SUBQUOTIENTS OF HECKE $C^*$-ALGEBRAS

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In [2], Bost and Connes studied a particular Hecke $C^*$-algebra $C\mathcal{Q}$ arising in number theory. The algebra $C\mathcal{Q}$ can be realised as a semigroup crossed product $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*$ by an endomorphic action $\alpha$ of the multiplicative semigroup $\mathbb{N}^*$ on the group $C^*$-algebra $C^*(\mathbb{Q}/\mathbb{Z})$ [7], and this realisation has provided useful insight into the analysis of $C\mathcal{Q}$ [5, 13]. Since individual elements of $\mathbb{Q}/\mathbb{Z}$ and $\mathbb{N}^*$ involve only finitely many primes, $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*$ is the direct limit of subalgebras $C^*(G_F) \rtimes_\alpha \mathbb{N}^F$, where $F$ is a finite set of primes, $G_F$ is the subgroup of $\mathbb{Q}/\mathbb{Z}$ in which the denominators have all prime factors in $F$, and $\mathbb{N}^F$ acts through the embedding $(n_p) \mapsto \prod_{p \in F} p^{n_p}$ of $\mathbb{N}^F$ in $\mathbb{N}^*$ (see Section 1). One can therefore hope to understand the Hecke algebra $C\mathcal{Q}$ in terms of the finite-prime analogues $C^*(G_F) \rtimes_\alpha \mathbb{N}^F$.

Our goal is to analyse the structure of these finite-prime analogues of the Bost-Connes algebra. We started this analysis in [10], where we described a composition series for the two-prime analogue and identified the subquotients in familiar terms: there is a large type I ideal, a commutative quotient isomorphic to $C(T^2)$, and the intermediate subquotient is isomorphic to a direct sum of Bunce-Deddens algebras. Here we describe a composition series for $C^*(G_F) \rtimes_\alpha \mathbb{N}^F$. Again there are a large type I ideal and a commutative quotient, and the intermediate subquotients are direct sums of simple $C^*$-algebras. We can describe the simple summands as ordinary crossed products by actions of $\mathbb{Z}^S$ for $S \subset F$. When $|S| = 1$, these actions are odometers and the crossed products are Bunce-Deddens algebras; when $|S| > 1$, the actions are an apparently new class of higher-rank odometer actions, and the crossed products are an apparently new class of classifiable AT-algebras.

We begin with a short section in which we describe the algebras we intend to study. In §2, we describe our composition series for the semigroup crossed product $C^*(G_F) \rtimes_\alpha \mathbb{N}^F$. It has $|F| + 1$ subquotients, and all but two of them are direct sums of algebras stably isomorphic to ordinary crossed products of the form $C(\mathcal{U}(\mathbb{Z}_{F \setminus S})) \rtimes \mathbb{Z}^S$, where $S \subset F$ and $\mathcal{U}(\mathbb{Z}_{F \setminus S})$ is the group of units in the ring $\prod_{p \in F \setminus S} \mathbb{Z}_p$. Our main tools are the analysis of invariant ideals in semigroup crossed products from [9] and some technical lemmas on sums and intersections of ideals in $C^*$-algebras. We also use the general results of [16] to see that the simple summands are classifiable.

In §3, we show that when $S = \{q\}$ is a singleton, $C(\mathcal{U}(\mathbb{Z}_{F \setminus S})) \rtimes \mathbb{Z}^S$ is a direct sum of finitely many Bunce-Deddens algebras; as in [10], the number of summands depends on the orders of $q$ in the finite groups $\prod_{p \neq q} \mathcal{U}(\mathbb{Z}/p\mathbb{Z})$ for large $l \in \mathbb{N}$. We then consider the case where $S = \{q, r\}$. By computing the $K$-theory of $C(\mathcal{U}(\mathbb{Z}_{F \setminus S})) \rtimes \mathbb{Z}^S$, we find that the simple summands are classifiable.

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we can see that they are not Bunce-Deddens algebras, for example. We expect these
summands to be even harder to recognise when \(|S| > 2\).

In §4, we use techniques like those of §2 to identify subquotients of the Bost-Connes
algebra \(C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*\). These include algebras stably isomorphic to \(C(\mathcal{U}(\mathbb{Z}_p \setminus S)) \rtimes \mathbb{Z}^S\)
when \(S\) is a cofinite subset of the set \(\mathcal{P}\) of all primes; in this case, though, these crossed
products are themselves simple, and even though the general theory of [16] no longer
applies, we can see using results from [1] that they are classifiable AT-algebras. We
finish with a purely number theoretic Appendix in which we identify the orders of
an odd integer in the groups \(\mathcal{U}(\mathbb{Z}/p^l\mathbb{Z})\) and their products. As in [10, Theorem 3.1],
these are needed when we want to identify the number of simple summands in the
various subquotients.

We thank N. Christopher Phillips for drawing our attention to [17], and for pointing
out that our simple direct summands are AT-algebras rather than general AH-
algebras.

1. Preliminaries

We denote by \(\mathbb{N}^*\) the semigroup of positive integers under multiplication, and by
\(\mathbb{N}\) the semigroup of nonnegative integers under addition. It was shown in [7, Proposition 2.1] that there is an action \(\alpha\) of \(\mathbb{N}^*\) by endomorphisms of \(C^*(\mathbb{Q}/\mathbb{Z})\) such that

\[
\alpha_n(\delta_r) = \frac{1}{n} \sum_{ns=r} \delta_s \quad \text{for } r \in \mathbb{Q}/\mathbb{Z} \text{ and } n \in \mathbb{N}^*.
\]

The corresponding semigroup crossed product \(C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*\) is isomorphic to the
Hecke \(C^*\)-algebra \(C_Q\) of Bost and Connes [7, Corollary 2.10]. We denote by \((i_A, i_{\mathbb{N}^*})\)
the canonical covariant representation of \((C^*(\mathbb{Q}/\mathbb{Z}), \mathbb{N}^*, \alpha)\) in \(C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*\).

Let \(F\) be a set of prime numbers. The rational numbers of the form \(\frac{1}{n} \sum_{p \in F} m_p\) form a subgroup of \(\mathbb{Q}\), whose image in \(\mathbb{Q}/\mathbb{Z}\) we denote by \(G_F\). The integrated form of the map \(r \mapsto \delta_r : G_F \to UC^*(\mathbb{Q}/\mathbb{Z})\) is a homomorphism \(i_F\) of \(C^*(G_F)\) into
\(C^*(\mathbb{Q}/\mathbb{Z})\); a standard duality argument shows that \(i_F\) is injective, so that we can identify \(C^*(G_F)\) with the subalgebra \(i_F(C^*(G_F))\) of \(C^*(\mathbb{Q}/\mathbb{Z})\). When \(n\) has all its prime factors in \(F\), \(\alpha_n\) leaves this subalgebra invariant, and hence composing \(\alpha\) with the map \((m_p)_{p \in F} \mapsto \prod_{p \in F} p^{m_p}\) gives an action of \(\mathbb{N}^F\) on \(C^*(G_F)\), which we also
denote by \(\alpha\). The pair \((i_F, i_{\mathbb{N}^*}|_{\mathbb{N}^F})\) is a covariant representation of \((C^*(G_F), \mathbb{N}^F, \alpha)\) in \(C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*\).

Since \(i_F\) is injective, we can deduce from the main theorem of [11] (or by minor modifications to the argument in §3 of [7]) that the corresponding homomorphism

\[
i_F \times i_{\mathbb{N}^*}|_{\mathbb{N}^F} : C^*(G_F) \rtimes_\alpha \mathbb{N}^F \to C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*
\]

is also an injection. We use this injection to identify \(C^*(G_F) \rtimes_\alpha \mathbb{N}^F\) with a subalgebra of \(C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*\).

The crossed product \(C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*\) is spanned by the elements of the form
\(i_A(\delta_r) i_{\mathbb{N}^*}(m) i_{\mathbb{N}^*}(n)^*\) [7, Lemma 3.2]. If \(F\) contains all the prime factors of \(m\), \(n\) and the denominator of \(r\), then \(i_A(\delta_r) i_{\mathbb{N}^*}(m) i_{\mathbb{N}^*}(n)^*\) lies in \(C^*(G_F) \rtimes_\alpha \mathbb{N}^F\). Thus \(C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*\) is the direct limit \(\bigcup_F C^*(G_F) \rtimes_\alpha \mathbb{N}^F\) over increasing finite subsets \(F\) of the set \(\mathcal{P}\) of prime numbers.
In the next section, we shall describe a composition series for \( C^*(G_F) \times_\alpha \mathbb{N}^F \) when \( F \) is a finite subset of \( \mathcal{P} \), and identify the subquotients in terms of ordinary crossed products \( C(X_S) \rtimes \mathbb{Z}^S \) associated to subsets \( S \) of \( F \). The underlying space \( X_S \) is the group of units \( \mathcal{U}(\mathbb{Z}_F^S) \) in the ring \( \mathbb{Z}_F^S := \prod_{p \in F \setminus S} \mathbb{Z}_p \); as an additive group, \( \mathbb{Z}_F^S \) is the dual group of \( G_{F \setminus S} \). The action of a prime \( q \in S \) on
\[
C(\mathcal{U}(\mathbb{Z}_F^S)) \subset C(\mathbb{Z}_F^S) \cong C^*(G_{F \setminus S}) \subset C^*(\mathbb{Q}/\mathbb{Z})
\]
induced by \( \alpha_q \) is multiplication by \( q \) on \( \mathcal{U}(\mathbb{Z}_F^S) \) (see \cite{10, Lemma 1.1}), which because \( q \) is a unit in \( \mathbb{Z}_F \) is an automorphism. Thus the action of \( \mathbb{N}^S \) on \( C(\mathcal{U}(\mathbb{Z}_F^S)) \) extends to an action \( \sigma \) of \( \mathbb{Z}^S \) such that
\[
\sigma(m_p)(f)(x) = f((\prod_{p \in S} p^{m_p} x)^{-1}) \text{ for } (m_p) \in \mathbb{N}^S.
\]
As a matter of notation, we shall view a crossed product \( A \rtimes_\beta G \) by an action of a group as the universal \( C^* \)-algebra generated by a copy of \( A \) and a unitary representation \( i_G : G \to U(A \rtimes_\beta G) \) satisfying the covariance relation \( \beta_s(a) = i_G(s) a i_G(s)^* \).

