

The Pigeonhole Principle

Recall that a function $f : A \rightarrow B$ is *one-to-one* (or 1-1) if different elements of A have different images in B . For such a function to exist, it is necessary that $|B| \geq |A|$. If $|A| > |B|$, then no function $f : A \rightarrow B$ can be 1-1 and, thus, some two elements of A have the same image. More generally, if A and B are finite sets with $|A| > k \cdot |B|$ then some element of B is the image of at least $k + 1$ elements of A .

Theorem 1. *If n pigeons are distributed among $k > 0$ pigeonholes, then some pigeonhole contains at least $\lceil \frac{n}{k} \rceil$ pigeons.*

Proof. We argue the contrapositive. Suppose that at most $\lceil \frac{n}{k} \rceil - 1$ pigeons are put into each pigeonhole. Then, since $\lceil \frac{n}{k} \rceil - 1 < \frac{n}{k} \leq \lceil \frac{n}{k} \rceil$, the total number of pigeons is at most

$$k(\lceil \frac{n}{k} \rceil - 1) < k(\frac{n}{k}) = n.$$

□

When $n = k + 1$ the statement becomes: *If $k + 1$ or more pigeons are distributed among k pigeonholes, then some one pigeonhole contains two or more pigeons.*

Exercise 1. *Prove that, for $t > 0$, given t boxes B_1, B_2, \dots, B_t , and t integers n_1, n_2, \dots, n_t , if at least $1 + \sum_{i=1}^t (n_i - 1)$ objects are distributed among these t boxes, then there exists i such that box B_i contains at least n_i objects.*

Exercise 2. *Prove that if n objects are distributed among $k > 0$ boxes, then some box contains at most $\lfloor \frac{n}{k} \rfloor$ objects.*

The Pigeonhole Principle is often used in the following equivalent form: *If n pigeons are selected from among those in $k > 0$ pigeonholes, then at least $\lceil \frac{n}{k} \rceil$ pigeons are selected from some pigeonhole.* When $n = k + 1$ the statement becomes: *If $k + 1$ or more pigeons are selected from those in k pigeonholes, then two or more pigeons are selected from some pigeonhole.*

There are two main steps involved in applying the Pigeonhole Principle. First, *decide what the pigeons are.* They will be the things that you'd like several of to have some special property. Then, *set up the pigeonholes* so that the pigeons in the same pigeonhole have the property you want. You want to have enough fewer pigeonholes than the number of pigeons to be selected, so that the "right" number of pigeons get selected from the same pigeonhole. (There can be a real art to this step.)

In order to get a less informal description of the two steps above, we need a definition. Let X be a set. A *partition of X* is a collection $\{X_1, X_2, \dots, X_k\}$ of non-empty, pairwise disjoint, subsets of X such that $X_1 \cup X_2 \cup \dots \cup X_k = X$. The number, k , of subsets in the collection is the *degree*, or *number of parts* in the partition.

Example 1. *Let $X = \{1, 2, \dots, 10\}$. Then*

- $\mathcal{P}_1 = \{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}, \{10\}\}$,
- $\mathcal{P}_2 = \{\{1\}, \{2\}, \{3\}, \{4, 5, 6, 7, 8, 9, 10\}\}$, and
- $\mathcal{P}_3 = \{\{4, 5, 6\}, \{10\}, \{7, 8, 9\}, \{1, 2, 3\}\}$

are all partitions of X into four parts. Further, $\mathcal{P}_1 = \mathcal{P}_3$ because the order that the subsets are listed does not matter.

Proving things with the Pigeonhole Principle.

- (1) *Define the pigeons.* These are the objects that you would like several of to have the same property.
- (2) *Define the pigeonholes.* These partition the set of pigeons into subsets so that collections of objects in the same subset all have the property you want.

Example 2. *Prove that if seven distinct numbers are selected from $\{1, 2, \dots, 11\}$, then some two of these numbers sum to 12.*

The pigeons are the numbers $1, 2, \dots, 11$. Define the pigeonholes by partitioning these numbers into the six sets $\{1, 11\}, \{2, 10\}, \{3, 9\}, \{4, 8\}, \{5, 7\}, \{6\}$, so two different numbers in the same subset sum to 12. Since seven numbers are selected from 6 subsets, the Pigeonhole Principle guarantees that two of them are selected from the same set. These two numbers sum to 12.

Note that the quantity seven is the best possible in the sense that it is possible to select six numbers from $\{1, 2, \dots, 11\}$ so that no two of the numbers selected sum to 12. One example of six such numbers is $1, 2, 3, 4, 5, 6$.

Exercise 3. *Prove the general version of Example ??: If $n+1$ numbers are selected from $\{1, 2, \dots, 2n-1\}$, then some two of these numbers sum to $2n$, and this result is best possible in the sense that it is possible to select n numbers so that no two of them sum to $2n$. Formulate and prove similar statements for collections of numbers selected from $\{1, 2, \dots, 2n\}$.*

Example 3. *Prove that if five points are selected from the interior of a 1×1 square, then there are two points whose distance is less than $\sqrt{2}/2$.*

The pigeons are the points in the interior of the square. Set up the pigeonholes by partitioning the unit square into four $1/2 \times 1/2$ subsquares by joining the midpoints of opposite sides (and assigning the points on the line segments to one of the adjacent subsquares). Since 5 points are selected, two of them are selected from the same subsquare, including its boundary. Since the points selected are on the interior of the square, the distance between them is less than the length of a diagonal of a subsquare, which is $\sqrt{2}/2$.

