

Spectral triples for subshifts

Antoine Julien
Norwegian University of Science and Technology,
Trondheim, Norway
antoine.julien@math.ntnu.no,

Ian F. Putnam,*
Department of Mathematics and Statistics,
University of Victoria,
Victoria, B.C., Canada V8W 3R4,
ifputnam@uvic.ca

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Abstract

We propose a construction for spectral triple on algebras associated with subshifts. One-dimensional subshifts provide concrete examples of \mathbb{Z} -actions on Cantor sets. The C^* -algebra of this dynamical system is generated by functions in $C(X)$ and a unitary element u implementing the action. Building on ideas of Christensen and Ivan, we give a construction of a family of spectral triples on the commutative algebra $C(X)$. There is a canonical choice of eigenvalues for the Dirac operator D which ensures that $[D, u]$ is bounded, so that it extends to a spectral triple on the crossed product.

We study the summability of this spectral triple, and provide examples for which the Connes distance associated with it on the commutative algebra is unbounded, and some for which it is bounded. We conjecture that our results on the Connes distance extend to the spectral triple defined on the noncommutative algebra.

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1 Introduction

Alain Connes introduced the notion of a spectral triple which consists of a separable, infinite dimensional complex Hilbert space \mathcal{H} , a $*$ -algebra of bounded linear operators on \mathcal{H} , A , and a self-adjoint unbounded operator D such that $(1 + D^2)^{-1}$ is a compact operator. The key relation is that for all a in A , $[D, a] = Da - aD$ is densely defined and extends to a bounded operator [9]. The motivating example is where A consists of the smooth functions on a compact manifold and D is some type of elliptic differential operator.

In spite of its geometric origins, there has been considerable interest in finding examples where the algebra A is the continuous functions on a compact, totally disconnected metric space with no isolated points. We refer to such a space as a Cantor set. The first example was given by Connes [8, 9, 10] but many other authors have also contributed: Sharp [27], Pearson–Bellissard [23], Christensen–Ivan [7], etc. Many of these results concern not just a Cantor set, but also some type of dynamical system on a Cantor set. Many such systems are closely related to aperiodic tilings and are important as mathematical models for quasicrystals.

We continue these investigations here by studying subshifts. Let \mathcal{A} be a finite set (called the alphabet). Consider $\mathcal{A}^{\mathbb{Z}}$ with the product topology and the homeomorphism σ which is the left shift: $\sigma(x)_n = x_{n+1}, n \in \mathbb{Z}$, for all x in $\mathcal{A}^{\mathbb{Z}}$. A subshift X is any non-empty, closed and shift-invariant subset of $\mathcal{A}^{\mathbb{Z}}$. It is regarded as a dynamical system for the map $\sigma|_X$.

Our aim here is to construct and study examples of spectral triples for the algebra of locally constant functions on X , which we denote $C_{\infty}(X)$. Besides, we let $C(X)$ denote the C^* -algebra of continuous functions on X and $C(X) \rtimes \mathbb{Z}$ be the crossed product C^* -algebra generated by $C(X)$ and a canonical unitary u . We will also consider the $*$ -subalgebra of the crossed product generated by $C_{\infty}(X)$ and u and study how our spectral triples extend to this algebra.

Our construction is just a special case of that given by Christensen and Ivan for AF-algebras [7]. Given an increasing sequence of finite-dimensional C^* -algebras

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$$

one considers their union and a representation constructed by the GNS method. The same sequence of sets provides an increasing sequence of finite-dimensional subspaces for the associated Hilbert spaces which is used to construct the eigenspaces for the operator D . The one other piece of data needed for our construction is a σ -invariant measure with support X , which we denote μ .

Our case of the subshift has some special features. The first is that the sequence of finite-dimensional subalgebras we construct is canonical, arising from the structure as a subshift. Secondly, the choice of eigenvalues for the operator D is constrained by the dynamics (see Theorem 3.2) in such a way that we find a canonical choice for D , which we denote by D_X .

One similarity with the situation of Christensen and Ivan is that our operator D_X is positive, meaning that, at least to this point, it has no interesting index

data, but is rather regarded as a noncommutative metric structure as in Rieffel's work [24, 25].

We mention one other result by Bellissard, Marcolli and Reihani [4] which is relevant and motivated our work. They begin with a spectral triple (\mathcal{H}, A, D) and an action α of the integers on A by automorphisms. Under the assumption that the action is quasi-isometric (that is, there is a constant $C \geq 1$ such that

$$C^{-1} \|[D, a]\| \leq \|[D, \alpha^n(a)]\| \leq C \|[D, a]\|,$$

for all a in A and n in \mathbb{Z} , they construct a spectral triple for $A \times \mathbb{Z}$ satisfying certain conditions. Moreover, they show that every spectral triple on $A \times \mathbb{Z}$ satisfying these conditions arises this way. While this is a very good result, it is somewhat disappointing since the hypothesis of quasi-isometric is very strong and quite atypical in dynamical systems of interest. In fact, subshifts are expansive. That is, if d is any metric on the shift space, there is a constant $\epsilon_0 > 0$ such that for every $x \neq y$, there is an integer n with $d(\sigma^n(x), \sigma^n(y)) \geq \epsilon_0$. In some sense, this is the opposite of quasi-isometric. Of course, our spectral triples do not satisfy the conditions given in [4]. The key point is that the representation of $A \times \mathbb{Z}$ used in [4] is the left regular one on $\mathcal{H} \otimes \ell^2(\mathbb{Z})$. Our representations are on the Hilbert space $L^2(X, \mu)$, which may be regarded as more natural from a dynamical perspective.

We also mention previous constructions of spectral triples for subshifts or tiling spaces (see also the review article [17]). Kellendonk, Lenz and Savinien [19, 18] built spectral triples for the commutative algebra of one-dimensional subshifts. Their construction is similar to ours in the sense that it is based on the “tree of words” (which is related to our increasing family of subalgebras). In their papers, however, the algebra is represented on a Hilbert space of the form $\ell^2(E)$, where E is a countable set built from a discrete approximation of the subshift. It is unclear how the shift action reflects on this set, and there doesn't seem to be an obvious extension to the noncommutative algebra. They can still relate combinatorial properties of the subshift (namely the property of being repulsive) to properties of the spectral triples. Their construction also seems to be topologically quite rich, as the K -homology class of their triple is in general not trivial. In a recent paper, Savinien extended some results to higher-dimensional tilings [26]. Previously, Whittaker [28] had proposed a spectral triple construction for some hyperbolic dynamical systems (Smale spaces), a family which contains self-similar tiling spaces. In Whittaker's construction, the triple is defined on a noncommutative algebra which corresponds to $C(X) \times \mathbb{Z}$ in our case. His representation is however quite specific to the Smale space structure, and it is unclear how our construction compares to his.

This paper is organized as follows. In the next section, we discuss the basics of subshifts. This includes an introduction to three important classes which will be our main examples. The third section gives the construction of the spectral triples and some of their basic properties. In the fourth section, we establish results on the summability of the spectral triples. Leaving the precise statements until later, various aspects of summability are closely linked with the

entropy of the subshift and also its complexity. In the fifth section, we discuss the Connes metric. At this point all of our results here deal with the algebra $C_\infty(X)$. We hope to extend these to $C_\infty(X) \times \mathbb{Z}$ in future work. Again leaving the precise statements until later, it turns out that the behaviour of the Connes metric depends on very subtle properties of the dynamics and varies quite wildly between our three classes of examples.

Summary of the main results

Given a subshift X with an invariant measure μ , we represent $C(X) \times \mathbb{Z}$ on $L^2(X, \mu)$. Let $C_\infty(X)$ be the algebra of locally constant functions on X . For any increasing sequence going to infinity $(\alpha_n)_{n \geq 0}$, we define a Dirac operator with these eigenvalues, so that $(C_\infty(X), L^2(X, \mu), D)$ is a spectral triple (Theorem 3.2). It extends to a spectral triple on $C_\infty(X) \times \mathbb{Z}$ if and only if $(\alpha_n - \alpha_{n-1})$ is bounded. This leads to the definition of a canonical operator D_X (with $\alpha_n = n$), which is studied in the rest of the paper (Definition 3.3).

The summability depends on the complexity of the subshift. The complexity counts the number of factors of length n appearing in the subshift. If the subshift has positive entropy $h(X)$ (which corresponds to the complexity function growing exponentially), the summability of e^{sD_X} depends on whether $s < h(X)$ or $s > h(X)$ (Theorem 4.1). In the case of zero entropy, the complexity may grow asymptotically like n^d for some number d . Whether D_X is s -summable then depends whether $s < d$ or $s > d$ (Theorem 4.2).

In the last section, we investigate the Connes metric for various subshifts. In this section, we focus on the spectral triple restricted to the commutative algebra $C_\infty(X)$. For (aperiodic, irreducible) shifts of finite type, the Connes metric is infinite (Theorem 5.8). For linearly recurrent subshifts, the Connes metric is finite and induces the weak-* topology (Theorem 5.10). This result is positive in the sense that it includes many examples of interest related to one-dimensional quasicrystals, including primitive substitution subshifts. For Sturmian subshifts, the results depend rather subtly (but probably not surprisingly) on the continued fraction expansion of the irrational number which parameterizes the subshift. We show that there is a large class of numbers $\theta \in (0, 1)$ such that the subshift with parameter θ has finite Connes metric and induces the weak-* topology (Theorem 5.14). In particular, this is the case for almost all $\theta \in (0, 1)$ (for the Lebesgue measure). However, we also exhibit Sturmian subshifts having infinite Connes metric (Theorem 5.13).

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2 Subshifts

Let \mathcal{A} be a finite set. We let $\sigma : \mathcal{A}^{\mathbb{Z}} \rightarrow \mathcal{A}^{\mathbb{Z}}$ denote the left shift map. That is, $\sigma(x)_n = x_{n+1}$, for x in $\mathcal{A}^{\mathbb{Z}}$ and n in \mathbb{Z} . Let $X \subset \mathcal{A}^{\mathbb{Z}}$ be a subshift; that is, it is a closed subset with $\sigma(X) = X$. Let μ be a σ -invariant probability measure on X with support equal to X . The system (X, σ) always admits invariant measures, but the hypothesis that it has full support is non-trivial. It is satisfied in most cases of interest.

For $n \geq 1$, we let X_n denote the words in X of length n . That is, we have

$$X_n = \{(x_1, x_2, \dots, x_n) \mid x \in X\}.$$

We say that such finite words are *admissible*. For convenience, we let X_0 be the set consisting of the empty word, which we denote ε . For w in X_n , we define the associated cylinder set

$$\begin{aligned} U(w) &= \{x \in X \mid (x_{-m}, \dots, x_m) = w\}, n = 2m + 1, \\ U(w) &= \{x \in X \mid (x_{1-m}, \dots, x_m) = w\}, n = 2m. \end{aligned}$$

We let $\xi(w)$ be the characteristic function of $U(w)$. The set $U(w)$ is clopen and $\xi(w)$ lies in both $C(X)$ and $L^2(X, \mu)$. We let $\mu(w) = \mu(U(w))$, for convenience. The fact that we assume the support of μ is X means that $\mu(w) \neq 0$, for any word w .

We define $\pi : X_n \rightarrow X_{n-1}$, for $n > 1$, by

$$\begin{aligned} \pi(w_{-m}, \dots, w_m) &= (w_{1-m}, \dots, w_m) && \text{if } n = 2m + 1, \\ \pi(w_{1-m}, \dots, w_m) &= (w_{1-m}, \dots, w_{m-1}) && \text{if } n = 2m. \end{aligned}$$

We extend this to $n = 1$ by defining $\pi(a)$ to be the empty word, for all a in X_1 . We also define $U(\varepsilon)$ to be X . Notice with this indexing, we have $U(\pi(w)) \supset U(w)$, for all w in X_n , $n > 1$.

