to r$.

    (b) Establish the validity of the argument

$$
\begin{array}{rl}
 & u \to r \\
 & (r \wedge s) \to (p \vee t) \\
 & q \to (u \wedge s) \\
 & \neg t \\
 & q \\
\hline
\therefore & p
\end{array}
$$

    (c) Consider the argument

$$
\begin{array}{rl}
 & u \to r \\
 & (r \wedge s) \to (p \vee t) \\
 & q \to (u \wedge s) \\
 & \neg t \\
\hline
\therefore & q \to p
\end{array}
$$

    Use part (a), and the fact that if b is true then $a \to b$ is true, to explain why the validity of this argument is established by your proof in (b).

43. Suppose that the integer a is a multiple of 3, and the integer b is a multiple of 4. Give a direct proof that ab is a multiple of 12.

44. Prove that if the integer n^2 is a multiple of 5, then the integer n is a multiple of 5. (Hint: prove the contrapositive using a proof by cases; there are 4 cases.)

45. Prove that $\sqrt{5}$ is irrational.