2. Finitely many primes

The object of this section is to prove the following theorem. For the definitions of \( \text{AT-algebra} \), real rank zero and stable rank one, see \cite{20} and the references given there.

**Theorem 2.1.** Let \( F \) be a finite set of primes. Then there is a composition series \( \{I_k \mid 1 \leq k \leq |F| \} \) of ideals in \( C^*(G_F) \times_\alpha \mathbb{N}^F \) such that

\[ \begin{align*}
(\text{a}) & \quad I_1 \cong C(\mathcal{U}(\mathbb{Z}_F), \mathcal{K}(l^2(\mathbb{N}^F))); \\
(\text{b}) & \quad I_{k+1}/I_k \cong \bigoplus_{S \subseteq F, |S| = k}(C(\mathcal{U}(\mathbb{Z}_F^S)) \times_\sigma \mathbb{Z}^S) \otimes \mathcal{K}(l^2(\mathbb{N}^F \setminus S)); \\
(\text{c}) & \quad (C^*(G_F) \rtimes \mathbb{N}^F)/I_F \cong C(\mathcal{U}(\mathbb{Z}_F)).
\end{align*} \]

Each \( C(\mathcal{U}(\mathbb{Z}_F^S)) \times_\sigma \mathbb{Z}^S \) is a finite direct sum of simple AT-algebras with real rank zero and a unique tracial state.

The proof of the theorem will occupy the rest of the section. We will need some notation and a number of preliminary results.

Under the Fourier transform \( C^*(G_F) \cong C(\mathbb{Z}_F) \) the action \( \alpha \) becomes
\[
\alpha_{(n_p)}(f)(x) = \begin{cases} f((\prod_{p \in F} p^{n_p} x)^{-1}) & \text{if } x \in (\prod_{p \in F} p^{n_p} x) \mathbb{Z}_F, \\ 0 & \text{otherwise} \end{cases}
\]
(see \cite{10, Lemma 1.1}). For \( S \subseteq F \), we set \( \mathbb{Z}_S := \{ a \in \mathbb{Z}_F \mid a_p = 0 \text{ for } p \in S \} \), and we write \( \mathbb{Z}_p \) for \( \mathbb{Z}_{\{p\}} \). The next lemma identifies \( C_0(\mathbb{Z}_F \setminus \mathbb{Z}_S) \) as the kind of ideal for which taking crossed products behaves well (see \cite{9}).

**Lemma 2.2.** For \( S \subseteq F \), \( C_0(\mathbb{Z}_F \setminus \mathbb{Z}_S) \) is an extendibly invariant ideal in \( C(\mathbb{Z}_F) \).

**Proof.** It suffices by \cite{9, Theorem 4.3} to show that for each \( n \in \mathbb{N}^F \), the endomorphism \( x \mapsto (\prod_{p \in F} p^{n_p} x) \) of \( \mathbb{Z}_F \) leaves both \( \mathbb{Z}_S \) and \( \mathbb{Z}_F \setminus \mathbb{Z}_S \) invariant. Certainly \( (\prod_{p \in F} p^{n_p}) \mathbb{Z}_S \) is contained in \( \mathbb{Z}_S \). If \( x \notin \mathbb{Z}_S \), then \( x_r \neq 0 \) for some \( r \in S \), \( \prod_{p \in F} p^{n_p} x_r \neq 0 \) for this \( r \), and \( (\prod_{p \in F} p^{n_p}) x \notin \mathbb{Z}_S \). \( \square \)
Theorem 1.7 of [9] now allows us to identify $C_0(\mathbb{Z}_F \setminus \mathbb{Z}_S) \times N^F$ with an ideal $J_S$ in $C(\mathbb{Z}_F) \rtimes_\alpha N^F$ such that $(C(\mathbb{Z}_F) \rtimes_\alpha N^F)/J_S = C(\mathbb{Z}_S) \rtimes N^F$; we write $J_p$ for $J_{\{p\}}$.

**Lemma 2.3.** $J_S = \bigoplus_{p \in S} J_p$.

**Proof.** Since $\mathbb{Z}_S = \bigcap_{p \in S} \mathbb{Z}_p$, we have $\mathbb{Z}_F \setminus \mathbb{Z}_S = \bigcup_{p \in S} \mathbb{Z}_F \setminus \mathbb{Z}_p$, and $C_0(\mathbb{Z}_F \setminus \mathbb{Z}_S) = \bigoplus_{p \in S} C_0(\mathbb{Z}_F \setminus \mathbb{Z}_p)$. It follows from Lemma 1.3 of [9] that if $I$, $J$ and $I + J$ are extendibly invariant ideals in $(A, P)$, then $(I + J) \times P = (I \times P) + (J \times P)$. Thus the result follows from Lemma 2.2. □

For $1 \leq k \leq |F|$, we define

$$I_k := \prod_{S \subseteq C F, |S| = k} J_S = \bigcap_{S \subseteq C F, |S| = k} J_S.$$  

(2.1)

It follows from [9, Lemma 1.3] that if $I$ and $J$ are extendibly invariant ideals in $(A, P)$, then

$$(I \times P)(J \times P) = (IJ) \times P,$$

and hence $I_k = C_0(\bigcap_{S \subseteq C F, |S| = k} (\mathbb{Z}_F \setminus \mathbb{Z}_S)) \rtimes N^F$. Therefore

$$I_1 = C_0(\bigcap_{p \notin F} (\mathbb{Z}_F \setminus \mathbb{Z}_p)) \rtimes N^F = C_0(\bigcap_{p \notin F} (\mathbb{Z}_p \setminus 0)) \rtimes N^F,$$

since $\mathbb{Z}_p \setminus 0$ is homeomorphic to $U(\mathbb{Z}_p) \times \mathbb{N}$ by [10, Lemma 2.3], part (a) of Theorem 2.1 follows from an argument similar to the one in the last paragraph of [10, page 176]. Similarly, we can prove part (c) by following the proof of (2.4) of [10], because $(C^*(G_F) \rtimes_\alpha N^F)/I_{|F|} = \mathbb{C} \rtimes N^F$.

To prove part (b) of Theorem 2.1, we need some lemmas. The first contains some general facts about families of ideals in $C^*$-algebras.

**Lemma 2.4.** Suppose that $I_1, \ldots, I_n$ are ideals in a $C^*$-algebra $B$.

(a) With $F_n = \{1, \ldots, n\}$, we have

$$\prod_{S \subseteq C F_n, |S| = k} (\sum_{i \in S} I_i) = \sum_{R \subseteq C F_n, |R| = n - k + 1} (\prod_{j \in R} I_j) \quad \text{for } 1 \leq k \leq n.$$  

(2.2)

(b) Suppose that $K$ is an ideal such that $I_i I_j \subset K$ for all $i, j$. Then $(\sum_{i=1}^n I_i)/K$ is naturally isomorphic to $\bigoplus_{i=1}^n (I_i/I_i \cap K)$.

**Proof.** We prove (a) by induction on $n$. The statement is trivial for $n = 1, 2$. Suppose it holds for $n - 1$. When $k = 1$, both sides of (2.2) are $\prod_{i=1}^n I_i$, so we assume $k \geq 2$. Writing the left-hand side LHS of (2.2) as $(\prod_{n \in S})(\prod_{n \notin S})$ and applying the inductive hypothesis to $F_{n-1}$ shows that

$$\text{LHS} = \left(\prod_{|S| = k, n \in S} (I_n + \sum_{i \in S \setminus \{n\}} I_i)\right) \left(\sum_{R \subseteq C F_{n-1}, |R| = n - k} (\prod_{j \in R} I_j)\right).$$

(2.3)

Because $I_n$ is an ideal and $I_n^2 = I_n$, the first term of (2.3) simplifies to give

$$\text{LHS} = \left(\prod_{S' \subseteq C F_{n-1}, |S'| = k - 1} (\sum_{i \in S'} I_i)\right) \left(\sum_{R \subseteq C F_{n-1}, |R| = n - k} (\prod_{j \in R} I_j)\right).$$
We can use the inductive hypothesis on \( F_{n-1} \) with \( k \) replaced by \( k-1 \) to get
\[
(2.4) \quad \text{LHS} = \left( I_n + \sum_{R' \subseteq F_{n-1}, |R'|=n-k+1} (\prod_{j \in R'} I_j) \right) \left( \sum_{R \subseteq F_{n-1}, |R|=n-k} (\prod_{j \in R} I_j) \right),
\]
which is contained in
\[
(2.5) \quad \sum_{R \subseteq F_{n-1}, |R|=n-k} (\prod_{j \in R \cup \{n\}} I_j) + \sum_{R' \subseteq F_{n-1}, |R'|=n-k+1} (\prod_{j \in R'} I_j).
\]
Since (2.5) is the same as the right-hand side RHS of (2.2), LHS \( \subseteq \) RHS. On the other hand, every element of every \( \prod_{j \in R'} I_j \) arises in (2.4) because we can pick \( R \subseteq R' \), so RHS \( \subseteq \) LHS.