Of course, the function π may be iterated and for all positive integers k , $\pi^k : X_n \rightarrow X_{n-k}$. This leads to the unfortunate notation $\pi^{n-m} : X_n \rightarrow X_m$, when $n > m$. For convenience, we let π_m be the map defined on the union of all X_n , $n > m$, which is π^{n-m} on X_n . Thus $\pi_m : \cup_{n>m} X_n \rightarrow X_m$. We also extend this function to be defined on X by

$$\pi_{2m+1}(x) = (x_{-m}, \dots, x_m), \quad \pi_{2m}(x) = (x_{1-m}, \dots, x_m).$$

Observe that if w is in X_n then $\#\pi^{-1}\{w\} > 1$ if and only if, for any w' with $\pi(w') = w$, we have $U(w) \supsetneq U(w')$. For $n \geq 1$, we denote by X_n^+ the w in X_n satisfying either condition. Such words are called (left- or right-) *special words*.

Of basic importance in the dynamics of subshifts is the *complexity* of the shift X , which is simply $\#X_n$, $n \geq 1$, regarded as a function on the natural numbers. Notice first that if $X = \mathcal{A}^{\mathbb{Z}}$ is the full shift, then the number of words of length n is simply $(\#\mathcal{A})^n$, for $n \geq 1$. More generally, if X is a somewhat large subset of $\mathcal{A}^{\mathbb{Z}}$, then we reasonably expect the growth of the complexity to

be exponential. To capture the base of the exponent, the *entropy* of X , denoted $h(X)$, is defined to be

$$h(X) = \lim_{n \rightarrow \infty} \frac{\log(\#X_n)}{n} = \inf \left\{ \frac{\log(\#X_n)}{n} ; n \geq 1 \right\}. \quad (1)$$

Of course, it is a non-trivial matter to see the limit exists, and equals the infimum. This ‘‘configuration entropy’’ or ‘‘patch-counting entropy’’ agrees with the topological entropy for the dynamical system. See [21, 3]. Secondly, there is some variation in the literature on dynamical systems as to the base with which the logarithm should be taken. We will use \log to mean natural logarithm throughout.

We also define $R(w)$ for any w in $\cup_{n \geq 1} X_n$

$$R(w) = \sup \left\{ \frac{\mu(w)}{\mu(w')} \mid \pi(w') = w \right\}.$$

Notice that $R(w) \geq 1$, with strict inequality if and only if w is in $\cup_n X_n^+$.

Lemma 2.1. *For any w in $X_n^+, n \geq 1$ and w' with $\pi(w') = w$, we have*

$$\left(\frac{\mu(w)}{\mu(w')} - 1 \right)^{-1} \leq R(w).$$

Proof. Let $\pi^{-1}\{w\} = \{w_1, \dots, w_I\}$, arranged so that $\mu(w_1) \leq \mu(w_2) \leq \dots \leq \mu(w_I)$. Since w is in X_n^+ , we know that $I \geq 2$. Since $U(w)$ is the disjoint union of the sets $U(w_i), 1 \leq i \leq I$, we have $\mu(w_1) + \mu(w_2) + \dots + \mu(w_I) = \mu(w)$. It follows that for any $1 \leq i \leq I$, we have

$$\left(\frac{\mu(w)}{\mu(w_i)} - 1 \right)^{-1} = \frac{\mu(w_i)}{\mu(w) - \mu(w_i)} \leq \frac{\mu(w)}{\mu(w_1)} = R(w).$$

□

2.1 Shifts of finite type

Let $G = (G^0, G^1, i, t)$ be a finite directed graph. That is, G^0, G^1 are finite sets (representing the vertices and edges, respectively) and i, t (meaning ‘initial’ and ‘terminal’) are maps from G^1 to G^0 .

The set of edges in the graph is our symbols, $\mathcal{A} = G^1$, and X^G consists of all doubly infinite paths in the graph, *i.e.*

$$X^G = \{(x_n)_{n \in \mathbb{Z}} \mid x_n \in G^1, t(x_n) = i(x_{n+1}), \text{ for all } n \in \mathbb{Z}\}.$$

This set X^G is a subshift of the set $(G^1)^{\mathbb{Z}}$ and is called a (nearest-neighbour) *subshift of finite type*, or simply *shift of finite type*, or SFT.

Associated to the graph G is an incidence matrix A . It is simplest to describe if we set $G^0 = \{1, 2, \dots, I\}$, for some positive integer I . In this case, A is an $I \times I$ matrix and

$$A_{i,j} = \#\{e \in G^1 \mid i(e) = i, t(e) = j\}, \quad 1 \leq i, j \leq I.$$

For any $n \geq 1$, $1 \leq i, j \leq I$, $A_{i,j}^n$ equals the number of paths of length n from i to j .

We say that G is *irreducible* if for each i, j , there exists n with $A_{i,j}^n > 0$. We say that G is aperiodic if X^G is infinite. For an irreducible, aperiodic shift of finite type, the matrix A has a unique positive eigenvalue of maximum absolute value called the Perron eigenvalue, which we denote by λ_G . We denote u_G and v_G (or simply u and v) respectively the left and right eigenvectors associated with λ_G , normalised such that $\sum_k v_k = 1$ and $\sum_k u_k v_k = 1$. The subshift has a unique invariant measure of maximal entropy (see for example [11]) which we refer to as the Parry measure. It is defined by

$$\mu(w_1, \dots, w_n) = u_{i(w_1)} v_{t(w_n)} \lambda^{-(n-1)}. \quad (2)$$

We remark that if G is an irreducible graph, then the entropy of the associated shift of finite type is $h(X^G) = \log(\lambda_G)$.

2.2 Linearly recurrent subshifts

Linear recurrence (or linear repetitivity) is a very strong regularity condition for a subshift. The name refers to estimates on the size of the biggest gap between successive repetitions of patterns of a given size. Equivalently, it refers to the biggest return times in small neighbourhoods of a given size in the dynamical system (X, \mathbb{Z}) .

Definition 2.2. *If w is a finite admissible word of a subshift X , a right return word to w of X is any word r such that: $-rw$ is admissible; $-w$ is a prefix of rw ; $-$ there are exactly two occurrences (possibly overlapping) of w in rw .*

Definition 2.3. *A minimal subshift X is linearly recurrent (or LR) if there exists a constant K such that for all $n \in \mathbb{N}$, any word $w \in X_n$ and any right return word r to w , we have*

$$|r| \leq K |w|.$$

It is a bound on the return time in the cylinders $U(w)$ for $w \in X_n$: for any $x \in U(w)$, there is $0 < k \leq K |w|$ such that $\sigma^k(x) \in U(w)$.

Here are a few important consequences of linear recurrence.

Proposition 2.4 (Durand [12, 13]). *Let X be a linearly recurrent subshift. Then (X, σ) is uniquely ergodic.*

Proposition 2.5 (Durand [12, 13]). *Let X be a linearly recurrent subshift with constant K , and μ be its unique invariant measure. Then for all $n \in \mathbb{N}$ and all $w \in X_n$,*

$$\frac{1}{K} \leq n\mu(w) \leq K. \quad (3)$$

This presentation is actually somewhat backwards. In [12], Durand proves that equation (3) holds for any invariant measure, and uses it in conjunction with a result of Boshernitzan [6] to prove unique ergodicity.

Proposition 2.6 (Durand [12, 13]). *Let (X, σ) be a linearly recurrent subshift with constant K . Then*

- *Any right return word r to $w \in X_n$ has length at least $|r| > |w|/K$;*
- *For any fixed $w \in X_n$, there are at most $K(K+1)^2$ return words to w .*

Also proved in [12], is the fact that any linearly recurrent subshift has sub-linear complexity in the sense that $\#X_n \leq Cn$ for some constant C .

Proposition 2.5 has an immediate consequence.

Lemma 2.7. *Let X be a linearly recurrent subshift. Then there exists a constant $C > 1$ such that for all $w \in X_n^+$,*

$$C^{-1} \leq R(w) \leq C.$$

Note that the class of linearly recurrent subshifts contains the class of primitive substitution subshifts. Many well-known examples, such as the Fibonacci or the Thue–Morse subshifts are given by primitive substitutions, hence are linearly recurrent, and are therefore covered by our results (see [15], for example for a survey on substitutions).

2.3 Sturmian systems

There are many different equivalent ways to define a Sturmian sequence (see [2], for example). We give here the definition as the coding of an irrational rotation of angle θ on a circle. A crucial ingredient to identify properties satisfied by a Sturmian sequence is the continued fraction expansion of the number θ . We remind the reader of a few of its properties. The results on continuous fraction, which are stated here without proof, can be found in the classic book by Khinchin¹ [20]. To fix notations, $[\theta]$ (integer part of θ) refers to the unique integer satisfying $\theta - 1 < [\theta] \leq \theta$, and $\{\theta\} := \theta - [\theta] \in [0, 1)$ refers to the fractional part of θ .

Let θ be a real number, which we assume irrational. Let $\theta_0 = \theta$ and $\theta_1 = \{\theta\}$, so that $\theta = a_0 + \theta_1$, with $a_0 \in \mathbb{Z}$ and $\{\theta\} \in [0, 1)$. Then there are a unique $a_1 \in \mathbb{N}$ and $0 < \theta_2 < 1$ such that

$$\theta = a_0 + \frac{1}{a_1 + \theta_2},$$

namely $a_1 = [1/\theta_1]$ and $\theta_2 = \{1/\theta_1\}$.

¹Also transcribed “Khinchine” in the French-language publications.

By iteration, there is a sequence a_2, a_3, \dots of positive integers such that for all n ,

$$\theta = a_0 + \frac{1}{a_1 + \frac{1}{\dots \frac{1}{a_n + \theta_{n+1}}}}$$

The number θ is entirely determined by its sequence of *partial quotients* $(a_n)_{n \geq 0}$, and we write $\theta = [a_0; a_1, a_2, \dots]$.

The finite continued fraction (truncating after the term a_n) is called the n -th convergent and is denoted $[a_0; a_1, \dots, a_n]$. It is a rational number, which has a unique representation as an irreducible fraction p_n/q_n .

It is a standard result that p_n and q_n satisfy the recurrence relations

$$\begin{cases} p_n = a_n p_{n-1} + p_{n-2} \\ q_n = a_n q_{n-1} + q_{n-2} \end{cases} \quad \text{with the convention} \quad \begin{cases} p_{-2} = 0, & p_{-1} = 1 \\ q_{-2} = 1, & q_{-1} = 0. \end{cases}$$

Lemma 2.8. *Let $0 < \theta < 1$ be irrational, so that $\theta = [0; a_1, a_2, \dots]$. Let $(\theta_n)_{n \geq 0}$ be defined as above. We define $\lambda_n(\theta)$ (or just λ_n) by $\lambda_n = \prod_{i=1}^n \theta_i$, for $n \geq 1$ (with the convention $\lambda_0 = 1$). Let p_n/q_n be the n -th convergent. Then the following holds.*

1. For all n , $\theta = (\theta_n^{-1} p_{n-1} + p_{n-2}) / (\theta_n^{-1} q_{n-1} + q_{n-2})$.
2. For all n , $q_n p_{n-1} - p_n q_{n-1} = (-1)^n$.
3. For all n , $\theta_n^{-1} q_{n-1} + q_{n-2} = \lambda_n^{-1}$.
4. For all n , $\lambda_{n+1} = (-1)^n (\theta q_n - p_n)$.