Exercise 4. *Prove that if four points are selected from the interior of a unit circle, then there are two points whose distance apart is less than $\sqrt{2}$.*

Example 4. *Prove that if 10 integers are selected from $\{1, 2, \dots, 18\}$, the selection includes integers a and b such that a divides b .*

The pigeons are the numbers between 1 and 18. To set up the pigeonholes, notice that the largest odd divisor of an integer between 1 and 18 is one of the nine numbers $1, 3, 5, 7, 9, 11, 13, 15, 17$. Set up the pigeonholes by partitioning the numbers between 1 and 18 into subsets so that numbers in the same subsets have the same largest odd divisor. Since 10 integers are selected, the Pigeonhole Principle guarantees that some two of them have the same largest odd divisor, t . Let these two numbers be a and b , where $a < b$. Then, $a = 2^r t$ and $b = 2^s t$, where $r < s$, so that $a2^{s-r} = b$. Since $s - r$ is a positive integer, it follows that $a|b$.

Exercise 5. *Prove that if $n + 1$ integers are selected from $\{1, 2, \dots, 2n\}$, then the selection includes integers a and b such that $a|b$.*

Example 5. Prove that if 11 integers are selected from among $\{1, 2, \dots, 20\}$, then the selection includes integer a and b such that $a - b = 2$.

The pigeons are the numbers between 1 and 20. Set up the pigeonholes by partitioning them into the 10 sets

$$\{3, 1\}, \{4, 2\}, \{7, 5\}, \{8, 6\}, \{11, 9\}, \{12, 10\}, \{15, 13\}, \{16, 14\}, \{19, 17\}, \{20, 18\}$$

so that pairs of numbers belonging to the same set differ by two. Since 11 integers are selected, two of them are selected from the same set. By the way the sets were defined, these two integers differ by two.

Exercise 6. Prove that if $n + 1$ integers are selected from among $\{1, 2, \dots, 2n\}$, then the selection includes integers a and b such that $b = a + 1$.

Exercise 7. Prove that if $n + 1$ integers are selected from among $\{1, 2, \dots, 2n\}$, then the selection includes integer a and b such that $\gcd(a, b) = 1$. (Hint: Exercise ??.)

We conclude our work on the Pigeonhole Principle with two harder examples.

Example 6. Over a 44 day period, Gary will train for triathlons at least once per day, and a total of 70 times in all. Show that there is a period of consecutive days during which he trains exactly 17 times.

It will take some work before the pigeons can be described. Let $x_0 = 0$ and, for $i = 1, 2, \dots, 44$, let x_i be the number of times Gary trains up to the end of day i . Then $0 = x_0 < x_1 < x_2 < x_3 < \dots < x_{44} = 70$. We need to find subscripts i and j such that $x_i + 17 = x_j$. This implies that Gary trains exactly 17 times in the period of days $i + 1, i + 2, \dots, j$. Therefore, we want one of $x_0, x_1, x_2, \dots, x_{44}$ to be equal to one of $x_0 + 17, x_1 + 17, \dots, x_{44} + 17$. The pigeons are the 90 numbers $x_0, x_1, x_2, \dots, x_{44}, x_0 + 17 < x_1 + 17 < \dots < x_{44} + 17$. Now, since

$$17 \leq x_0 + 17 < x_1 + 17 < \dots < x_{44} + 17 = 87,$$

we know that the 90 numbers $x_0, x_1, \dots, x_{44}, x_1 + 17, x_2 + 17, \dots, x_{44} + 17$ can take on at most 88 different values. By the Pigeonhole Principle, some two of them must be equal. The inequalities imply that no two of x_0, x_1, \dots, x_{44} can be equal, and no two of $x_0 + 17, x_1 + 17, \dots, x_{44} + 17$ can be equal. Hence one of x_0, x_1, \dots, x_{44} must equal one of $x_1 + 17, x_2 + 17, \dots, x_{44} + 17$, which is what we wanted.

Exercise 8. Over a 30 day period, Peter will walk the dog at least once per day, and a total of 45 times in all. Prove that there is a period of consecutive days in which he walks the dog exactly 14 times.

Example 7. A party is attended by $n \geq 2$ people. Prove that there will always be two people in attendance who have the same number of friends at the party. (Assume that the relation “is a friend of” is symmetric, that is, if x is a friend of y then y is a friend of x .)

Each person either is, or is not, a friend of each of the the other $n - 1$ people in attendance. Thus, the possible values for the number of friends a person can have in attendance at the party are $0, 1, \dots, n - 1$. However, it can not be the case that there is someone at the party with 0 friends and someone else with $n - 1$ friends: if a person is friends with everyone then (since “is a friend of” is symmetric) everyone at the party has at least one friend there. Thus, the possible values for the

number of friends a person can have in attendance at the party are $0, 1, \dots, n - 2$ or $1, 2, \dots, n - 1$. In either case, there are n numbers (of friends among the people in attendance) that can take on at most $n - 1$ different values. By the Pigeonhole Principle, two of the numbers are equal. Thus, some two people in attendance who have the same number of friends at the party.

Exercise 9. *Ten baseball teams are entered in a round-robin tournament (meaning that every team plays every other team exactly once) in which ties are not allowed. Prove that if no team loses all of its games, then some two teams finish the tournament with the same number of wins.*