To prove (b), note that the map \( \phi_i : a + I_i \cap K \mapsto a + K \) is an injection of \( I_i/(I_i \cap K) \) into \( (\sum_{i=1}^n I_i)/K \), and
\[
\phi_i(a + I_i \cap K) = ab + K = 0 \quad \text{for } i \neq j
\]
because \( ab \in I_i I_j \subset K \). So the \( \phi_j \) combine to give an injection \( \phi \) of \( \bigoplus (I_i/I_i \cap K) \) into \( (\sum_{i=1}^n I_i)/K \), which is clearly surjective. \( \square \)

**Lemma 2.5.** The ideals \( I_k \) of \( C^*(G_F) \rtimes_\alpha \mathbb{N}^F \) defined in (2.1) satisfy
\[
I_{k+1}/I_k = \bigoplus_{S \subseteq F, |S|=k} \left( \bigcap_{p \notin S} J_{S \cup \{p\}} \right)/J_S.
\]

**Proof.** Lemma 2.4 (a) gives \( I_{k+1} = \sum_{R \subseteq F, |R|=n-k} (\prod_{p \in R} J_p) \). The product of any two ideals \( \prod_{p \in R} J_p \) with \( |R| = n-k \) has at least \( n-k+1 \) factors \( J_p \), and hence is contained in \( I_k = \sum_{R \subseteq F, |R|=n-k+1} (\prod_{p \in R} J_p) \). Thus part (b) of Lemma 2.4 gives
\[
(2.6) \quad I_{k+1}/I_k = \bigoplus_{R \subseteq F, |R|=n-k} \frac{\prod_{p \in R} J_p}{I_k \cap (\prod_{p \in R} J_p)}.
\]

Now
\[
I_k \cap (\prod_{p \in R} J_p) = \sum_{|T|=n-k+1} (\prod_{q \notin T} J_q)(\prod_{p \in R} J_p);
\]
each of these summands has at least one factor \( J_q \) for \( q \notin R \), and is then contained in \( J_q(\prod_{p \in R} J_p) \). Using \( I \cap J = IJ \) again gives
\[
I_k \cap (\prod_{p \in R} J_p) = \sum_{q \notin \bar{R}} J_q(\prod_{p \in \bar{R}} J_p) = (\sum_{q \notin \bar{R}} J_q)(\prod_{p \in \bar{R}} J_p),
\]
and using the isomorphism \( (I + J)/I = J/(I \cap J) \) and Lemma 2.3 gives
\[
\frac{\prod_{p \in R} J_p}{I_k \cap (\prod_{p \in R} J_p)} = \frac{J_{F \setminus R} + (\prod_{p \in R} J_p)}{J\setminus R}.
\]
Finally we observe that
\[
J_{F \setminus R} + (\prod_{p \in R} J_p) = \prod_{p \in R} (J_{F \setminus R} + J_p) = \prod_{p \in R} J_{(F \setminus R) \cup \{p\}}
\]
and write \( S = F \setminus R \) to deduce the result. \( \square \)

**Lemma 2.6.** The ideals \( J_S \) in \( C^*(G_F) \rtimes_\alpha \mathbb{N}^F \) satisfy
\[
\left( \bigcap_{p \in F \setminus S} J_{S \cup \{p\}} \right)/J_S \cong \left( C^*(U(\mathbb{Z}_F \setminus S)) \rtimes_\sigma \mathbb{Z}^S \right) \otimes K(\ell^2(\mathbb{N}^{F \setminus S})).
\]
Thus a standard argument shows that as claimed. □

Proposition 2.8. Let $\Gamma$ be a finite quotient of $\Z^k$. Then $\Gamma \rtimes \Z^k$ is isomorphic to $C(\overline{\Gamma^k}, M_{|\Gamma|}(\C))$. 

Proof. We first realise $(\bigcap_{p \in F \setminus S} J_{S \cup \{p\}}) / J_S$ as a semigroup crossed product:

$$\bigcap_{p \in F \setminus S} J_{S \cup \{p\}} = C_0(\bigcap_{p \in F \setminus S} (\Z_F \setminus \Z_{S \cup \{p\}})) \times \N^F = C_0(\Z_F \setminus (\bigcup_{p \in F \setminus S} \Z_{S \cup \{p\}})) \times \N^F.$$ 

Thus

$$(\bigcap_{p \in F \setminus S} J_{S \cup \{p\}}) / J_S = C_0(\Z_S \setminus (\bigcup_{p \in F \setminus S} \Z_{S \cup \{p\}})) \times \N^F = C_0((\prod_{p \in F \setminus S} \Z_p \setminus \{0\}) \times (\prod_{p \in S} \{0\})) \times \N^F.$$ 

The arguments of Corollary 2.4 and Lemma 2.5 of [10] show that this last crossed product is isomorphic to $(C(U(\Z_{F \setminus S})) \rtimes_\sigma \Z^S) \otimes K(I^F(\N^{F\setminus S})).$ □

Part (b) of Theorem 2.1 follows immediately from Lemmas 2.5 and 2.6.

To finish the proof of Theorem 2.1, it remains to prove the statements about the structure of $C(U(\Z_{F \setminus S})) \rtimes_\sigma \Z^S$. Corollary A.6 implies that $H := \overline{\Z^S}$ has finite index in $U(\Z_{F \setminus S})$. The argument at the end of the proof of [10, Theorem 3.1] shows that $C(U(\Z_{F \setminus S})) \rtimes_\sigma \Z^S$ is a finite direct sum of algebras isomorphic to $C(H) \rtimes_\sigma \Z^S$, which is simple because $\Z^S$ acts minimally and freely on $H$. Since $H$ is an open and closed subset of $U(\Z_{F \setminus S})$, it is totally disconnected, and it follows from [16, Theorem 6.11] that $C(H) \rtimes_\sigma \Z^S$ has real rank zero and stable rank one.

The space $U(\Z_{F \setminus S})$ is the inverse limit of the finite groups $U(\Z/(\prod_{p \in F \setminus S} p^p)\Z)$ over $l = (l_p) \in \N^{F \setminus S}$. The diagonally embedded copy of $N$ is cofinal in $\N^{F \setminus S}$, and hence

$$(2.7) \quad U(\Z_{F \setminus S}) = \varprojlim U(\Z/(\prod_{p \in F \setminus S} p^n)\Z).$$

Let $\pi_n$ denote the canonical surjection of $U(\Z_{F \setminus S})$ onto $U(\Z/(\prod_{p \in F \setminus S} p^n)\Z)$.

Lemma 2.7. Let $H_n := \pi_n(H) \subset U(\Z/(\prod_{p \in F \setminus S} p^n)\Z)$ and let $\Z^S$ act on $H_n$ via the embedding $(n_q) \mapsto \prod_{q \in S} q^{n_q}$ of $\Z^S$ in $\Z$. Then there are $C^*$-subalgebras $A_n$ of $C(H) \rtimes_\sigma \Z^S$ such that $A_n \cong C(H_n) \rtimes_\sigma \Z^S$ and $C(H) \rtimes_\sigma \Z^S = \bigcup A_n$.

Proof. The homomorphism $\pi_n$ induces an injection $\pi_n^* \circ \pi_n$ of $C(H_n)$ into $C(H)$, and then $C(H) = \bigcup_{n \in \N} \pi_n^*(C(H_n))$. On $\Z^S \subset H$, $\pi_n$ is reduction modulo $\prod_{p \in F \setminus S} p^n$, so $\pi_n^*$ converts the action $\sigma$ into the canonical action of $\Z^S$ by multiplication on $H_n$. Thus $\pi_n^*$ induces a homomorphism $\pi_n^* \rtimes id$ of $C(H_n) \rtimes \Z^S$ into $C(H) \rtimes_\sigma \Z^S$. The homomorphism $\pi_n^*$ is faithful on $C(H_n)$ and intertwines the dual actions, and hence a standard argument shows that $\pi_n^* \rtimes id$ is faithful on $C(H_n) \rtimes \Z^S$ (see, for example, [10, Lemma 4.2]). Since $\bigcup_n \pi_n^*(C(H_n))$ is dense in $C(H)$, we therefore have

$$C(H) \rtimes_\sigma \Z^S = \bigcup_{n \in \N} \pi_n^* \rtimes id(C(H_n) \rtimes \Z^S),$$

as claimed. □

We can identify the subalgebras $A_n$ explicitly.

Proposition 2.8. Let $F$ be a finite quotient of $\Z^k$. Then $C(F) \rtimes \Z^k$ is isomorphic to $C(\overline{\Gamma^k}, M_{|\Gamma|}(\C))$. 
Proof. Let \( H \) be the subgroup of \( \mathbb{Z}^k \) with \( F = \mathbb{Z}^k / H \). Then \( H \) is itself a free abelian group of rank \( k \), and hence has the form \( A \mathbb{Z}^k \) for some \( A \in M_k(\mathbb{Z}) \cap GL_k(\mathbb{Q}) \). The matrix \( A \) has a Smith normal form: there are matrices \( P, Q \in GL_k(\mathbb{Z}) \) such that \( B := P^{-1} A Q^{-1} \) is diagonal [12, §3.22]. Then \( H = A \mathbb{Z}^k = P B Q \mathbb{Z}^k = P B \mathbb{Z}^k \cong B \mathbb{Z}^k = b_1 \mathbb{Z} \oplus \cdots \oplus b_n \mathbb{Z} \). In other words, multiplying by \( P^{-1} \) gives an automorphism of \( \mathbb{Z}^k \) which carries \( H \) into \( \bigoplus b_i \mathbb{Z} \).