Proof. The first two points are classic (see for example Theorem 2 and the proof of Theorem 7 in [20]). The third point is proved by induction. For $n = 1$, this is the equality $\theta_1^{-1} = \lambda_1^{-1}$. Assume it holds for fixed n . Then

$$\begin{aligned} \theta_{n+1}^{-1} q_n + q_{n-1} &= \theta_{n+1}^{-1} ((a_n + \theta_{n+1}) q_{n-1} + q_{n-2}) \\ &= \theta_{n+1}^{-1} (\theta_n^{-1} q_{n-1} + q_{n-2}) \\ &= \theta_{n+1}^{-1} \lambda_n^{-1} = \lambda_{n+1}^{-1}. \end{aligned}$$

For the fourth point, we start apply points 1, 2 and 3 in succession:

$$\begin{aligned} (-1)^n (\theta q_n - p_n) &= (-1)^n \frac{q_n (\theta_{n+1}^{-1} p_n + p_{n-1}) - p_n (\theta_{n+1}^{-1} q_n + q_{n-1})}{\theta_{n+1}^{-1} q_n + q_{n-1}} \\ &= (-1)^n \frac{q_n p_{n-1} - p_n q_{n-1}}{\theta_{n+1}^{-1} q_n + q_{n-1}} = (-1)^{2n} \lambda_{n+1} = \lambda_{n+1}. \end{aligned}$$

□

Definition 2.9. *Sturmian sequences of irrational parameter $0 < \theta < 1$ are defined as the sequences $w \in \{0, 1\}^{\mathbb{Z}}$ such that*

$$w_n = \begin{cases} 0 & \text{if } \{n\theta - x\} \in I, \\ 1 & \text{otherwise,} \end{cases}$$

where x is a real number, $\{y\} = y - \lfloor y \rfloor$ is the fractional part of the real number y , and I is either $[0, 1 - \theta)$ or $(0, 1 - \theta]$

Let X^θ be the set of all Sturmian sequences of parameter θ .

It is well known that X^θ is a closed shift-invariant subspace of the full shift. By irrationality of θ , it is aperiodic and minimal.

Proposition 2.10. *If X^θ is a Sturmian subshift with irrational θ , then $\#X_n^\theta = n + 1$. It implies that $\#(X_n^\theta)^+ = 1$ for all n .*

It is well known that the subshifts X^θ are uniquely ergodic, and the frequency of the letter 1 is equal to the length of the interval I , i.e. θ (the frequency of the letter 1 is defined as the measure of the cylinder set associated with the word $1 \in X_1^\theta$ for the unique invariant measure).

The measure of the cylinder sets for a Sturmian subshift is very well controlled in terms of the partial quotients. This fact can be seen as a consequence of the three-distances theorem. We use the three distance theorem the way it is presented in [1] (noting that our λ_{n+1} is their η_n , by point 4 of Lemma 2.8). The relationship between this theorem and the frequencies of Sturmian sequences is established in [5].

Theorem 2.11 ([1, 5]). *Let X^θ be a Sturmian subshift of parameter $0 < \theta < 1$ irrational. If $kq_n + q_{n-1} \leq m < (k+1)q_n + q_{n-1}$ for some $n \geq 1$ and $0 < k \leq a_{n+1}$, then the frequencies $\mu(w)$ ($w \in X_m^\theta$) are in the set:*

$$\left\{ (\lambda_n - k\lambda_{n+1}), \lambda_{n+1}, (\lambda_n - (k-1)\lambda_{n+1}) \right\}.$$

If $m = (k+1)q_n + q_{n-1} - 1$, the third frequency does not occur.

Remark in particular that if $m = q_n + q_{n-1} - 1 = (a_n + 1)q_{n-1} + q_{n-2} - 1$, then the $\mu(w)$ is either λ_n or $\lambda_{n-1} - a_n\lambda_n = \lambda_{n+1}$ for $w \in X_m^\theta$.

3 Spectral triples

From this point on, we assume that a subshift (X, σ) is given. In time, we will assume that it is a shift of finite type, a linearly recurrent subshift, or a Sturmian subshift.

We consider the crossed product C^* -algebra $C(X) \times \mathbb{Z}$, which is generated by $f \in C(X)$ and a unitary u satisfying $uf = f \circ \sigma^{-1}u$, for all f in $C(X)$. This is represented on $L^2(X, \mu)$ by

$$(f\xi)(x) = f(x)\xi(x), \quad (u\xi)(x) = \xi(\sigma^{-1}(x)),$$

for ξ in $L^2(X, \mu)$, f in $C(X)$ and x in X .

Let $C_0 = \mathbb{C}1$ be the constant functions and, recalling that $\xi(w)$ denotes the characteristic function of the set $U(w)$, $w \in X_n$, $n \geq 1$, let $C_n = \text{span}\{\xi(w) \mid w \in X_n\}$, for $n \geq 1$. Notice that $C_0 \subset C_1 \subset C_2 \cdots$, each C_n is a finite-dimensional subspace of $L^2(X, \mu)$ and a finite-dimensional $*$ -subalgebra of $C(X)$. Moreover, the union of the C_n is dense in both, with the appropriate norms. For convenience, let $C_{-1} = \{0\}$. Let P_n denote the orthogonal projection of $L^2(X, \mu)$ on C_n , for each non-negative n . We let $C_\infty(X) = \cup_{n=1}^\infty C_n$. We will denote by $C_\infty(X) \times \mathbb{Z}$ the $*$ -algebra of operators on $L^2(X, \mu)$ generated by $C_\infty(X)$ and u .

As we indicated above, our main interest is in the crossed product $C(X) \times \mathbb{Z}$. We have here a specific representation of this C^* -algebra, but it is a reasonable question to ask if it is faithful. First, the representation of $C(X)$ on $L^2(X, \mu)$ is faithful since we assume our measure μ has support equal to X . Secondly, we use the fact that, if our system is topologically free, that is, for every n in \mathbb{Z} , the set $\{x \in X \mid \sigma^n(x) = x\}$ has empty interior in X , then every non-trivial, closed, two-sided ideal in $C(X) \times \mathbb{Z}$ has a non-trivial intersection with $C(X)$. (See [14] for a precise statement.) In our case, since the representation of $C(X)$ is faithful, so is that of $C(X) \times \mathbb{Z}$, provided our system is topologically free. We make this assumption implicitly from now on and remark that it holds in all of our examples.

We define an operator D as follows. First, we choose a strictly increasing sequence of positive real numbers, α_n , $n \geq 0$, and define the domain of our operator

$$\mathcal{D} = \left\{ \sum_{n \geq 0} \xi_n \mid \xi_n \in C_n \cap C_{n-1}^\perp, \sum_n (\alpha_n \|\xi_n\|)^2 < \infty \right\}.$$

We define

$$D\xi = \alpha_n \xi, \quad \xi \in C_n \cap C_{n-1}^\perp.$$

Lemma 3.1. *Let $n \geq 0$. For all f in C_n , we have*

1. $fC_m \subset C_m$, for all $m \geq n$,
2. $f\mathcal{D} \subset \mathcal{D}$,
3. $[D, f]|C_n^\perp = 0$.

Proof. For the first item, we know that $C_n \subset C_m$ whenever $m \geq n$ and that C_m is a unital subalgebra so that $fC_m \in C_n C_m \subset C_m C_m = C_m$. For the second, suppose that ξ is in $C_m \cap C_{m-1}^\perp$, for some $m > n$. C_m and C_{m-1} are invariant under C_n . Hence so is C_{m-1}^\perp and $C_m \cap C_{m-1}^\perp$. The second part follows from this. On the space $C_m \cap C_{m-1}^\perp$, D is a scalar and hence commutes with f . Taking direct sums over all $m > n$ yields the result. \square

Theorem 3.2. *Let X be a subshift.*

1. $(C_\infty(X), L^2(X, \mu), D)$ is a spectral triple if and only if the sequence α_n , $n \geq 0$ tends to infinity.

2. $[D, u]$ extends to a bounded linear operator on $L^2(X, \mu)$ if and only if $\alpha_n - \alpha_{n-1}, n \geq 1$ is bounded.
3. $(C_\infty(X) \times \mathbb{Z}, L^2(X, \mu), D)$ is a spectral triple if and only if the sequence $\alpha_n, n \geq 0$ tends to infinity and $\alpha_n - \alpha_{n-1}, n \geq 1$ is bounded.

We use here Connes' definition of a spectral triple (or K -cycle) [10, Chapter IV-2] (see also [16, Def. 9.16]). There is a variety of conditions that may be required on a spectral triple depending on the context. For example, in the context of quantum spaces, Rieffel [24, 25] requires that a certain metric induced by D on the state space of the algebra coincides with the weak-* topology. We make no such assumption at this stage, however we shall investigate this Connes metric in Section 5.

We remark also that D is defined to be a positive operator. It follows that the associated Fredholm module has a trivial class in K -homology and carries no index data.

Proof. We see from the third part of Lemma 3.1, that for any f in $C_\infty(X)$, f maps the domain of D to itself and from the final part, $[D, f]$ obviously extends to a bounded linear operator on $L^2(X, \mu)$. The final condition is that $(1 + D^2)^{-1}$ is compact is evidently equivalent to its eigenvalues converging to zero, and also to the sequence $\alpha_n, n \geq 0$ tending to infinity.

Fix a positive integer n and let \mathcal{P}_n be the partition of X given by the cylinder sets: $\{U(w) \mid w \in X_n\}$. It can easily be seen that the partition

$$\sigma(\mathcal{P}_n) = \{\sigma(U(w)) \mid w \in X_n\}$$

is clearly finer than \mathcal{P}_{n-2} while it is coarser than \mathcal{P}_{n+2} . (It is probably easiest to check the cases of n even and odd separately.) From this it follows that $C_{n-2} \subset uC_n \subset C_{n+2}$. Hence, we also have $C_{n-2}^\perp \supset uC_n^\perp \supset C_{n+2}^\perp$ as well. Let ξ be any vector in $C_n \cap C_{n-1}^\perp$. We know that $u\xi$ is in $C_{n+2} \cap C_{n-3}^\perp$ and on this space, which is invariant under D , we have $\alpha_{n-2} \leq D \leq \alpha_{n+2}$. Now, we compute

$$[D, u]\xi = Du\xi - uD\xi = Du\xi - \alpha_n u\xi = (D - \alpha_n)u\xi,$$

and the conclusion follows. \square

In particular cases, we will consider the following spectral triple.

Definition 3.3. Let X be a subshift and let μ be a σ invariant probability measure with support X . The spectral triple $(C_\infty(X), L^2(X, \mu), D_X)$ is defined as above, associated with the sequence $\alpha_n = n$.

Theorem 3.4. Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ and $X' \subseteq \mathcal{A}'^{\mathbb{Z}}$ be subshifts with invariant probability measures μ and μ' , respectively. If there exists a homeomorphism $h : X \rightarrow X'$ such that $h \circ \sigma|_X = \sigma|_{X'} \circ h$ and $h^*(\mu') = \mu$, then $v\xi = \xi \circ h$ is a unitary operator from $L^2(X', \mu')$ to $L^2(X, \mu)$ such that

1. $vC_\infty(X')v^* = C_\infty(X)$,

2. $vu_{X'}v^* = u_X$,
3. v maps the domain of $D_{X'}$ onto the domain of D_X and $vD_{X'}v^* - D_X$ is bounded.

Proof. That v is unitary and the first two parts of the conclusion all follow trivially from the hypotheses. For the last part, we use the result of Curtis–Lyndon–Hedlund [22, Theorem 6.2.9] that h arises from a sliding block code. That is, there exist positive integers M, N and function $h_0 : X_{N+M+1} \rightarrow \mathcal{A}'$ such that

$$h(x)_n = h_0(x_{n-M} \dots x_{n+N}),$$

for all x in X . We lose no generality in assuming that $M = N$. It then easily follows that for any w in X_n , $n > N$, $h(U(w))$ is contained in $U'(w')$, where w' is in X_{n-2N} and is given by $w'_i = h_0(w_{i-N} \dots w_{i+N})$, for all appropriate i . It follows that

$$vC'_{n-2N}v^* \subseteq C_n.$$

A similar result also holds for h^{-1} and we may assume without loss of generality, it uses the same N . This means that

$$vC'_n v^* \supseteq C_{n-2N},$$

as well. From these two facts, it is easy to see that $\|vD_{X'}v^* - D_X\| \leq 2N$. \square

4 Summability

The goal of this section is to study the summability of the operator D_X as introduced in Definition 3.3, for a shift space X . For $p \geq 1$, the spectral triple D_X is said to be p -summable if $(1 + D_X^2)^{-p/2}$ is trace class; and θ -summable if for all $t > 0$, $e^{-tD_X^2}$ is trace class.

Recall the definitions from Section 2, in particular Equation (1). Also note that in our construction earlier, $\dim(C_n) = \#X_n$, for all $n \geq 1$, while $\dim(C_n \cap C_{n-1}^\perp) = \#X_n - \#X_{n-1}$, for all $n \geq 0$.

Theorem 4.1. *Let $X \subset \mathcal{A}^{\mathbb{Z}}$ be a subshift and D_X be defined as in Definition 3.3.*

1. *If $s > h(X)$, then e^{-sD_X} is trace class.*
2. *If $s > 0$ is such that e^{-sD_X} is trace class, then $s \geq h(X)$.*

Proof. For the first part, choose $s > s' > h(X)$. There exists $n_0 \geq 1$ such that $\frac{1}{n} \log(\#X_n) < s'$, for all $n \geq n_0$. Hence, we have $\#X_n \leq e^{ns'}$ and $e^{-sn}\#X_n \leq e^{n(s'-s)}$, for $n \geq n_0$. Since $s' < s$ the series $e^{n(s'-s)}$ is a summable geometric series. It follows that $e^{-sn}\#X_n$ is also summable. Also, we have

$$\mathrm{Tr}(e^{-sD_X}) = \sum_{n=0}^{\infty} e^{-sn} \dim(C_n \cap C_{n-1}^\perp) \leq \sum_{n=0}^{\infty} e^{-sn} \dim(C_n).$$

Since $\dim(C_n) = \#X_n$, this completes the proof of the first part.

Conversely, suppose that e^{-sD_X} is trace class. This operator is positive and its trace is

$$\mathrm{Tr}(e^{-sD_X}) = \sum_{n=0}^{\infty} e^{-sn} (\dim(C_n) - \dim(C_{n-1})).$$

An Abel transform (summation by part) on the partial sum gives:

$$\sum_{n=0}^N e^{-sn} (\dim(C_n) - \dim(C_{n-1})) \geq \sum_{n=0}^{N-1} (1 - e^{-s}) e^{-sn} \dim(C_n).$$

Since the left-hand side converges, the series $\sum e^{-sn} \dim(C_n)$ is convergent and hence the terms go to zero. Thus, we have

$$0 = \lim_{n \rightarrow \infty} e^{-sn} \dim(C_n) = \lim_{n \rightarrow \infty} e^{-sn + \log(\dim(C_n))} = \lim_{n \rightarrow \infty} \left(e^{-s + \frac{1}{n} \log(\#X_n)} \right)^n$$

We conclude that for n sufficiently large, $e^{-s + \frac{1}{n} \log(\#X_n)}$ is less than 1. On the other hand we know that $\frac{1}{n} \log(\#X_n)$ tends to $h(X)$. The conclusion follows. \square

Theorem 4.2. *Let $X \subset \mathcal{A}^{\mathbb{Z}}$ be a subshift and D_X be defined as in Definition 3.3.*

1. *If, for some constants $C, s_0 \geq 1$, $\#X_n \leq Cn^{s_0}$, for all $n \geq 1$, then D_X is s -summable for all $s > s_0$.*
2. *If D_X is s -summable for some $s > 0$, then there exists $C \geq 1$ such that $\#X_n \leq Cn^s$, for $n \geq 1$.*

Proof. For the first part, let $s > s_0$ and compute

$$\begin{aligned} \mathrm{Tr}((1 + D_X^2)^{-s/2}) &= \lim_{N \rightarrow \infty} \sum_{n=0}^N (1 + n^2)^{-s/2} (\dim(C_n) - \dim(C_{n-1})) \\ &= \lim_{N \rightarrow \infty} \left[\sum_{n=1}^N \left((1 + n^2)^{-s/2} - (1 + (n+1)^2)^{-s/2} \right) \dim(C_n) \right. \\ &\quad \left. + (1 + N^2)^{-s/2} \dim(C_N) \right] \end{aligned}$$

The second term $(1 + N^2)^{-s/2} \dim(C_N)$ tends to zero from the hypothesis. For the first term, we denote

$$S_N := \sum_{n=1}^N \left((1 + n^2)^{-s/2} - (1 + (n+1)^2)^{-s/2} \right) \dim(C_n)$$

for convenience. Consider the function $f(x) = -(1 + x^2)^{-s/2}$. Its derivative is $f'(x) = sx(1 + x^2)^{-s/2-1}$ which is positive and decreasing for $x \geq 1$. By an

application of the mean value theorem to this function on the interval $[n, n+1]$, we obtain

$$(1+n^2)^{-s/2} - (1+(n+1)^2)^{-s/2} \leq sn(1+n^2)^{-s/2-1} \leq sn^{-s-1}.$$

We multiply by $\dim(C_n)$ and sum. Since $s > s_0$, we obtain

$$S_N \leq s \sum_{n=1}^N n^{-s-1} \dim(C_n) \leq s \sum_{n=1}^N Cn^{s_0-s-1} < \infty.$$

For the second statement, assume D_X is s -summable for a fixed s , and let $\beta_n, n \geq 0$ be any ℓ^1 sequence. We denote $r_n := \sum_{m \geq n} |\beta_m|$. Define a bounded operator B on $L^2(X, \mu)$ by setting it equal to r_n on the space $C_n \cap C_{n-1}^\perp$, for all $n \geq 0$. Notice that B is positive and commutes with D_X . The operator $(1 + D_X^2)^{-s/2} B$ is then a positive, trace class operator so, for every $N \geq 1$, we have (line 2 to line 3 is a summation by parts):

$$\begin{aligned} \infty &> \text{Tr} \left((1 + D_X^2)^{-s/2} B \right) \\ &\geq \sum_{n=0}^N (1+n^2)^{-s/2} r_n (\dim(C_n) - \dim(C_{n-1})) \\ &\geq \sum_{n=0}^{N-1} \dim(C_n) [r_n (1+n^2)^{-s/2} - r_{n+1} (1+(n+1)^2)^{-s/2}] \\ &\geq \sum_{n=0}^{N-1} \dim(C_n) [(r_n - r_{n+1})(1+n^2)^{-s/2}] = \sum_{n=0}^{N-1} \dim(C_n) [|\beta_n| (1+n^2)^{-s/2}]. \end{aligned}$$

We have shown that for any $(\beta_n)_{n \geq 0} \in \ell^1$, the sequence $(\beta_n \dim(C_n) (1+n^2)^{-s/2})_{n \geq 0}$ is also in ℓ^1 . It follows that $(\dim(C_n) (1+n^2)^{-s/2})_{n \geq 0}$ is bounded. The conclusion follows after noting that $(1+n^2)^{s/2} \leq Cn^s$, for some constant C and all $n \geq 1$. □

5 The Connes metric

In this section, we develop some tools to analyze the Connes metric and apply them to our specific cases of interest. These results are very technical so it may be of some value to indicate, at least vaguely, what is going on. It seems that a number of factors influence the behaviour of the Connes metric for a subshift X . The two most important ones are:

1. For a given x in X , the sequence $\{n \in \mathbb{N} \mid \pi_n(x) \in X_n^+\}$. That is, in looking at the sequence of words $\pi_n(x)$, how often do we have strict containment $U(\pi_n(x)) \supsetneq U(\pi_{n+1}(x))$? Generally speaking, large gaps between such values of n will tend to help keep the Connes metric finite. For example, in linearly recurrent (in particular substitutive) subshifts, this sequence tends to grow exponentially (see the discussion after Lemma 5.11).

2. For the various w , the ratio $\mu(\pi(w))/\mu(w)$. These ratios remaining bounded seem to help keep the Connes metric finite.

Our main interest is in the operator D_X whose eigenvalues are the natural numbers. However, we will treat the more general case of D for most of our estimates, partly so that we can see the exact rôle of the eigenvalues α_n . We will assume that this sequence is increasing and unbounded.

5.1 Technical preliminaries

For w in X_n^+ , we introduce a space of functions

$$F_w = \text{span} \{ \xi(w') \mid \pi(w') = w \} \cap C_n^\perp.$$

In fact, we could also make the same definition for any w in X_n , but F_w is non-zero if and only if w is in X_n^+ . This is a finite-dimensional space and it will be convenient to have an estimate comparing the supremum norm with the L^2 -norm for its elements.

Lemma 5.1. *For w in X_n^+ and f in F_w , we have*

$$\|f\|_2 \leq \mu(w)^{\frac{1}{2}} \|f\|_\infty \leq R(w)^{\frac{1}{2}} \|f\|_2.$$

Proof. The first inequality is trivial. For the second, since f is in F_w , $f = \sum_{\pi(w')=w} a_{w'} \xi(w')$ and so $\|f\|_\infty = |a_{w'}|$, for some w' with $\pi(w') = w$. Then we have

$$\|f\|_2^2 = \int_{U(w)} f^2 d\mu \geq |a_{w'}|^2 \mu(w') = \mu(w') \|f\|_\infty^2 \geq \mu(w) R(w)^{-1} \|f\|_\infty^2.$$

Multiplying by $R(w)$ and taking square roots yields the result. □

We are next going to introduce a special collection of functions. At the simplest level they provide spanning sets for our subspaces F_w . More importantly and subtly, their interactions with the operator D are quite simple.

Suppose that $\pi(w)$ is in X_n^+ , let

$$\eta(w) = \frac{\mu(\pi(w))}{\mu(w)} \xi(w) - \xi(\pi(w)).$$

Notice that we could make the same definition for any w , but if $\pi(w)$ is not in X_n^+ , for some n , then $\eta(w) = 0$. Also, if w is in X_n^+ , the vectors $\{ \eta(w') \mid \pi(w') = w \}$ span F_w , but are linearly dependent since their sum is zero. Also observe that for any w in X_n , the sequence of vectors $\eta(\pi_{m+1}(w))$, for those $m < n$ with $\pi_m(w)$ in X_m^+ are an orthogonal set.

We summarize some properties of $\eta(w)$.

Lemma 5.2. *Suppose that $\pi(w)$ is in X_n^+ .*

1. $\eta(w)$ is in C_{n+1} .
2. $\eta(w)$ is in C_n^\perp .
3. $\eta(w)^2 = \left(\frac{\mu(\pi(w))}{\mu(w)} - 2\right)\eta(w) + \left(\frac{\mu(\pi(w))}{\mu(w)} - 1\right)\xi(\pi(w))$.
4. $\|\eta(w)\|_2^2 = \mu(\pi(w))\left(\frac{\mu(\pi(w))}{\mu(w)} - 1\right)$.
5. If we consider $\eta(w)$ in $C(X)$ and $\eta(w) + \mathbb{C}$ in the quotient space $C(X)/\mathbb{C}$, then $\|\eta(w) + \mathbb{C}\|_\infty = \frac{\mu(\pi(w))}{2\mu(w)}$.