Thus

\[
C(F) \rtimes \mathbb{Z}^k \cong C(\prod_{i=1}^k (\mathbb{Z} / b_i \mathbb{Z})) \rtimes \mathbb{Z}^k \cong \bigotimes_{i=1}^k (C(\mathbb{Z} / b_i \mathbb{Z}) \rtimes \tau \mathbb{Z}),
\]

where \( \tau \) is the action of \( \mathbb{Z} \) by translation.

By [14, Corollary 2.5], \( C(\mathbb{Z} / b \mathbb{Z}) \rtimes \tau \mathbb{Z} \) is isomorphic to the induced algebra

\[
\text{Ind}_{(b \mathbb{Z})}^\mathbb{Z} (C(\mathbb{Z} / b \mathbb{Z}) \rtimes \tau (\mathbb{Z} / b \mathbb{Z}), \hat{\tau}),
\]

which is described in terms of a generator \( \beta \) of the dual action \( \hat{\tau} \) as the mapping torus

\[
MT(\beta) = \{ f : [0, 1] \rightarrow C(\mathbb{Z} / b \mathbb{Z}) \rtimes \tau (\mathbb{Z} / b \mathbb{Z}) \mid f(1) = \beta(f(0)) \}.
\]

Since \( C(\mathbb{Z} / b \mathbb{Z}) \rtimes \tau (\mathbb{Z} / b \mathbb{Z}) \cong B(\mathbb{Z} / b \mathbb{Z}) = M_{b_i}(\mathbb{C}) \), the automorphism \( \beta \) is inner, and there is a continuous path \( \beta_t \) in \( \text{Aut} M_{b_i}(\mathbb{C}) \) such that \( \beta_0 = \text{id} \) and \( \beta_1 = \beta \). Now \( \phi(f)(t) = \beta_t^{-1}(f(t)) \) defines an isomorphism \( \phi \) of (2.8) onto \( C(\mathbb{T}, M_{b_i}(\mathbb{C})) \). We therefore have

\[
C(F) \rtimes \mathbb{Z}^k \cong \bigotimes_{i=1}^k C(\mathbb{T}, M_{b_i}(\mathbb{C})) \cong C(\mathbb{T}^k, M_{\prod_i b_i}(\mathbb{C})),
\]

and the result follows on observing that \( \prod_i |b_i| = |\det B| = |\det A| = |F| \).

It follows from Proposition 2.8 and the decomposition \( C(H) \rtimes_{\sigma} \mathbb{Z}^S = \bigcup A_n \) that \( C(H) \rtimes_{\sigma} \mathbb{Z}^S \) is an AH-algebra\(^1\). The K-theory of \( C(H_n) \rtimes_{\sigma} \mathbb{Z}^S \) is torsion free and this property is preserved under inductive limits, so \( C(H) \rtimes_{\sigma} \mathbb{Z}^S \) has torsion free K-theory.

Thus, since \( C(H) \rtimes_{\sigma} \mathbb{Z}^S \) is a simple AH-algebra with real rank zero and no dimension growth, it is an AT-algebra by [17, Lemma 7.5].

(\text{The long story here, basically rewritten from [17, Lemma 7.5], is as follows: Denote} \( A = C(H) \rtimes_{\sigma} \mathbb{Z}^S \). Then \( K_1(A) \) is a weakly unperforated Riesz group. We need a modified version of Elliott’s “range of the invariant” theorem, Theorem 8.3 from \textit{On the classification of C*-algebras of real rank zero}, J. reine angew. Math. \textbf{443} (1993). This modified theorem says that there is a simple AT-algebra \( B \) with real rank zero and with the same ordered \( K \)-theory as that of \( A \). The modification in the proof of Elliott’s Theorem 8.3 consists of noticing that since \( K_1(A) \) is torsion free, all the algebras in the direct system constructed by Elliott must have primitive ideal space the circle or a point; in other words, no dimension-drop algebras occur, and the limit \( B \) will be an AT-algebra. Since all simple AH-algebras with real rank zero and no dimension growth have tracial rank zero by Proposition 2.6 of H. Lin, \textit{Tracially AF C*-algebras}, Trans. Amer. Math. Soc. \textbf{353} (2001), 693-722, it follows that \( A \) has tracial rank zero. In particular, \( B \) has tracial rank zero and hence \( A \cong B \) by the

\(^1\text{To see that an inductive limit} \bigcup A_n \text{ is an AH-algebra, it suffices to show that each } A_n \text{ is a corner in a matrix algebra } M_N(C(X)), \text{ or, equivalently, that } A_n \text{ is a homogeneous algebra with vanishing Dixmier-Douady class. Since the Dixmier-Douady class } \delta(A) \text{ of an } m\text{-homogeneous algebra satisfies } m\delta(A) = 0, \text{ and } H^3(\mathbb{T}^k, \mathbb{Z}) \text{ has no torsion, it suffices to prove that each } A_n \text{ is a homogeneous algebra with spectrum } \mathbb{T}^k. \text{ In our situation we could prove this in several ways. However, Proposition 2.8 makes the stronger statement that } A_n \text{ is isomorphic to } M_m(C(\mathbb{T}^k)). \)
classification theorem for $C^*$-algebras with tracial rank zero: this is Theorem 5.2 of H. Lin, Classification of simple $C^*$-algebras of tracial topological rank zero, preprint.)

We also use the decomposition $C(H) \rtimes \mathbb{Z}^S = \bigcup A_n$ to prove that $C(H) \rtimes \mathbb{Z}^S$ has a unique tracial state. Let $\mu$ denote the Haar measure on $H \subset \mathcal{U}(\mathbb{Z}_F \setminus S)$. The action $\sigma$ permutes the cylinder sets $\{\pi_n^{-1}(m) \mid m \in H_n\}$, so every invariant probability measure agrees with $\mu$ on cylinder sets. Since the characteristic functions of such sets span a dense subspace of $C(H)$, it follows that $\mu$ is the only invariant probability measure, and $C(H) \rtimes \mathbb{Z}^S$ has a unique tracial state by [3, Corollary VIII.3.8].

This completes the proof of Theorem 2.1.

3. The structure of $C(\mathcal{U}(\mathbb{Z}_F \setminus S)) \rtimes \sigma \mathbb{Z}^S$

3.1. When $S$ contains just one prime. We consider $C(\mathcal{U}(\mathbb{Z}_F \setminus S)) \rtimes \sigma \mathbb{Z}^S$ when $S = \{q\}$. To simplify the notation, we relabel $F \setminus \{q\}$ as $F$. The following result generalises [10, Theorem 3.1] in two directions: to sets $F$ with $|F| > 1$ and to sets $F$ containing the even prime $2$. If $l = (l_p) \in \mathbb{N}^F$ is a multi-index, we write $o_l(q)$ for the order of $q$ in $\prod_{p \in F} \mathcal{U}(\mathbb{Z}/p^l_p \mathbb{Z})$.

**Theorem 3.1.** Suppose $F$ is a finite set of primes and $q$ is a prime which does not belong to $F$. Then there are a multi-index $K = (K_p) \in \mathbb{N}^F$ and $d \in \mathbb{N}$ such that

$$o_K + 1(q) = d(\prod_{p \in F} p^{l_p}) \text{ for every } l \in \mathbb{N}^F,$$

and $C(\mathcal{U}(\mathbb{Z}_F)) \rtimes \sigma \mathbb{Z}$ is the direct sum of $\left(\prod_{p \in F} (p-1)p^{K_p-1}\right)/d$ copies of a Bunce-Deddens algebra with supernatural number $d(\prod_{p \in F} p^\infty)$.

The existence of $K$ and $d$ satisfying (3.1) is established in Proposition A.5. We saw in §2 that $C(\mathcal{U}(\mathbb{Z}_F)) \rtimes \sigma \mathbb{Z}$ is the direct sum of copies of the simple algebra $C(H) \rtimes \sigma \mathbb{Z}$, where $H$ is the closure of $q^\mathbb{Z}$ in $\mathcal{U}(\mathbb{Z}_F)$. It remains to prove that $C(H) \rtimes \sigma \mathbb{Z}$ is a Bunce-Deddens algebra and to calculate the index $[\mathcal{U}(\mathbb{Z}_F) : H]$, which is the number of simple direct summands.

Let $\{n_k\}$ be integers with $n_k \geq 2$, and let $X_k = \{0, 1, \ldots, n_k - 1\}$. Addition by 1 with carry over is a homeomorphism of the totally disconnected space $X := \prod_{k \geq 0} X_k$ called an odometer action, and the resulting crossed product $C(X) \rtimes \tau \mathbb{Z}$ is a Bunce-Deddens algebra with supernatural number $n := \prod_{k \geq 0} n_k$ (see [3, Chapter VIII.4]).

Our claim that $C(H) \rtimes \sigma \mathbb{Z}$ is a Bunce-Deddens algebra will follow from the next proposition, which generalises [10, Proposition 3.6].

**Proposition 3.2.** Suppose $\{G_l \in \mathbb{N}\}$ are finite groups and $G = \varprojlim(G_l, \pi_l)$. Fix $g \in G$ and let $L$ denote the closed subgroup of $G$ generated by $g$. Consider the action $\sigma : \mathbb{Z} \to \text{Aut} C(G)$ such that $\sigma_n(f)(x) = f(g^{-n}x)$. Let $o_l(g)$ denote the order of $\pi_l(g)$ in $G_l$, and let

$$d_l := \begin{cases} o_0(g) & \text{if } l = 0 \\ o_l(g)/o_{l-1}(g) & \text{if } l \geq 1. \end{cases}$$

Then $C(L) \rtimes \sigma \mathbb{Z}$ is a Bunce-Deddens algebra with supernatural number $\prod_{l \geq 0} d_l$. 