Proof. The first part is clear. For the second, $\eta(w)$ is clearly supported on $\xi(\pi(w))$ so it is perpendicular to $\xi(w')$, for all $w' \neq \pi(w)$ in X_n . In addition, we compute

$$\langle \eta(w), \xi(\pi(w)) \rangle = \frac{\mu(\pi(w))}{\mu(w)}\mu(w) - \mu(\pi(w)) = 0.$$

For the third part, use the fact that $\xi(w)$ and $\xi(\pi(w))$ are idempotents and their product is the former, so that

$$\begin{aligned} \eta(w)^2 &= \left(\frac{\mu(\pi(w))^2}{\mu(w)^2} - 2\frac{\mu(\pi(w))}{\mu(w)}\right)\xi(w) + \xi(\pi(w)) \\ &= \left(\frac{\mu(\pi(w))}{\mu(w)} - 2\right)\frac{\mu(\pi(w))}{\mu(w)}\xi(w) + \xi(\pi(w)) \\ &= \left(\frac{\mu(\pi(w))}{\mu(w)} - 2\right)(\eta(w) + \xi(\pi(w))) + \xi(\pi(w)) \\ &= \left(\frac{\mu(\pi(w))}{\mu(w)} - 2\right)\eta(w) + \left(\frac{\mu(\pi(w))}{\mu(w)} - 1\right)\xi(\pi(w)). \end{aligned}$$

For the fourth part, we know already that $\frac{\mu(\pi(w))}{\mu(w)}\xi(w) = \eta(w) + \xi(\pi(w))$. Moreover, the two terms on the right are orthogonal, so we have

$$\left(\frac{\mu(\pi(w))}{\mu(w)}\right)^2 \mu(w) = \|\eta(w)\|_2^2 + \mu(\pi(w)),$$

and the conclusion follows.

For the last part, the function $\eta(w)$ takes on three values: 0, $a = \frac{\mu(\pi(w))}{\mu(w)} - 1 > 0$ (on $U(w)$) and $b = -1 < 0$. For any such function, the minimum norm on $\eta(w) + \mathbb{C}$ is obtained at $\eta(w) - \frac{a+b}{2}$ and equals $a - \frac{a+b}{2} = \frac{a-b}{2}$. The rest is a simple computation. \square

We next want to summarize some of the properties of the functions $\eta(w)$ as operators. In particular, their commutators with D are of a simple form.

Lemma 5.3. *Suppose that w is in X_{n+1} with $\pi(w)$ in X_n^+ . Considering $\eta(w)$ as an element of $C(X)$, we have the following.*

1. $\eta(w)C_n \subset \mathbb{C}\eta(w) \subset C_{n+1} \cap C_n^\perp$.
2. $[\eta(w), D] = \eta(w) \otimes \zeta(w)^* - \zeta(w) \otimes \eta(w)^*$, where $\zeta(w) = \mu(\pi(w))^{-1}(D - \alpha_{n+1})\xi(\pi(w))$.
3. $\|[\eta(w), D]\| = \|\eta(w)\|_2 \|\zeta(w)\|_2$.
4. $\zeta(w)$ is in C_n and is non-zero.
5. $\|[\eta(w), D]\| \leq \left(\frac{\mu(\pi(w))}{\mu(w)}\right)^{1/2} \alpha_{n+1}$.

Proof. For the first point, the support of $\eta(w)$ is contained in $U(\pi(w))$. So for any w' in X_n , $\eta(w)\xi(w') = 0$ unless $w' = \pi(w)$, in which case $\eta(w)\xi(w') = \eta(w) \in C_{n+1} \cap C_n^\perp$.

For the second point, we know that $DC_n \subset C_n$ and $D(C_{n+1} \cap C_n^\perp)$ is a subspace of $(C_{n+1} \cap C_n^\perp)$. Combining this with the first part above, we see that $D\eta(w)$ and $\eta(w)D$ both map C_n into $(C_{n+1} \cap C_n^\perp)$. We also know that $[\eta(w), D]|_{C_{n+1}^\perp} = 0$. In other words, $[\eta(w), D](I - P_{n+1}) = 0$. From above, we have $P_n[\eta(w), D]P_n = 0$. In addition, we compute

$$\begin{aligned}
& (P_{n+1} - P_n)[\eta(w), D](P_{n+1} - P_n) \\
&= (P_{n+1} - P_n)(\eta(w)D - D\eta(w))(P_{n+1} - P_n) \\
&= (P_{n+1} - P_n)\eta(w)D(P_{n+1} - P_n) - (P_{n+1} - P_n)D\eta(w)(P_{n+1} - P_n) \\
&= (P_{n+1} - P_n)\eta(w)\alpha_{n+1}(P_{n+1} - P_n) \\
&\quad - (P_{n+1} - P_n)\alpha_{n+1}\eta(w)(P_{n+1} - P_n) = 0
\end{aligned}$$

From all this—writing $I = P_n + (P_{n+1} - P_n) + (I - P_{n+1})$ —, we have

$$[\eta(w), D] = (P_{n+1} - P_n)[\eta(w), D]P_n + P_n[\eta(w), D](P_{n+1} - P_n)$$

Since $[\eta(w), D]$ is skew adjoint, the second term is the opposite of the first. From part 1, the first is a rank one operator of the form $\eta(w) \otimes \zeta(w)^*$, for some $\zeta(w)$ in C_n .

To find $\zeta(w)$, the simplest thing is to apply $[\eta(w), D]$ to $\eta(w)$ so that

$$\begin{aligned}
\|\eta(w)\|_2^2 \zeta(w) &= -[\eta(w), D](\eta(w)) \\
&= -\eta(w)D(\eta(w)) - D(\eta(w)^2) \\
&= -\alpha_{n+1}\eta(w)^2 - D(\eta(w)^2) \\
&= -(\alpha_{n+1} - D)(\eta(w)^2) \\
&= -(\alpha_{n+1} - D) \left[\left(\frac{\mu(\pi(w))}{\mu(w)} - 2 \right) \eta(w) + \left(\frac{\mu(\pi(w))}{\mu(w)} - 1 \right) \xi(\pi(w)) \right] \\
&= 0 + \left(\frac{\mu(\pi(w))}{\mu(w)} - 1 \right) (D - \alpha_{n+1})\xi(\pi(w)).
\end{aligned}$$

Now using the fact that $\|\eta(w)\|_2^2 = \left(\frac{\mu(\pi(w))}{\mu(w)} - 1\right) \mu(\pi(w))$, we have

$$\begin{aligned}\zeta(w) &= \|\eta(w)\|_2^{-2} \left(\frac{\mu(\pi(w))}{\mu(w)} - 1\right) (D - \alpha_{n+1})\xi(\pi(w)) \\ &= \mu(\pi(w))^{-1} (D - \alpha_{n+1})\xi(\pi(w)).\end{aligned}$$

For the third part, having expressed $[\eta(w), D] = \eta(w) \otimes \zeta(w)^* - \zeta(w) \otimes \eta(w)^*$, we note that the first term maps C_n to C_n^\perp and the second maps C_n^\perp to C_n . Hence the norm is equal to the supremum of the norms of the two terms which is simply $\|\eta(w)\|_2 \|\zeta(w)\|_2$.

The first statement of the fourth part follows from the fact that $\xi(\pi(w))$ is in C_n , which is invariant under D , and the formula for $\zeta(w)$. The second part follows because $D - \alpha_{n+1}$ is invertible on C_n as $\alpha_{n+1} > \alpha_1, \alpha_2, \dots, \alpha_n$.

For the last part, we have

$$\begin{aligned}\|[\eta(w), D]\| &= \|\eta(w)\|_2 \|\zeta(w)\|_2 \\ &= \left(\frac{\mu(\pi(w))}{\mu(w)} - 1\right)^{1/2} \mu(\pi(w))^{1/2} \mu(\pi(w))^{-1} \|(D - \alpha_{n+1})\xi(\pi(w))\|_2 \\ &\leq \left(\frac{\mu(\pi(w))}{\mu(w)}\right)^{1/2} \mu(\pi(w))^{-1/2} \|(D - \alpha_{n+1})\xi(\pi(w))\|_2.\end{aligned}$$

It remains only to note that $\mu(\pi(w))^{1/2} = \|\xi(\pi(w))\|$ and that $\xi(\pi(w))$ is a vector in C_n , with $0 \leq D|_{C_n} \leq \alpha_{n+1}$ so that $\|D - \alpha_{n+1}\| \leq \alpha_{n+1}$. \square

A nice simple consequence of this result is that the only functions in C_∞ commuting with D are the scalars. We introduce now the notation Q_w for the orthogonal projection of $L^2(X, \mu)$ onto F_w . It will be convenient to break a function $f \in L^2(X, \mu)$ into its components $Q_w(f)$. Since the commutator of D with the functions $\eta(w')$ has a fairly simple form and Q_w is the projection onto the span of the $\eta(w')$ for $\pi(w') = w$, we can hope that the quantities $[D, Q_w(f)]$ are accessible to computations. Note that $Q_w(f) = \xi(w)(P_{n+1}(f) - P_n(f))$, for any f in $L^2(X, \mu)$.

Theorem 5.4. *Let X be a subshift and assume the sequence α_n is strictly increasing. If f is in $C_\infty(X)$ and $[f, D] = 0$, then f is in $C_0 = \mathbb{C}1$.*

Proof. Begin by adding a scalar to f so that f is in C_0^\perp which clearly alters neither hypothesis nor conclusion. Suppose that f is in C_n . We will prove by induction on n that $f = 0$. The case $n = 0$ is straightforward, since f lies in both C_0 and C_0^\perp . Now assume $n \geq 1$. We may write

$$f = \sum_{0 \leq m < n} \sum_{\pi(w) \in X_m^+} \beta(w) \eta(w),$$

where the $\beta(w)$ are scalars.

Fix w_0 in X_{n-1}^+ and let w_1 be in X_n with $\pi(w_1) = w_0$. We will compute $[f, D]\eta(w_1)$, which is zero by hypothesis. This will be done term by term in the sum above for f , using part 2 of Lemma 5.3. First, suppose that w is in X_m with $m < n$. In this case, both $\eta(w)$ and $\zeta(w)$ lie in C_{n-1} and hence their inner products with $\eta(w_1)$ are zero so this term contributes nothing.

Secondly, if w is in X_n with $\pi(w) \neq w_0$, then $\eta(w)$ is supported on $U(\pi(w))$ while $\eta(w_0)$ is supported on $U(w_0)$. As these sets are disjoint, the vectors are orthogonal. Since $\zeta(w)$ is in C_{n-1} , it is also orthogonal to $\eta(w_0)$.

This leaves us to consider w with $\pi(w) = w_0$. For each such w , $\zeta(w)$ is again in C_{n-1} and is orthogonal to $\eta(w_1)$. We conclude that

$$[f, D]\eta(w_1) = \sum_{\pi(w)=w_0} \beta(w) \langle \eta(w_1), \eta(w) \rangle \zeta(w).$$

Observe from the formula in part 2 that $\zeta(w)$ depends only on $\pi(w) = w_0$ and is non-zero. We conclude that

$$\sum_{\pi(w)=w_0} \beta(w) \langle \eta(w_1), \eta(w) \rangle = 0.$$

Since $\eta(w)$ and $\eta(w_1)$ are real-valued, we also have

$$\begin{aligned} 0 &= \sum_{\pi(w)=w_0} \beta(w) \langle \eta(w), \eta(w_1) \rangle = \left\langle \sum_{\pi(w)=w_0} \beta(w) \eta(w), \eta(w_1) \right\rangle \\ &= \langle Q_{w_0}(f), \eta(w_1) \rangle. \end{aligned}$$

If we now vary w_1 over $\pi^{-1}\{w_0\}$, this forms a spanning set for F_{w_0} . We conclude that $Q_{w_0}(f) = 0$. As this holds for every w_0 in X_n^+ , we conclude that f actually lies in C_{n-1} . By induction hypothesis $f = 0$. \square

It is a good time to observe the following fairly simple consequence of these estimates. We will use this later to give examples of Sturmian systems where the Connes metric is infinite.

Theorem 5.5. *Let X be a subshift and let μ be a σ -invariant probability measure with support X . If the set $\{(n+1)^{-2} \mu(\pi(w))/\mu(w) \mid n \geq 1, \pi(w) \in X_n^+\}$ is unbounded, the Connes metric associated with $(C_\infty(X), L^2(X, \mu), D_X)$ is infinite.*

Proof. It follows from the last part of Lemma 5.3 that the functions

$$f_w = (n+1)^{-1} \left(\frac{\mu(\pi(w))}{\mu(w)} \right)^{-1/2} \eta(w)$$

all satisfy $\|[f_w, D]\| \leq 1$. On the other hand, from part 5 of Lemma 5.2, we have

$$\|f_w + \mathbb{C}\|_\infty = (2(n+1))^{-1} \left(\frac{\mu(\pi(w))}{\mu(w)} \right)^{1/2},$$

which, by hypothesis, is unbounded. \square

The next results will be used as tools in showing the the Connes metric is finite and induces the weak-* topology for some subshifts. The point is really to get estimates on how a function may move vectors in $C_n \cap C_{n-1}^\perp$ to $C_m \cap C_{m-1}^\perp$, for values $n > m$. The second is the important estimate; we include the first because it seems convenient to divide it into two steps.

Lemma 5.6. *Let f be in $\cup_n C_n$. Let w be in X_n^+ and $m < n$ with $\pi_m(w)$ in X_m^+ . We have*

$$\langle [D, f]\eta(\pi_{m+1}(w)), Q_w(f) \rangle = (\alpha_{n+1} - \alpha_{m+1}) \|Q_w(f)\|_2^2 \left(\frac{\mu(\pi_m(w))}{\mu(\pi_{m+1}(w))} - 1 \right).$$

Proof. For convenience let $w' = \pi_{m+1}(w)$. Observe that the function $Q_w(f)$ is in $C_{n+1} \cap C_n^\perp$ while $\eta(w')$ is in $C_{m+1} \cap C_m^\perp$. Therefore, we have

$$\begin{aligned} \langle [D, f]\eta(w'), Q_w(f) \rangle &= \langle Df\eta(w'), Q_w(f) \rangle - \langle fD\eta(w'), Q_w(f) \rangle \\ &= \langle f\eta(w'), DQ_w(f) \rangle - \langle \alpha_{m+1}f\eta(w'), Q_w(f) \rangle \\ &= \langle f\eta(w'), \alpha_{n+1}Q_w(f) \rangle - \langle \alpha_{m+1}f\eta(w'), Q_w(f) \rangle \\ &= (\alpha_{n+1} - \alpha_{m+1}) \langle f\eta(w'), Q_w(f) \rangle. \end{aligned}$$

Now we observe that $Q_w(f)$ is supported on $U(w) \subset U(w')$ (since $w' = \pi_{m+1}(w)$) and $\eta(w')$ is identically $\left(\frac{\mu(\pi(w'))}{\mu(w')} - 1 \right)$ on the latter. Therefore, we have

$$\begin{aligned} \langle [D, f]\eta(w'), Q_w(f) \rangle &= (\alpha_{n+1} - \alpha_{m+1}) \langle f\eta(w'), Q_w(f) \rangle \\ &= (\alpha_{n+1} - \alpha_{m+1}) \left(\frac{\mu(\pi(w'))}{\mu(w')} - 1 \right) \langle f, Q_w(f) \rangle \\ &= (\alpha_{n+1} - \alpha_{m+1}) \left(\frac{\mu(\pi(w'))}{\mu(w')} - 1 \right) \langle Q_w(f), Q_w(f) \rangle. \end{aligned}$$

□

Proposition 5.7. *Let f be in $\cup_n C_n$. Let w be in X_n^+ and $m < n$ with $\pi_m(w)$ in X_m^+ . If $\|[D, f]\| \leq 1$, then*

$$\|Q_w(f)\|_\infty \leq R(w)^{1/2} R(\pi_m(w))^{1/2} (\alpha_{n+1} - \alpha_{m+1})^{-1} \mu(w)^{-1/2} \mu(\pi_{m+1}(w))^{1/2}.$$

Proof. Let $w' = \pi_{m+1}(w)$. From the hypothesis and the last Lemma, we have

$$\begin{aligned} \|\eta(w')\|_2 \|Q_w(f)\|_2 &\geq |\langle [D, f]\eta(w'), Q_w(f) \rangle| \\ &= (\alpha_{n+1} - \alpha_{m+1}) \|Q_w(f)\|_2^2 \left(\frac{\mu(\pi(w'))}{\mu(w')} - 1 \right), \end{aligned}$$

and hence $\|\eta(w')\|_2 \geq (\alpha_{n+1} - \alpha_{m+1}) \|Q_w(f)\|_2 \left(\frac{\mu(\pi(w'))}{\mu(w')} - 1 \right)$. Now we invoke our estimate for $\|\eta(w')\|_2$ from Lemma 5.2 and recall from Lemma 2.1 that

$\left(\frac{\mu(\pi(w'))}{\mu(w')} - 1\right)^{-1} \leq R(\pi(w')) = R(\pi_m(w))$, so we have

$$\begin{aligned}
\|Q_w(f)\|_\infty &\leq \mu(w)^{-1/2} R(w)^{1/2} \|Q_w(f)\|_2 \\
&\leq \mu(w)^{-1/2} R(w)^{1/2} \|\eta(w')\|_2 (\alpha_{n+1} - \alpha_{m+1})^{-1} \left(\frac{\mu(\pi(w'))}{\mu(w')} - 1\right)^{-1} \\
&= \mu(w)^{-1/2} R(w)^{1/2} \mu(w')^{1/2} \left(\frac{\mu(\pi(w'))}{\mu(w')} - 1\right)^{1/2} \\
&\quad (\alpha_{n+1} - \alpha_{m+1})^{-1} \left(\frac{\mu(\pi(w'))}{\mu(w')} - 1\right)^{-1} \\
&\leq \mu(w)^{-1/2} R(w)^{1/2} \mu(w')^{1/2} (\alpha_{n+1} - \alpha_{m+1})^{-1} \left(\frac{\mu(\pi(w'))}{\mu(w')} - 1\right)^{-1/2} \\
&= \mu(w)^{-1/2} R(w)^{1/2} \mu(\pi_{m+1}(w))^{1/2} (\alpha_{n+1} - \alpha_{m+1})^{-1} R(\pi_m(w))^{-1/2}.
\end{aligned}$$

The conclusion follows. \square

5.2 Shifts of finite type

Theorem 5.8. *Let G be an aperiodic, irreducible graph, X^G be its associated shift of finite type and μ be the Parry measure. The Connes metric from $(C_\infty(X^G), L^2(X^G, \mu), D_{X^G})$ is infinite.*

The proof will involve some technical lemmas. Let us first lay down some basic tools. Recall that we write our vertex set $G^0 = \{1, 2, \dots, I\}$. Also recall that u_G, v_G are the left and right Perron eigenvectors for the incidence matrix A with Perron eigenvalue λ_G , appropriately normalised. For convenience, we shorten these to u, v and λ respectively. The invariant measure is defined by Equation (2).

Since G is aperiodic there exists a vertex j with $\#t^{-1}\{j\} > 1$. Let us assume that $j = 1$. It follows then that for any word $w = e_1 e_2 \dots e_n$ of even length with $i(e_1) = 1$, w is in X_n^+ .

From the fact that G is irreducible, we may find a word $w = e_1 e_2 \dots e_p$ with $i(e_1) = 1 = t(e_p)$. For each $k \geq 1$, we define $w_k = e_p w^{2k}$. Note that w_k is in X_{2kp+1} and $\pi(w_k) = w^{2k}$, which is in X_{2kp}^+ .

We note that $\mu(w_k) = \|\xi(w_k)\|_2^2 = v_{i(e_p)} u_1 \lambda^{-2kp}$ and that $\mu(\pi(w_k)) = \|\xi(\pi(w_k))\|_2^2 = v_1 u_1 \lambda^{-2kp+1}$. It follows that

$$1 < \frac{\mu(\pi(w_k))}{\mu(w_k)} = v_1 v_{i(e_p)}^{-1} \lambda. \quad (4)$$

This is constant in k and we denote its value by a for convenience.

For $K \geq 1$, consider the function $f_K = \sum_{k=1}^K k^{-1} \eta(w_k)$. Notice this function is in C_0^\perp , so it takes on both positive and negative values. Secondly, from the

definition, we see that

$$f_K|U(w_k) = \sum_{k=1}^K k^{-1} \left(\frac{\mu(\pi(w_k))}{\mu(w_k)} - 1 \right) = (a-1) \sum_{k=1}^K k^{-1}.$$

Since f_K takes on negative values, we conclude that in the quotient $C(X)/\mathbb{C}$, we have $\|f_K + \mathbb{C}\|_\infty \geq \frac{a-1}{2} \sum_{k=1}^K k^{-1}$. The obvious conclusion being that these norms go to infinity as K does. We will now show that $\|[f_K, D]\|$ is bounded and the conclusion follows from [25, Theorem 2.1].

It follows from part 3 of Lemma 5.3 that

$$\begin{aligned} [f_K, D] &= \sum_{k=1}^K k^{-1} (\eta(w_k) \otimes \zeta(w_k)^* - \zeta(w_k) \otimes \eta(w_k)^*) \\ &= \left(\sum_{k=1}^K k^{-1} \zeta(w_k) \otimes \eta(w_k)^* \right)^* - \sum_{k=1}^K k^{-1} \zeta(w_k) \otimes \eta(w_k)^*. \end{aligned}$$

Let $T_K = \sum_{k=1}^K k^{-1} \zeta(w_k) \otimes \eta(w_k)^*$. It suffices to prove that $\|T_K\|$ is bounded, independent of K .

First notice that T_K is zero on the orthogonal complement of the span of the $\eta(w_k)$, $1 \leq k \leq K$. Secondly, this set is orthonormal. Therefore, in order to control $\|T_K\|$, it suffices to evaluate the norm of products $\|T_K \eta\|$ for vectors η of the form

$$\eta = \sum_{k=1}^K \beta_k \mu(\pi(w_k))^{-1/2} (a-1)^{-1/2} \eta(w_k).$$

It follows from Lemma 5.2 and Equation (4) that $\|\eta\|_2^2 = \sum_{k=1}^K |\beta_k|^2$.