Proof. Let \( X := \prod_{l \geq 0} [0, 1, \ldots, d_l - 1] \). The argument in the proof of [10, Proposition 3.6] shows that the continuous maps \( h_t : X \to G_t \) given by
\[
(3.3) \quad h_t(a_n) = \pi_t(g_{a_0+a_1d_0+\cdots+a_id_0\ldots d_{i-1}})
\]
combine to give an equivariant homeomorphism \( h : X \to L \) which induces the required isomorphism. \( \square \)

Our subgroup \( H \) of \( \mathcal{U}(\mathbb{Z}_F) \) is the inverse limit \( \lim \pi_i(H) \), where \( \pi_i : \mathcal{U}(\mathbb{Z}_F) \to \mathcal{U}(\mathbb{Z}/(\prod_{p \in F} p^{K_p+1})\mathbb{Z}) \) is the canonical surjection. Then Proposition 3.2 and (3.1) imply that \( C(H) \rtimes_{\sigma} \mathbb{Z}^2 \) is a Bunce-Deddens algebra with supernatural number \( d(\prod_{p \in F} p)^{\infty} \) for \( d = o_K(q) \). By Corollary A.6, we have that
\[
(3.4) \quad |\mathcal{U}(\mathbb{Z}_F) : H| = (\prod_{p \in F} (p - 1)p^{K_p-1})/d,
\]
which finishes the proof of Theorem 3.1.

3.2. When \( S \) consists of two primes. We now analyse \( C(\mathcal{U}(\mathbb{Z}_F) \times \mathbb{Z}^S) \) when \( S = \{q, r\} \). For simplicity, we consider only the case \( F = \{p, q, r\} \), so that we are interested in the crossed product \( C(\mathcal{U}(\mathbb{Z}_p)) \rtimes_{\sigma} \mathbb{Z}^2 \), where
\[
\sigma_{m,n}(f)(x) = f(q^{-m}r^{-n}x).
\]

Theorem 3.3. The \( C^* \)-algebra \( C(\mathcal{U}(\mathbb{Z}_p)) \rtimes_{\sigma} \mathbb{Z}^2 \) is a finite direct sum of copies of a simple \( AT \)-algebra \( A \) which has real rank zero, a unique tracial state and \( K \)-theory satisfying two short exact sequences:
\[
\begin{align*}
0 & \longrightarrow \mathbb{Z}[p^{-1}] \longrightarrow K_0(A) \longrightarrow \mathbb{Z} \longrightarrow 0 \\
0 & \longrightarrow \mathbb{Z} \longrightarrow K_1(A) \longrightarrow \mathbb{Z}[p^{-1}] \longrightarrow 0.
\end{align*}
\]

Everything except the assertion about \( K \)-theory was proved in Theorem 2.1; the simple \( C^* \)-algebra \( A \) is \( C(H) \rtimes_{\sigma} \mathbb{Z}^2 \), where \( H \) is the closure of \( q\mathbb{Z}^r \) in \( \mathcal{U}(\mathbb{Z}_p) \). We aim to analyse \( C(H) \rtimes_{\sigma} \mathbb{Z}^2 \) by writing it as an iterated crossed product \( (C(H) \rtimes_{\sigma} \mathbb{Z}) \rtimes \mathbb{Z} \rtimes \mathbb{Z} \). The inside crossed product is not simple unless \( q\mathbb{Z} \) is dense in \( H \), and it is helpful to reduce to this case using the following lemma.

Lemma 3.4. Let \( H_q \) denote the closure of \( q\mathbb{Z} \) in \( \mathcal{U}(\mathbb{Z}_p) \). Then \( H_q \) has finite index \( I(q) \) in \( H \), and hence is an open and closed subset of \( H \). The inclusion of \( C(H_q) \) in \( C(H) \) induces an isomorphism of \( C(H_q) \rtimes_{\sigma} (\mathbb{Z} \times I(q)\mathbb{Z}) \) onto the corner \( \chi_{H_q}(C(H) \rtimes_{\sigma} \mathbb{Z}^2)\chi_{H_q} \).

Proof. Corollary A.6 implies that \( H_q \) has finite index in \( \mathcal{U}(\mathbb{Z}_p) \), so it certainly has finite index in \( H \). The inclusion of \( C(H_q) \) in \( C(H) \) and the map
\[
(m, I(q)n) \mapsto \chi_{H_q}i_{\mathbb{Z}^2}(m, I(q)n)\chi_{H_q}
\]
form a covariant representation of \( (C(H_q), \mathbb{Z} \times I(q)\mathbb{Z}, \sigma) \) in \( \chi_{H_q}(C(H) \rtimes_{\sigma} \mathbb{Z}^2)\chi_{H_q} \), and hence give a homomorphism
\[
\phi : C(H_q) \rtimes_{\sigma} (\mathbb{Z} \times I(q)\mathbb{Z}) \to \chi_{H_q}(C(H) \rtimes_{\sigma} \mathbb{Z}^2)\chi_{H_q}.
\]
We can identify \( (\mathbb{Z} \times I(q)\mathbb{Z})^\wedge \) with \( \mathbb{T}^2/(\mathbb{Z} \times I(q)\mathbb{Z})^\perp = \mathbb{T}^2/(1 \times C_I(q)) \), where \( C_n \) denotes the group of \( n \)th roots of unity, and then \( \phi \) carries the dual action \( \sigma_{[w,z]} \) into \( \sigma_{w,z} \).
Thus every \( \chi_{H_q}(f_{i\mathbb{Z}^2}(m,n))\chi_{H_q} = (f\chi_{H_q})i_{\mathbb{Z}^2}(m,n)\chi_{H_q} = i_{\mathbb{Z}^2}(m,n)\sigma_{m,n}^{-1}(f\chi_{H_q})\chi_{H_q} \).

Since the support of \( \sigma_{m,n}^{-1}(f\chi_{H_q}) \) is contained in \( q^{-m-r-n}H_q = r^{-n}H_q \), we have

\[
\sigma_{m,n}^{-1}(f\chi_{H_q})\chi_{H_q} = \begin{cases} 
\sigma_{m,n}^{-1}(f\chi_{H_q}) & \text{if } I(q) \text{ divides } n \\
0 & \text{otherwise,}
\end{cases}
\]

and

\[
\chi_{H_q}(f_{i\mathbb{Z}^2}(m,n))\chi_{H_q} = \begin{cases} 
i_{\mathbb{Z}^2}(m,n)\sigma_{m,n}^{-1}(f\chi_{H_q}) & \text{if } I(q) \text{ divides } n \\
0 & \text{otherwise}
\end{cases}
= \begin{cases} 
\phi((f\chi_{H_q})i_{\mathbb{Z}^2}(m,n)) & \text{if } I(q) \text{ divides } n \\
0 & \text{otherwise.}
\end{cases}
\]

Thus every \( \chi_{H_q}(f_{i\mathbb{Z}^2}(m,n))\chi_{H_q} \) is in the range of \( \phi \), and \( \phi \) is surjective. \( \square \)

**Corollary 3.5.** Define \( \gamma : \mathbb{Z} \to \text{Aut}(C(H_q) \rtimes_{\sigma q} \mathbb{Z}) \) by

\[
(3.6) \quad \gamma_m(f_{i\mathbb{Z}}(n)) = \sigma_{I(q)m}^{r}(f_{i\mathbb{Z}}(n)).
\]

Then \( (C(H_q) \rtimes_{\sigma q} \mathbb{Z}) \rtimes_{\sigma} \mathbb{Z} \) is isomorphic to a full corner in \( C(H) \rtimes_{\sigma} \mathbb{Z}^2 \).

**Proof.** Theorem 4.1 of [15] gives \( C(H_q) \rtimes_{\sigma} (\mathbb{Z} \times I(q)\mathbb{Z}) \cong (C(H_q) \rtimes_{\sigma q} \mathbb{Z}) \rtimes I(q)\mathbb{Z} \), so the result follows from Lemma 3.4 on replacing \( I(q)\mathbb{Z} \) by the isomorphic group \( \mathbb{Z} \). \( \square \)

The analysis in §3.1 shows that \( C(H_q) \rtimes_{\sigma q} \mathbb{Z} \) is a Bunce-Deddens algebra. The \( K \)-theory of Bunce-Deddens algebras is well-known. To state the version we need, recall that if \( n = (n_k)_{k \geq 0} \) is a sequence with \( n_k \geq 2 \), then \( \mathbb{Z}[n^{-1}] \) denotes the set of rational numbers with denominator \( N_k = \prod_{i=0}^{k} n_i \) for some \( k \geq 0 \).

**Proposition 3.6.** Suppose \( n = (n_k)_{k \geq 0} \), \( X_k = \{0, \ldots, n_k - 1\} \), \( X = \prod X_k \) and \( \tau : \mathbb{Z} \to \text{Aut}(C(X)) \) is the associated odometer. Then there are isomorphisms \( \phi_0 : K_0(C(X) \rtimes_{\tau} \mathbb{Z}) \to \mathbb{Z}[n^{-1}] \) such that \( \phi_0([\chi_{J(a_0, \ldots, a_k)}]) = N_k^{-1} \) for each cylinder set \( J(a_0, \ldots, a_k) \), and \( \phi_1 : K_1(C(X) \rtimes_{\tau} \mathbb{Z}) \to \mathbb{Z} \) such that \( \phi_1(i_{\mathbb{Z}}(1)) = 1 \).