We also have

$$\begin{aligned} T_K \eta &= \left[\sum_{k=1}^K k^{-1} \zeta(w_k) \otimes \eta(w_k)^* \right] \sum_{k'=1}^K \beta_{k'} \mu(\pi(w_{k'}))^{-1/2} (a-1)^{-1/2} \eta(w_{k'}) \\ &= \sum_{k=1}^K \sum_{k'=1}^K k^{-1} \beta_{k'} \mu(\pi(w_{k'}))^{-1/2} (a-1)^{-1/2} \langle \eta(w_{k'}), \eta(w_k) \rangle \zeta(w_k) \\ &= \sum_{k=1}^K k^{-1} \beta_k \mu(\pi(w_k))^{-1/2} (a-1)^{-1/2} \mu(\pi(w_k)) (a-1) \zeta(w_k) \\ &= \sum_{k=1}^K k^{-1} \beta_k \mu(\pi(w_k))^{1/2} (a-1)^{1/2} \zeta(w_k). \end{aligned}$$

Lemma 5.9. *For all $k \geq 1$, we have*

1. $P_{2p(k-1)} \xi(\pi(w_k)) = \lambda^{-2p} \xi(\pi(w_{k-1}))$,
2. $\xi(\pi(w_k)) - \lambda^{-2p} \xi(\pi(w_{k-1}))$ is in $C_{2pk} \cap C_{2p(k-1)}^\perp$,

$$3. \|\xi(\pi(w_k)) - \lambda^{-2p}\xi(\pi(w_{k-1}))\|_2^2 = u_1 v_1 (\lambda - \lambda^{1-2p}) \lambda^{-2pk}.$$

$$4. \xi(\pi(w_k)) = \sum_{j=1}^k \lambda^{-2p(k-j)} [\xi(\pi(w_j)) - \lambda^{-2p}\xi(\pi(w_{j-1}))] + \lambda^{-2pk}.$$

Proof. The vectors $\xi(w), w \in X_{2p(k-1)}$ form an orthonormal set in $C_{2p(k-1)}$. The only one of these with a non-zero inner product with $\xi(\pi(w_k))$ is $\xi(\pi(w_{k-1}))$. Hence we have

$$\begin{aligned} P_{2p(k-1)}\xi(\pi(w_k)) &= \frac{\langle \xi(\pi(w_k)), \pi(\xi(w_{k-1})) \rangle}{\langle \xi(\pi(w_{k-1})), \pi(\xi(w_{k-1})) \rangle} \xi(\pi(w_{k-1})) \\ &= \frac{\mu(\xi(\pi(w_k)))}{\mu(\xi(\pi(w_{k-1})))} \xi(\pi(w_{k-1})) = \lambda^{-2p}\xi(\pi(w_{k-1})). \end{aligned}$$

The second part follows immediately from the first. The third follows from the orthogonality of $\lambda^{-2p}\xi(\pi(w_{k-1}))$ and $\xi(\pi(w_k)) - \lambda^{-2p}\xi(\pi(w_{k-1}))$, so we have

$$\begin{aligned} \|\xi(\pi(w_k)) - \lambda^{-2p}\xi(\pi(w_{k-1}))\|_2^2 &= \|\xi(\pi(w_k))\|_2^2 - \|\lambda^{-2p}\xi(\pi(w_{k-1}))\|_2^2 \\ &= u_1 v_1 \lambda^{-2kp+1} - \lambda^{-4p} u_1 v_1 \lambda^{-2(k-1)p+1} \\ &= u_1 v_1 (\lambda - \lambda^{1-2p}) \lambda^{-2kp}. \end{aligned}$$

The last also follows from the first and an easy induction argument, which we omit. \square

In the following computation, we denote $\xi(\pi(w_j)) - \lambda^{-2p}\xi(\pi(w_{j-1}))$ by ξ_j , for $1 \leq j \leq K$. We compute

$$\begin{aligned} T_K \eta &= \sum_{k=1}^K k^{-1} \beta_k \mu(\pi(w_k))^{1/2} (a-1)^{1/2} \zeta(w_k) \\ &= \sum_{k=1}^K k^{-1} \beta_k \mu(\pi(w_k))^{-1/2} (a-1)^{1/2} (2pk+1-D) \xi(\pi(w_k)) \\ &= \sum_{k=1}^K k^{-1} \beta_k (u_1 v_1 \lambda^{-2pk+1})^{-1/2} (a-1)^{1/2} (2pk+1-D) \xi(\pi(w_k)) \\ &= \left(\frac{a-1}{u_1 v_1 \lambda} \right)^{1/2} \sum_{k=1}^K k^{-1} \beta_k \lambda^{pk} (2pk+1-D) \\ &\quad \left[\sum_{j=1}^k \lambda^{-2p(k-j)} (\xi(\pi(w_j)) - \lambda^{-2p}\xi(\pi(w_{j-1}))) + \lambda^{-2pk} \right] \\ &= \left(\frac{a-1}{u_1 v_1 \lambda} \right)^{1/2} \sum_{k=1}^K \sum_{j=1}^k k^{-1} \beta_k \lambda^{-pk} \lambda^{2pj} (2pk+1-D) \xi_j \\ &\quad + \left(\frac{a-1}{u_1 v_1 \lambda} \right)^{1/2} \sum_{k=1}^K k^{-1} \beta_k \lambda^{pk} (2pk+1-D) \lambda^{-2pk}. \end{aligned}$$

Let us begin by considering the second term on the right which is simply a constant function considered as a vector in $L^2(X, \mu)$. Ignoring the constants in front of the sum, the norm of this vector is bounded by

$$\begin{aligned} \sum_{k=1}^K |\beta_k| k^{-1} (2pk+1) \lambda^{-pk} &\leq \sum_{k=1}^K |\beta_k| (2p+1) \lambda^{-pk} \\ &\leq (2p+1) \left(\sum_{k=1}^K |\beta_k|^2 \right)^{1/2} \left(\sum_{k=1}^K \lambda^{-2pk} \right)^{1/2} \\ &\leq (2p+1) \left(\frac{\lambda^{-2p}}{1-\lambda^{-2p}} \right)^{1/2} \|\eta\|_2. \end{aligned}$$

Now we turn to the first term. For any given j , the vector ξ_j is in $C_{2pj} \cap C_{2p(j-1)}^\perp$ and on this space $1+2p(j-1) \leq D \leq 2pj$. For values of $k \geq j$, we have $0 \leq 2pk+1-D \leq 2p(k-j+1)$ and, in particular, $\|2pk+1-D\| \leq 2p(k-j+1)$. Fixing k for the moment, the norm of the vector $\sum_{j=1}^k \lambda^{-pk} \lambda^{2pj} (2pk+1-D) \xi_j$ is bounded by:

$$\begin{aligned} \sum_{j=1}^k \lambda^{2pj} \lambda^{-pk} (k-j+1) (u_1 v_1 (\lambda - \lambda^{1-2p}) \lambda^{-2pk})^{1/2} \\ \leq C \sum_{j=1}^k \lambda^{-2p(k-j)} (k-j+1) = C \sum_{j=1}^k j \lambda^{-2p(j-1)} \end{aligned}$$

where C gathers the constants and the equality is a change of variable. Now, by deriving the power series $\sum x^n$ and letting $x = \lambda^{-2p}$ we get

$$\sum_{j=1}^k j \lambda^{-2p(j-1)} \leq \frac{1}{(1-\lambda^{-2p})^2}.$$

If we now sum over k and apply the Cauchy-Schwarz inequality, we get

$$\begin{aligned} \left\| \sum_{k=1}^K \sum_{j=1}^k k^{-1} \beta_k \lambda^{-pk} \lambda^{2pj} (2pk+1-D) \xi_j \right\| &\leq C' \sum_{k=1}^K |\beta_k| k^{-1} \\ &\leq C' \|\beta\|_2 \left(\sum_{k \geq 1} k^{-2} \right)^{1/2} \leq C'' \|\beta\|_2. \end{aligned}$$

Putting all of this together we have shown that $\|T_K \eta\|_2$ is bounded above by a constant, independent of K , times $\|\eta\|_2$. This completes the proof of Theorem 5.8.

5.3 Linearly recurrent subshifts

Theorem 5.10. *Let X be a linearly recurrent subshift and let μ be its unique invariant measure. Then the spectral triple*

$$(C_\infty(X), L^2(X, \mu), D_X)$$

has finite Connes distance and the topology induced on the state space of $C(X)$ is the weak- topology.*

The proof will occupy the remainder of the subsection, and use Proposition 5.7 in a crucial way.

The general idea of the proof goes along these lines: starting from a function $f \in C_\infty(X)$ with $\|[f, D]\| \leq 1$, Proposition 5.7 allows to control $\|Q_w(f)\|_\infty$ where $w \in X_n^+$, in terms of n . Since any such function can be written up to a constant term as $f = \sum_{w \in X^+} Q_w(f)$, this allows to control the norm of f in the algebra $C(X)/\mathbb{C}$. Eventually, Rieffel's criterion [24, 25] is applied.

First, this lemma is an immediate consequence of Proposition 2.5.

Lemma 5.11. *Let X be a linearly recurrent subshift. There are $C_1, C_2 > 0$ such that for all n and all $w \in X_n$, $C_1 \leq R(w) \leq C_2$.*

Let us consider now $w \in X_n^+$. Any such w is the prefix of a word of the form rw where r is a return word to w . Therefore, all sufficiently long right-extensions of w start by a word of the form rw , with r some return-word to be picked in a finite set. Besides, the length of rw is always at least $(1 + 1/K)n$. Therefore, w has at most $K(K + 1)^2$ right-extensions of length $(1 + 1/K)n$. As a conclusion, there are at most $K(K + 1)^2$ indices $n < k \leq (1 + 1/K)n$ for which the right-extension of length k of w (say w') is a special word.

A similar result holds for left-extensions. The conclusion is then that for any $w \in X_n$, the number of indices $n < k \leq n(1 + 1/K)$ for which there is a $w' \in \pi_n^{-1}(\{w\}) \cap X_k^+$ is bounded above, and this bound is uniform in n (it only depends on K).

Consider now a sequence $f_i \in C_\infty$, such that $\|[D, f_i]\| \leq 1$ for all i . The goal is to extract a converging subsequence from the image of the sequence $(f_i)_i$ in the quotient algebra $C(X)/\mathbb{C}$. If we assume that all the f_i belong to C_0^\perp (which is possible up to adding a suitable scalar multiple of the identity), it is then sufficient to prove that the sequence $(f_i)_i$ has a convergent subsequence in $C(X)$.

Since $f_i \in C_0^\perp$, we can write

$$f_i = \sum_{w \in X^+} Q_w(f_i).$$

For a fixed w in X_n^+ , $n \geq 1$, we may apply Proposition 5.7 with $m = 0$ to conclude that

$$\begin{aligned} \|Q_w(f_i)\|_\infty &\leq R(w)^{1/2} R(\varepsilon)^{1/2} (n + 1)^{-1} \mu(w)^{-1/2} \\ &\leq C_0 (n + 1)^{-1} \mu(w)^{-1/2}, \end{aligned} \tag{5}$$

where C_0 is a constant which does not depend on w . In particular, this is bounded independent of i . Since the vector space F_w is finite-dimensional, any bounded sequence must have a convergent subsequence.

Enumerate $X^+ = \{w_1, w_2, w_3, \dots\}$. Let $(f_n^1)_{n \geq 1}$ be a subsequence of $(f_n)_{n \geq 1}$ such that $Q_{w_1}(f_n^1)$ converges. Inductively define a sequence $(f_n^k)_{n \geq 1}$ as a subsequence of $(f_n^{k-1})_{n \geq 1}$ such that $(Q_{w_k}(f_n^k))_n$ converges.

We now prove that the sequence $(f_n^n)_{n \geq 1}$ converges in $C(X)$. It is enough to show that it is Cauchy. The goal is to show that $\|f_n^n(x) - f_m^m(x)\|_\infty$ goes to zero uniformly in x when n, m tend to infinity.

To prepare this, consider a function f and evaluate $\sum_{w \in X^+} Q_w(f)(x)$ for $x \in X$. For a given x , infinitely many terms of the sum are zero. It is possible to rewrite it: if $f \in C_0^\perp$ and $\|[D, f]\| \leq 1$,

$$|f(x)| = \left| \sum_{n \in \mathbb{N}} \mathbf{1}_{\{\pi_n(x) \in X_n^+\}}(n) Q_{\pi_n(x)}(f)(x) \right|.$$

In particular,

$$\left| f(x) - \sum_{n=1}^N \sum_{w \in X_n^+} Q_w(f)(x) \right| \leq \sum_{n > N} \mathbf{1}_{\{\pi_n(x) \in X_n^+\}}(n) \|Q_{\pi_n(x)}(f)\|_\infty. \quad (6)$$

By the argument above, the cardinality of the set

$$\{n \in [(1 + 1/K)^k, (1 + 1/K)^{k+1}] \mid \pi_n(x) \in X_n^+\}$$

is bounded above by $K(K+1)^2$, independently of k and x . Call I_k the interval $[(1 + 1/K)^k, (1 + 1/K)^{k+1})$. One has

$$\left| f(x) - \sum_{n=1}^N \sum_{w \in X_n^+} Q_w(f)(x) \right| \leq \sum_{k \geq k_0} K(K+1)^2 \max \{ \|Q_{\pi_n(x)}(f)\|_\infty : n \in I_k \text{ such that } \pi_n(x) \in X_n^+ \}. \quad (7)$$

Now, it turns out that if $n \in I_k$, then $\mu(w)^{-1/2} \leq K^{1/2}(1 + 1/K)^{(k+1)/2}$, and $1/n \leq (1 + 1/K)^{-k}$, so using the bound of Equation (5) with $w = \pi_n(x)$ together with Lemma 2.7, we get

$$\|Q_{\pi_n(x)}(f)\|_\infty \leq C\alpha^k, \quad \text{for } \alpha = \left(1 + \frac{1}{K}\right)^{-1/2} < 1.$$

It is crucial to notice that both constants C and α are completely independent of f and x (as long as $\|[D, f]\| \leq 1$). In conclusion, the convergence in (7) is geometric, and $\sum_{w \in X^+} Q_w(f)$ converges uniformly to f on X . In particular, its norm converges.

Back to the sequence $(f_n^n)_{n \geq 1}$. Let $\varepsilon > 0$. First, choose M such that $\left\| \sum_{n \geq M} \sum_{w \in X_n^+} Q_w(f) \right\|_\infty$ is smaller than $\varepsilon/4$ (independent of f). Next, number the elements of $\bigcup_{i < M} X_i^+ := \{w_1, \dots, w_L\}$. For each $1 \leq l \leq L$, pick a

number I_l such that for all $n, m \geq I_l$,

$$\|Q_{w_l}(f_n^l) - Q_{w_l}(f_m^l)\|_\infty \leq \frac{\varepsilon}{2L}.$$

Pick $I = \max\{I_l \mid 1 \leq l \leq L\}$. Then, for all $n, m \geq I$, we can compute

$$\begin{aligned} \|f_n^n - f_m^m\|_\infty &= \left\| \sum_{w \in X^+} Q_w(f_n^n - f_m^m) \right\|_\infty \\ &\leq \left\| \sum_{l=1}^L Q_{w_l}(f_n^n - f_m^m) \right\|_\infty + \left\| \sum_{\substack{n \geq M \\ w \in X_n^+}} Q_w(f_n^n) \right\|_\infty + \left\| \sum_{\substack{n \geq M \\ w \in X_n^+}} Q_w(f_m^m) \right\|_\infty \\ &\leq \sum_{l=1}^L \left\| Q_{w_l}(f_n^n) - Q_{w_l}(f_m^m) \right\|_\infty + \left\| \sum_{\substack{n \geq M \\ w \in X_n^+}} Q_w(f_n^n) \right\|_\infty + \left\| \sum_{\substack{n \geq M \\ w \in X_n^+}} Q_w(f_m^m) \right\|_\infty \\ &\leq L \frac{\varepsilon}{2L} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon. \end{aligned}$$

Therefore, the sequence is Cauchy, so it converges in $C(X)$.

5.4 Sturmian systems

In this section, it is understood that the subshift in question is $X = X^\theta$. The following result is a consequence of Theorem 2.11.

Lemma 5.12. *Let X^θ be a Sturmian subshift with $0 < \theta < 1$ irrational. Let $\theta = [0; a_1, a_2, \dots]$ be the continued fraction expansion of θ . Then:*

1. *Assume $q_n + q_{n-1} - 1 \leq m < q_{n+1} + q_n - 1$. Then for all $w \in X_m^+$, $R(w) \leq a_{n+1} + 1$.*
2. *For any n , there exist $q_n + q_{n-1} \leq m < q_{n+1} + q_n$ and w in X_m such that $\mu(\pi(w))/\mu(w) \geq a_{n+1}$.*
3. *Let $x \in X$. Then, for all n ,*

$$\#\{m \in \mathbb{N} \cap [q_n + q_{n-1}, q_{n+1} + q_n) \mid \pi_m(x) \in X_m^+\} \leq a_{n+1} + 2$$

4. $q_n \geq 2^{(n-1)/2}$.

Proof. Point 1 is obtained by direct inspection of the formulas in Theorem 2.11. If $w \in X_m^+$ with $q_n + q_{n-1} \leq m < q_{n+1} + q_n - 1$, then the biggest quotient of the form $\mu(w)/\mu(w')$ for $\pi(w') = w$ is of the form

$$\frac{\lambda_n - k'\lambda_{n+1}}{\lambda_{n+1}} \leq \frac{\lambda_n}{\lambda_{n+1}} < a_{n+1} + 1$$

for some $0 \leq k' \leq a_{n+1}$. If $m = q_n + q_{n-1} - 1 = (a_n + 1)q_{n-1} + q_{n-2}$, such a quotient is of the form $\lambda_n/\lambda_{n+1} \leq a_{n+1} + 1$.

Similarly for point 2, observe that the frequencies associated with words of length $q_n + q_{n-1} - 1$ are λ_{n+1} and λ_n , while the frequencies associated with words of length $q_n + q_{n-1}$ are λ_n , λ_{n+1} and $\lambda_n - \lambda_{n+1}$. Since $\#X_m^+ = 1$ for all m , let w' be the unique word in $X_{q_n+q_{n-1}-1}^+$. By inspection, $\mu(w') = \lambda_n$ and it has two extensions of frequencies λ_{n+1} and $\lambda_n - \lambda_{n+1}$. Let w be its extension of frequency λ_{n+1} . Then $\mu(w')/\mu(w) = \theta_{n+1}^{-1} \geq a_{n+1}$.

Point 3 can also be deduced from Theorem 2.11: $w \in X_m^+$ if and only if $\mu(w) \neq \mu(w')$ for $w' \in \pi^{-1}(\{w\})$. Now, $\mu(w)$ can only take $a_{n+1} + 2$ values if $w \in X_m$ and $q_n + q_{n-1} \leq m < q_{n+1} + q_n$. This proves the result.

Point 4 is classic (see [20, Theorem 12]). \square

Let us begin with the less positive result, simply because it is easier to prove.

Theorem 5.13. *There exists $0 < \theta < 1$ irrational such that the spectral triple $(C_\infty(X^\theta), L^2(X^\theta, \mu), D_{X^\theta})$ has infinite Connes metric.*

Proof. Choose $a_1 = 1$. Define a_n inductively by $a_{n+1} = n(q_n + q_{n-1})^2$. Letting m and w be as in part 2 of Lemma 5.12, we have

$$(m+1)^{-2} \frac{\mu(\pi(w))}{\mu(w)} \geq (m+1)^{-2} a_{n+1} \geq (q_{n+1} + q_n)^{-2} a_{n+1} = n.$$

The conclusion follows from Theorem 5.5. \square

On the more encouraging side, we prove the following.

Theorem 5.14. *Let $0 < \theta < 1$ be irrational with $\theta = [0; a_1, a_2, \dots]$. If there exist constants $C \geq 1$, $s \geq 1$ such that $a_j \leq Cj^s$, then the spectral triple $(C_\infty(X^\theta), L^2(X^\theta, \mu), D_{X^\theta})$ has finite Connes metric and the topology induced on the state space of $C(X^\theta)$ is the weak-* topology.*

This result has an immediate consequence.

Corollary 5.15. *For almost all $\theta \in (0, 1)$ (for the Lebesgue measure), the spectral triple $(C_\infty(X^\theta), L^2(X^\theta, \mu), D_{X^\theta})$ has finite Connes metric and the topology induced on the state space of $C(X^\theta)$ is the weak-* topology.*

Proof. By a result of Khinchin [20, Theorem 30], if φ is a positive function defined on the natural numbers, then the estimate $a_n = O(\varphi(n))$ holds for almost all $\theta \in (0, 1)$ if and only if $\sum_{n \geq 1} (\varphi(n))^{-1}$ converges. In this case, we consider the function $\varphi(n) = n^s$ for $s > 1$. It follows that for almost all $\theta \in (0, 1)$, the numbers a_n satisfy $a_n \leq Cn^s$. In particular, almost all $\theta \in (0, 1)$ satisfy the hypotheses of Theorem 5.14. \square

We now turn to the proof of Theorem 5.14. It is actually quite similar to that of Theorem 5.10 for linearly recurrent subshifts.

As for linearly recurrent subshift, if $f \in C_0^+$, we estimate

$$\left| f(x) - \sum_{n=1}^N \sum_{w \in X_n^+} Q_w(f)(x) \right| \leq \sum_{n > N} \mathbf{1}_{\{\pi_n(x) \in X_n^+\}}(n) \|Q_{\pi_n(x)}(f)\|_\infty. \quad (8)$$

Now, we partition the positive integers into intervals $[q_k + q_{k-1} - 1, q_{k+1} + q_k - 1)$. For $w \in X_n^+$ with n in this interval, we can use Proposition 5.7 together with points 1 and 2 of Lemma 5.12 to get the following estimate. This estimate holds for any m in $[q_{k-2} + q_{k-3} - 1, q_{k-1} + q_{k-2} - 1)$ such that $\pi_m(w) \in X_m^+$.

$$\begin{aligned} \|Q_w(f)\|_\infty &\leq R(w)^{1/2} R(\pi_m(w))^{1/2} (n-m)^{-1} \mu(w)^{-1/2} \mu(\pi_{m+1}(w))^{1/2} \\ &\leq (a_{k+1} a_k)^{1/2} (n-m)^{-1} \mu(w)^{-1/2} \mu(\pi_{m+1}(w))^{1/2}. \end{aligned}$$

Now, given the intervals chosen,

$$(n-m)^{-1} \leq (q_k - q_{k-2})^{-1} = (a_k q_{k-1})^{-1} \leq 2^{-(k/2-1)} \leq \alpha^k$$

for some $\alpha < 1$. Furthermore, it results directly from Theorem 2.11 that $\mu(w)^{-1/2} \leq \lambda_{k+1}^{-1/2}$, and $\mu(\pi_{m+1}(w))^{1/2} \leq \lambda_{k-2}^{1/2}$. Therefore, we have

$$\|Q_w(f)\|_\infty \leq (a_{k+1} a_k)^{1/2} \alpha^k (\theta_{k-1} \theta_k \theta_{k+1})^{-1/2}.$$

It is straightforward from the definition that $\theta_k^{-1} \leq a_k + 1$. So using the assumption that $a_k = O(k^s)$, we can write

$$\|Q_w(f)\|_\infty = O(k^{5s/2} \alpha^k).$$

In other words, this is bounded above by a geometric series in k . The rest is similar to the proof for linearly recurrent subshifts: using point 3 of Lemma 5.12, we prove that the expressions in Equation (8) are dominated by

$$\sum_{k \geq k_0} k^s (k^{5s/2} \alpha^k),$$

for some $k_0 = k_0(N)$. It is essentially the remainder of a geometric series, which does not depend on x . Therefore the convergence of $\sum_w Q_w(f)$ to f is uniform. The rest of the proof is identical as before.

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