**Proof.** Because \( K_1(C(X)) = 0 \), the Pimsner-Voiculescu sequence for the system \((C(X), \mathbb{Z}, \tau)\) reduces to

\[
0 \to K_1(C(X) \rtimes_{\tau} \mathbb{Z}) \xrightarrow{\delta} K_0(C(X)) \xrightarrow{\text{id} - \tau_*} K_0(C(X)) \xrightarrow{\text{id}} K_0(C(X) \rtimes_{\tau} \mathbb{Z}) \to 0.
\]

Now let \( C_k = \{J(a_0, \ldots, a_k)\} \) be the set of cylinder sets of length \( k + 1 \), and note that \( C(X) = \bigcup_{k=1}^{\infty} A_k \), where \( A_k = \text{span}\{\chi_J \mid J \in C_k\} \). Each \( \chi_J \) for \( J \in C_k \) is the sum of \( n_{k+1} \) basis elements of \( A_{k+1} \), so the maps \( [\chi_{J(a_0, \ldots, a_k)}] \mapsto N_k^{-1} \) of \( A_k \) into \( \mathbb{R} \) combine to give a homomorphism \( \psi_0 \) of \( K_0(C(X)) \cong \lim_k K_0(A_k) \) into \( \mathbb{R} \) with range \( \mathbb{Z}[n^{-1}] \). Since the generating automorphism \( \tau = \tau_1 \) permutes \( C_k \), the kernel of \( \psi_0 \) is the range of \( \text{id} - \tau_* \), and hence \( \psi_0 \) induces the required isomorphism \( \phi_0 \) of \( K_0(C(X) \rtimes_{\sigma} \mathbb{Z}) \) onto \( \mathbb{Z}[n^{-1}] \). To verify the statement about \( K_1 \), recall that \( \delta \) is the coboundary map for the Toeplitz extension of \( C(X) \rtimes_{\tau} \mathbb{Z} \) (see [18, §2]), and compute the index of \( i_{\mathbb{Z}}(1) \) in \( K_0(C(X) \otimes \mathcal{K}) \cong K_0(C(X)) \). \( \square \)
Proof of Theorem 3.3. We saw in the proof of Proposition 3.2 and in the paragraph following it that the homeomorphism \( h \) of \( \prod_{k \geq 0} X_k \) onto the subgroup \( H_q \) of \( \mathcal{U}(\mathbb{Z}_p) \) satisfies
\[
\pi_k(h(\{a_n\})) = \pi_k(q^{a_0 + a_1 \sigma(q) + \cdots + a_k \sigma(q)p^{k-1}}) \quad \text{for } k \geq 0,
\]
and hence carries \( J(a_0, \ldots, a_k) \) onto \( Z(q^{a_0 + a_1 \sigma(q) + \cdots + a_k \sigma(q)p^{k-1}}) \), where
\[
Z_k(n) = \{ x \in \mathcal{U}(\mathbb{Z}_p) \mid \pi_k(x) = \pi_k(n) \}.
\]
So we deduce from Proposition 3.6 that there is an isomorphism \( \phi_0 \) of \( K_0(C(H_q) \rtimes_{\sigma} \mathbb{Z}) \) onto \( \frac{1}{o_p(q)} \mathbb{Z}[p^{-1}] \) such that
\[
\phi_0([\chi_{Z_k}(m)]) = \frac{1}{o_p(q)} \frac{1}{p^k}
\]
for every integer \( m \) which lies in \( H_q \).

Multiplying by the unit \( r^{-l(q)} \) carries \( Z_k(m) \) into \( Z_k(r^{-l(q)}m) \), and hence \( \phi_0 \circ (\gamma_l)_* = \phi_0 \). Thus \( (\gamma_l)_* \) is the identity on \( K_0(C(H_q) \rtimes_{\sigma} \mathbb{Z}) \). It is also the identity on \( K_1(C(H_q) \rtimes_{\sigma} \mathbb{Z}) \), and hence the Pimsner-Voiculescu sequence for \( ((C(H_q) \rtimes_{\sigma} \mathbb{Z}), \mathbb{Z}, \gamma) \) collapses to the two short exact sequences
\[
0 \rightarrow \mathbb{Z}[p^{-1}] \rightarrow K_0(C(H_q) \rtimes \mathbb{Z}^2) \rightarrow \mathbb{Z} \rightarrow 0
\]
\[
0 \rightarrow \mathbb{Z} \rightarrow K_1(C(H_q) \rtimes \mathbb{Z}^2) \rightarrow \frac{1}{o_p(q)} \mathbb{Z}[p^{-1}] \rightarrow 0.
\]
From this and Corollary 3.5 we can deduce (3.5); since the isomorphism induced by Corollary 3.5 scales the class of [1], we have removed the factor \( o_p(q)^{-1} \) by a further scaling to ensure that the final statement does not depend on the order of factors in our decomposition. \( \square \)

Remark 3.7. The number of simple summands in Theorem 3.3 is \( |\mathcal{U}(\mathbb{Z}_p) : H| \), and we can compute this using [10, Lemma 3.7]. For example, if \( p \) is odd and \( l \) is large, we have from (A.1) that
\[
|\pi_l(H)| = [o_p(q), o_p(r)] = [p^{l-L_p(q)}o_p(q), q^{l-L_p(r)}o_p(r)] = p^{l-min(L_p(q),L_p(r))}[o_p(q), o_p(r)];
\]
thus we deduce
\[
|\mathcal{U}(\mathbb{Z}_p) : H| = |\mathcal{U}(\mathbb{Z}/p^l\mathbb{Z}) : \pi_l(H)| = \frac{(p-1)p^{min(L_p(q),L_p(r))-1}}{|[o_p(q), o_p(r)]|}.
\]
We could carry out a similar analysis when \( |F| > 1 \), though it would not be so easy to work out some of the indices explicitly.

Remark 3.8. Theorem 2.1 implies in particular that \( C(H) \rtimes_{\sigma} \mathbb{Z}^2 \) satisfies the hypotheses of the classification theorem of Elliott for AT-algebras [20, Theorem 3.2.6]. We can tell from the computation of \( K \)-theory in Theorem 3.3 that \( C(H) \rtimes_{\sigma} \mathbb{Z}^2 \) is not a Bunce-Deddens algebra, but it is still closely related to an odometer. The homeomorphism of \( \prod_{k \geq 0} X_k \) onto \( H_q \) identifies the action of the first copy of \( \mathbb{Z} \) (multiplication by \( q \) on \( H_q \)) with an odometer (addition of 1 with carry over). The action of the second copy of \( \mathbb{Z} \) (multiplication by \( r \) on \( H_q \)) also acts as a permutation on each \( X_k \):
it moves $X_0$ around in a different order, and this action carries over into $X_1$ when the marker in $X_0$ returns to the starting point. So we can think of the action of $\mathbb{Z}^2$ as two independent odometers on the same set. We can normalise the scale so that either copy of $\mathbb{Z}$ acts by addition of 1 with carry over, but not so that both act this way at once.

4. The Bost-Connes algebra

The Hecke $C^*$-algebra $C_Q$ of Bost and Connes [2] is isomorphic to the semigroup crossed product $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\sigma \mathbb{N}^*$. The Fourier transform takes $C^*(\mathbb{Q}/\mathbb{Z})$ onto the algebra of continuous functions on the compact group $\mathcal{Z} := \prod_{p \in \mathcal{P}} \mathbb{Z}_p$ and carries $\alpha$ into the action given by (see [6, §3.1])

$$\alpha_n(f)(x) = \begin{cases} f(x/n) & \text{if } n \text{ divides } x \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.2 is valid with $F$ replaced by $\mathcal{P}$ and $\mathbb{Z}_F$ by $\mathcal{Z}$. Thus for $S \subset \mathcal{P}$, an application of [9, Theorem 1.7] gives that $J_S := C_0(\mathbb{Z} \setminus \mathcal{Z}_S) \rtimes_0 \mathbb{N}^*$ is an ideal of $C_Q = C(\mathcal{Z}) \rtimes \mathbb{N}^*$, with quotient isomorphic to $C(\mathcal{Z}_S) \rtimes \mathbb{N}^*$. Choose $a \in \mathcal{Z}$ such that $a_p = 0 \iff p \in S$. Then $\{\mathbb{Q}_a^* \cap \mathcal{Z}\}$ has closure $\mathcal{Z}_S$, so $C_0(\mathbb{Z} \setminus \mathcal{Z}_S)$ is the kernel of the representation $\pi_a$ considered in [8, page 440], and it follows from [10, Lemma 4.2] that $J_S$ is the kernel of the representation $\pi_a \times \nu$ described in [8, page 440]. We can now deduce that $S \mapsto J_S$, as $S$ runs through the proper subsets of $\mathcal{P}$, is the parametrisation of $(\text{Prim } C_Q) \setminus \overline{\mathbb{Q}_+^*}$ given in [8, Theorem 2.8].

Theorem 4.1. Suppose that $S$ is a proper subset of $\mathcal{P}$.

(a) If $\mathcal{P} \setminus S$ is infinite, then $J_S = \bigcap_{p \notin S} J_{S \cup \{p\}}$.

(b) If $0 < |\mathcal{P} \setminus S| < \infty$, then

$$\left(\bigcap_{p \notin S} J_{S \cup \{p\}}\right)/J_S \cong C(\mathcal{U}(\mathbb{Z}_F \setminus S)) \rtimes_{\sigma} \mathbb{Z}^S \otimes K(I^2(\mathbb{N}^{\mathcal{P} \setminus S})).$$

(c) $C_Q/J_\mathcal{P}$ is isomorphic to $C^*(\mathbb{Q}_+^*) = C(\overline{\mathbb{Q}_+^*})$.

Moreover, $C(\mathcal{U}(\mathbb{Z}_F \setminus S)) \rtimes_{\sigma} \mathbb{Z}^S$ is a simple AT-algebra with real rank zero and a unique tracial state.

It follows from [8, §2] that every basic open neighbourhood of $J_S$ has the form

$$U_G = \{J_T \mid T \subset \mathcal{P}, T \cap G = \emptyset\}$$

for some finite subset $G$ of $\mathcal{P} \setminus S$. When $\mathcal{P} \setminus S$ is infinite, there are always lots of $J_{S \cup \{p\}}$ in $U_G$, and thus $J_S \in \{J_{S \cup \{p\}} \mid p \notin S\}$; this says precisely that $\bigcap_{p \notin S} J_{S \cup \{p\}} \subset J_S$. The other inclusion is trivial, and (a) follows. Part (c) is true because $Z_\mathcal{P} = \{0\}$. To prove (b) we just need to replace $F$ by $\mathcal{P}$ and $\mathbb{Z}_F$ by $\mathcal{Z}$ in the proof of Lemma 2.6.

It remains to prove the statements about $C(\mathcal{U}(\mathbb{Z}_F \setminus S)) \rtimes_{\sigma} \mathbb{Z}^S$. The Chinese Remainder Theorem implies that $\mathbb{Z}$ is dense in $\mathbb{Z}_F \setminus S$, and hence $\mathbb{Z}^S = \mathbb{Z} \cap \mathcal{U}(\mathbb{Z}_F \setminus S)$ is dense in $\mathcal{U}(\mathbb{Z}_F \setminus S)$. Thus $\mathbb{Z}^S$ acts minimally and freely on $\mathcal{U}(\mathbb{Z}_F \setminus S)$, and $C(\mathcal{U}(\mathbb{Z}_F \setminus S)) \rtimes_{\sigma} \mathbb{Z}^S$ is simple. However, since $|S| = \infty$, we cannot apply the results of [16] to conclude that $C(\mathcal{U}(\mathbb{Z}_F \setminus S)) \rtimes_{\sigma} \mathbb{Z}^S$ has real rank zero and stable rank one, as we did in Section 2 for $C(H) \rtimes_{\sigma} \mathbb{Z}^S$. Instead we aim to use Theorems 1 and 2 of [1], and to do this we
need to show that \( C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes_\sigma \mathbb{Z}^S \) is an AH-algebra with the extra property of slow dimension growth.

Since \( \mathcal{P} \setminus S \) is finite, we have as in (2.7) that \( \mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S}) \) is the inverse limit of the finite groups \( \mathcal{U}(\mathbb{Z}/(\prod_{p \in \mathcal{P}\setminus S} p^{l_p})) \mathbb{Z} \) over \( l = (l_p) \in \mathbb{N}^{\mathcal{P}\setminus S} \). Hence \( \mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S}) = \lim F_n \), where

\[
F_n := \mathcal{U}(\mathbb{Z}/(\prod_{p \in \mathcal{P}\setminus S} p^n) \mathbb{Z}).
\]

We denote the canonical surjection of \( \mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S}) \) onto \( F_n \) by \( \pi_n \). The analogue of Lemma 2.7 for \( F_n \) and the canonical action of \( \mathbb{Z}^S \) by multiplication on \( F_n \) implies that \( C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes_\sigma \mathbb{Z}^S \) is the closed union of \( C^* \)-subalgebras isomorphic to \( C(F_n) \rtimes \mathbb{Z}^S \).

Towards applying Proposition 2.8, we note that the infinite direct sum \( \mathbb{Z}^S \) is the union of the subgroups \( \mathbb{Z}^E \) associated to finite subsets \( E \) of \( S \). Thus by using an argument similar to that in Lemma 2.7 we have that \( C(F_n) \rtimes \mathbb{Z}^S \) is the closed union of subalgebras isomorphic to \( C(F_n) \rtimes \mathbb{Z}^E \). However, by choosing a particular sequence \( E_n \) of finite subsets of \( S \), we can show that \( C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes_\sigma \mathbb{Z}^S \) has slow dimension growth.

Indeed, since \( \mathbb{Z}/(\prod_p p^n) \mathbb{Z} \cong \prod_p \mathbb{Z}/p^n \mathbb{Z} \), we have \( F_n \cong \prod_{p \in \mathcal{P}\setminus S} \mathcal{U}(\mathbb{Z}/p^n \mathbb{Z}). \) Thus \( F_n \) is a product of at most \( |\mathcal{P}\setminus S| + 1 \) cyclic groups (the +1 allows for the possibility that \( 2 \in \mathcal{P}\setminus S \)), and hence has a generating set \( \{x_{n,i}\} \) with at most \( |\mathcal{P}\setminus S| + 1 \) elements.

By Dirichlet’s Theorem, there are primes \( q_{n,i} \) such that

\[
q_{n,i} \equiv x_{n,i} \pmod{\prod_{p \in \mathcal{P}\setminus S} p^n},
\]

and each \( q_{n,i} \) belongs to \( S \) because it is a unit modulo \( \prod_{p \in \mathcal{P}\setminus S} p^n \). Now let \( E'_n := \{q_{n,i}\} \), list the primes in \( S \) as \( \{r_n | n \in \mathbb{N}\} \), and take

\[
E_n := (\bigcup_{m \leq n} E'_m) \cup \{r_1, \ldots, r_n\}.
\]

We then have \( \pi_n(\mathbb{Z}^{E_n}) = F_n, E_m \subset E_n \) for \( m \leq n \), and \( \bigcup E_n = S \).

We have now realised \( C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes_\sigma \mathbb{Z}^S \) as the closure of an increasing union \( \bigcup_{n \in \mathbb{N}} B_n \) in which \( B_n \) is isomorphic to the crossed product \( C(F_n) \rtimes \mathbb{Z}^{E_n} \) by a transitive action of \( \mathbb{Z}^{E_n} \). By an argument identical to the one at the end of Section 2 we conclude that \( C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes \mathbb{Z}^S \) has a unique tracial state.

We prove next that \( C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes \mathbb{Z}^S \) is an AH-algebra with real rank zero. Proposition 2.8 implies that \( B_n \cong C(F_n) \rtimes \mathbb{Z}^{E_n} \cong C(T|E_n|, M_{|E_n|}(\mathbb{C})) \). But

\[
\frac{|E_n|}{|F_n|} \leq \frac{n(|\mathcal{P}\setminus S| + 2)}{\prod_p (p - 1)p^{n-1}} \to 0 \text{ as } n \to \infty,
\]

and thus the sequence \( B_n \cong C(F_n) \rtimes \mathbb{Z}^{E_n} \) of subalgebras of \( C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes \mathbb{Z}^S \) has slow dimension growth. It now follows from [1, Theorem 1] that \( C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes \mathbb{Z}^S \) has topological stable rank one. Since the projections in \( C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes \mathbb{Z}^S \) trivially separate the unique tracial state, [1, Theorem 2] implies that \( C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes \mathbb{Z}^S \) has real rank zero. The K-groups of \( C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes \mathbb{Z}^S \) are inductive limits of torsion free groups, and hence are themselves torsion free, so it follows as in Section 2 that \( C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes \mathbb{Z}^S \) is an AT-algebra.

This completes the proof of Theorem 4.1.
Appendix A. The orders of a prime in groups of units

For $p$ prime and $m \in \mathbb{N}$ such that $(m, p) = 1$, we denote by $o_p(m)$ the order of $m$ in $\mathbb{U}(\mathbb{Z}/p^l\mathbb{Z})$. It was shown in [10, Theorem 3.1] that if $p$ is odd, there is a positive integer $L_p(m)$ such that

\begin{equation}
\label{eq:A1}
o_p(m) = \begin{cases} o_p(m) & \text{if } 1 \leq l \leq L_p(m) \\
p^{l-L_p(m)}o_p(m) & \text{if } l > L_p(m); \end{cases}
\end{equation}

the proof uses that the groups $\mathbb{U}(\mathbb{Z}/p^l\mathbb{Z})$ are cyclic. We will now show how to modify the arguments of [10, §3] to obtain an analogue of \eqref{eq:A1} for $p = 2$, in which case $\mathbb{U}(\mathbb{Z}/2^l\mathbb{Z})$ are no longer cyclic.

**Proposition A.1.** If $m$ is an odd integer and $m \equiv 1 \pmod{4}$, then there exists a positive integer $K = L_2(m)$ such that

\begin{equation}
\label{eq:A2}
o_2(m) = \begin{cases} 1 & \text{if } 1 \leq l \leq K \\
2^{l-K} & \text{if } l > K; \end{cases}
\end{equation}

if $m \equiv 3 \pmod{4}$, then there exists a positive integer $L = L_2(m)$ such that

\begin{equation}
\label{eq:A3}
o_2(m) = \begin{cases} 1 & \text{if } l = 1 \\
2 & \text{if } 1 < l \leq L \\
2^{l-(L-1)} & \text{if } l > L.
\end{cases}
\end{equation}

To prove Proposition A.1 we use general properties of cyclic groups as in [10, §3]. We begin with a lemma.

**Lemma A.2.** Suppose $l \geq 3$. Then the group $\{n \in \mathbb{U}(\mathbb{Z}/2^l\mathbb{Z}) \mid n \equiv 1 \pmod{4}\}$ is the cyclic subgroup $\langle 5 \rangle$ of $\mathbb{U}(\mathbb{Z}/2^l\mathbb{Z})$ generated by 5.

**Proof.** Theorem 2' of [4, Chapter 4.1] says that $|\langle 5 \rangle| = 2^{l-2}$. For $k \geq 0$ we have

$$5^k = (4 + 1)^k = \sum_{n=0}^{k} \binom{k}{n} 4^n = 4 \sum_{n=1}^{k} \binom{k}{n} 4^{n-1} + 1,$$

so $5^k \equiv 1 \pmod{4}$. Hence, if $n \equiv 5^k \pmod{2^l}$ for some $0 \leq k < 2^{l-2}$, then $n \equiv 1 \pmod{4}$. Since the order of $\{n \in \mathbb{U}(\mathbb{Z}/2^l\mathbb{Z}) \mid n \equiv 1 \pmod{4}\}$ is also $2^{l-2}$, the result follows.

**Corollary A.3.** An element of $\mathbb{U}(\mathbb{Z}/2^l\mathbb{Z})$ is congruent to 3 (mod 4) if and only if it is congruent to $-5^k \pmod{2^l}$ for some $k$ satisfying $0 \leq k < 2^{l-2}$.

**Corollary A.4.** Suppose $m \in \mathbb{Z}$ satisfies $m \equiv 1 \pmod{4}$. Then for every $l > 0$ we have

\begin{equation}
\label{eq:A4}
o_2(m) = \begin{cases} o_{2^{l+1}}(m) & \text{if } 2 \text{ does not divide } o_{2^{l+1}}(m) \\
o_{2^{l+1}}(m)/2 & \text{if } 2 \text{ divides } o_{2^{l+1}}(m). \end{cases}
\end{equation}

**Proof.** Since a number is coprime to $2^l$ if and only if it is coprime to $2^{l+1}$, the reduction map $\pi : \mathbb{U}(\mathbb{Z}/2^{l+1}\mathbb{Z}) \to \mathbb{U}(\mathbb{Z}/2^l\mathbb{Z})$ is a surjective homomorphism. Lemma A.2 implies that $m \equiv 5^r \pmod{2^{l+1}}$, where $r = o_{2^{l+1}}(5)/o_{2^{l+1}}(m) = 2^{l-1}/o_{2^{l+1}}(m)$. Hence, by
applying \([10, \text{Lemma 3.2}]\) to the restriction of \(\pi\) to a homomorphism of \(\langle 5 \rangle_{l+1}\) onto \(\langle 5 \rangle_l\), we have

\[
o_{2^l}(m) = o(\pi(5^l))
= \begin{cases} 
2^{l-1}/(2^{l-1}/o_{2^{l+1}}(m), 2^{l-1}) & \text{if } 2^{l-1} \text{ divides } 2^{l-1}/o_{2^{l+1}}(m) \\
2^{l-1}/(2(2^{l-1}/o_{2^{l+1}}(m), 2^{l-1})) & \text{if } 2^{l-1} \text{ does not divide } 2^{l-1}/o_{2^{l+1}}(m),
\end{cases}
\]

which simplifies to (A.4).

**Proof of Proposition A.1.** Suppose first that \(m \equiv 1 \pmod{4}\). For fixed \(N\), there exists an \(l \in \mathbb{N}\) satisfying \(m N < 2^l\). Then \(o_{2^l}(m) > N\) and hence the sequence \(\{o_{2^l}(m) \mid l \in \mathbb{N}\}\) must be unbounded. In particular, \(\{o_{2^l}(m)\}\) is not a constant sequence. Let \(K\) be the first integer such that \(o_{2^K}(m) < o_{2^{K+1}}(m)\). Then \(o_{2^l}(m) = o_{2^l}(m) = 1\) for \(1 \leq l \leq K\), and by Corollary A.4 we have \(o_{2^{K+1}}(m) = 2o_{2^l}(m) = 2\). Since \(o_{2^{K+1}}(m)\) divides \(o_{2^l}(m)\) for all \(l > K\), it follows that \(2\) divides \(o_{2^l}(m)\) for all \(l > K\).

We now apply Corollary A.4 \(l - K\) times to deduce that \(o_{2^l}(m) = 2^{l-K}o_{2^K}(m) = 2^{l-K}\).

Now suppose that \(m \equiv 3 \pmod{4}\). Certainly \(o_{2^l}(m) = 1\). For \(l > 1\), Corollary A.3 tells us that \(m \equiv -5^k\) for some \(0 \leq k < 2^{l-2}\). Thus \(m^2 \equiv 5^{2k} \pmod{2^l}\), and therefore \(m^2 \in \langle 5 \rangle_l\). Let \(L\) be the first integer such that \(o_{2^L}(m^2) < o_{2^{L+1}}(m^2)\). Applying Corollary A.4 to \(m^2\) and repeating the argument of the preceding paragraph gives (A.3) because \(o_{2^l}(m) = 2o_{2^l}(m^2)\).

We now need to extend these results to cover actions on \(U(\mathbb{Z}_F)\) for an arbitrary finite set \(F\) of primes. We write \(F = \{p_1, \ldots, p_n\}\) and fix a prime \(q\) which is not in \(F\). We denote by \(o_{(l_1, \ldots, l_n)}(q)\) the order of \((q, \ldots, q)\) in \(\prod_{i=1}^n U(\mathbb{Z}/p_i^{l_i}\mathbb{Z})\).

**Proposition A.5.** There exist positive integers \(K_1, \ldots, K_n\) and \(d\) such that

\[
o_{(K_1+l_1, \ldots, K_n+l_n)}(q) = dp_1^{l_1} \cdots p_n^{l_n}
\]

for every \((l_1, \ldots, l_n) \in \mathbb{N}^F\).

**Proof.** Suppose first that \(p_1, \ldots, p_n\) are distinct odd primes, and let \(L_{p_i}(q)\) be as in (A.1). Let

\[
z_i := \max\{z \mid p_i^z \text{ divides } o_{p_i}(q) \text{ for some } j \in \{1, \ldots, n\}\},
\]

and define \(K_i := L_{p_i}(q) + z_i \) and \(d := [o_{p_1}(q), \ldots, o_{p_n}(q)]\), where \([r_1, \ldots, r_n]\) is the least common multiple of the integers \(r_i\). In general, if \(g_i\) are elements of order \(r_i\) in finite groups \(G_i\), then the order of \((g_1, \ldots, g_n)\) in \(G_1 \times \cdots \times G_n\) is \([r_1, \ldots, r_n]\). Thus from the properties of \(L_{p_i}(q)\) we obtain

\[
o_{(K_1+l_1, \ldots, K_n+l_n)}(q) = [p_1^{(K_1+l_1)-L_{p_1}(q)} o_{p_1}(q), \ldots, p_n^{(K_n+l_n)-L_{p_n}(q)} o_{p_n}(q)]
= [p_1^{z_1+l_1} o_{p_1}(q), \ldots, p_n^{z_n+l_n} o_{p_n}(q)]
= p_1^{l_1} \cdots p_n^{l_n} o_{p_1}(q), \ldots, o_{p_n}(q)],
\]

which is (A.5).

Now suppose that \(2 \in F\), say \(p_1 = 2\). If \(q \equiv 1 \pmod{4}\), we let

\[
z_i := \max\{z \mid p_i^z \text{ divides } o_{p_i}(q) \text{ for some } j \in \{2, \ldots, n\}\},
\]

and define \(K_i := L_{p_i}(q) + z_i \) and \(d := [o_{p_1}(q), \ldots, o_{p_n}(q)]\), where \([r_1, \ldots, r_n]\) is the least common multiple of the integers \(r_i\). In general, if \(g_i\) are elements of order \(r_i\) in finite groups \(G_i\), then the order of \((g_1, \ldots, g_n)\) in \(G_1 \times \cdots \times G_n\) is \([r_1, \ldots, r_n]\). Thus from the properties of \(L_{p_i}(q)\) we obtain

\[
o_{(K_1+l_1, \ldots, K_n+l_n)}(q) = [p_1^{(K_1+l_1)-L_{p_1}(q)} o_{p_1}(q), \ldots, p_n^{(K_n+l_n)-L_{p_n}(q)} o_{p_n}(q)]
= [p_1^{z_1+l_1} o_{p_1}(q), \ldots, p_n^{z_n+l_n} o_{p_n}(q)]
= p_1^{l_1} \cdots p_n^{l_n} o_{p_1}(q), \ldots, o_{p_n}(q)],
\]

which is (A.5).
and define \( K_1 := L_2(q) + z_1 \), \( K_i := L_{p_i}(q) + z_i \) for \( i > 1 \), and \( d := [o_{p_2}(q), \ldots, o_{p_n}(q)] \).

Reasoning as in (A.6) gives (A.5).

If \( q \equiv 3 \pmod{4} \), we let
\[
z_1 = \max(1, \max\{z \mid 2^z \text{ divides } o_{p_j}(q) \text{ for some } j \in \{2, \ldots, n\}\}),
\]
\[
z_i = \max\{z \mid p_i^z \text{ divides } o_{p_j}(q) \text{ for some } j \in \{2, \ldots, n\}\}
\]
for \( i > 1 \), and define \( K_1 := L_2(q) + z_1 - 1 \), \( K_i := L_{p_i}(q) + z_i \) for \( i > 1 \) and \( d := [2, o_{p_2}(q), \ldots, o_{p_n}(q)] \). Again, reasoning as in (A.6) gives (A.5).□

**Corollary A.6.** The closure \( H \) of \( q^n/\mathcal{U}(\mathbb{Z}_F) \) in \( \mathcal{U}(\mathbb{Z}_F) \) is a subgroup of finite index
\[
|\mathcal{U}(\mathbb{Z}_F) : H| = \left(\prod_{i=1}^{n}(p_i - 1)p_i^{K_i - 1}\right)/d.
\]

*Proof.* Apply Proposition A.5 to \( \mathbf{l} = (l, l, \ldots, l) \) to see that \( |\pi_\mathbf{l}(H)| = d\left(\prod_{i=1}^{n}p_i^l\right) \) for large \( l \), and the result follows from [10, Lemma 3.7]. □

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