THE TWO-PRIME ANALOGUE OF THE HECKE $C^*$-ALGEBRA
OF BOST AND CONNES

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Abstract. Let $p$ and $q$ be distinct odd primes. We analyse a semigroup crossed
product $C^*(G_{p,q}) \rtimes N^2$ similar to the semigroup crossed product which models the
Hecke $C^*$-algebra of Bost and Connes. We describe a composition series of ideals
in $C^*(G_{p,q}) \rtimes N^2$, and show that the structure of one of the subquotients reflects
interesting number-theoretic information about the multiplicative orders of $q$ in the
rings $\mathbb{Z}/p^l\mathbb{Z}$.

In [3], Bost and Connes introduced and studied a Hecke $C^*$-algebra $\mathcal{C}_Q$ which has
many fascinating connections with number theory. It was shown in [11] that $\mathcal{C}_Q$
can be realised as a crossed product $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes N^*$ by an endomorphic action $\alpha$
of the multiplicative semigroup $N^*$ of positive integers, and this realisation gives a
great deal of insight into the Bost-Connes analysis (see [9]). Here we fix two odd
primes $p$ and $q$, and analyse the semigroup crossed product $C^*(G_{p,q}) \rtimes N^2$
associated to the subgroup $G_{p,q} := \{n/p^kq^l : n \in \mathbb{Z}\}/\mathbb{Z}$ of $\mathbb{Q}/\mathbb{Z}$ and the restriction of $\alpha$
to the subsemigroup $\{p^kq^l\} \subset N^*$, which is isomorphic to the additive semigroup $N^2$.
This crossed product still exhibits rich connections with number theory, though of a
somewhat different nature: it has a subquotient, for example, whose ideal structure
encodes the multiplicative orders of $q$ in the rings $\mathbb{Z}/p^l\mathbb{Z}$.

We begin our analysis by passing to the Fourier transform of our dynamical sys-
tem, which involves the algebras of continuous functions on the spaces of $p$-adic and
$q$-adic integers. We describe our dynamical system $(C^*(G_{p,q}), N^2, \alpha)$ and its Fourier
transform in §1. Next we construct a composition series for $C^*(G_{p,q}) \rtimes N^2$ using gen-
eral results about invariant ideals and tensor products of semigroup crossed products
which have been worked out in [13]. Our main structure theorem is Theorem 2.2,
which is proved in §2 and §3. Theorem 3.1, which gives a detailed description of an
ordinary crossed product $C(\mathcal{U}(\mathbb{Z}_p)) \rtimes \mathbb{Z}$ arising in our analysis, is interesting in its
own right: it shows, for example, that $C(\mathcal{U}(\mathbb{Z}_p)) \rtimes \mathbb{Z}$ is simple if and only if $q$ is a
primitive root modulo $p^l$ for all $l$, which happens if and only if it is primitive modulo
$p^l$ for any single $l > 1$ (see Remark 3.8). In the last section, we describe the topology


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on the primitive ideal space of $C^*(G_{p,q}) \rtimes_\alpha \mathbb{N}^2$, thus completely determining its ideal structure.

1. The dynamical system and its Fourier transform

Let $p$ and $q$ be distinct odd primes. We consider the additive group

$$\mathbb{Z}[p^{-1}, q^{-1}] = \{rp^{-k}q^{-l} : r, k, l \in \mathbb{Z}\}$$

and its quotient $G_{p,q} := \mathbb{Z}[p^{-1}, q^{-1}]/\mathbb{Z}$. We write $\alpha$ for the action of $\mathbb{N}^2$ by endomorphisms of the group $C^*$-algebra $C^*(G_{p,q})$ which is characterised on the canonical generating unitaries $\{\delta_r : r \in G_{p,q}\}$ by

$$\alpha_{m,n}(\delta_r) = \frac{1}{p^mq^n} \sum_{\{s \in G_{p,q} : p^mq^n s = r\}} \delta_s; \quad (1.1)$$

we can see that there is such an action either by modifying [11, Proposition 2.1] or by applying the general method of [14, §1] to the action of $\mathbb{N}^2$ on $\mathbb{Z}$ defined by $\eta_{m,n}(k) = p^mq^n k$ (see [14, Example 1.2]). As in [10, Proposition 2.1], the action satisfies

$$\alpha_{k,l}(1)\alpha_{m,n}(1) = \alpha_{k\lor m, l\lor n}(1). \quad (1.2)$$

A covariant representation of the dynamical system $(C^*(G_{p,q}), \mathbb{N}^2, \alpha)$ consists of a nondegenerate representation $\pi$ of $C^*(G_{p,q})$ and a representation $V$ of $\mathbb{N}^2$ by isometries on the same space such that

$$\pi(\alpha_{m,n}(a)) = V_{m,n}\pi(a)V_{m,n}^* \quad \text{for } a \in C^*(G_{p,q}) \text{ and } (m, n) \in \mathbb{N}^2; \quad (1.3)$$

the relation (1.2) then implies that the isometric representation $V$ is Nica covariant, in the sense that $V_{k,l}V_{i,l}V_{m,n}V_{m,n}^* = V_{k\lor m, i\lor n}V_{k\lor m, i\lor n}^*$. One can see that the system has nontrivial covariant representations by modifying the constructions in [11], or by applying [14, Lemma 1.7]. Thus there is a crossed product $(C^*(G_{p,q}) \rtimes_\alpha \mathbb{N}^2, i_A, i_S)$, which is a universal $C^*$-algebra for covariant representations of the system (see [10, Proposition 2.1]). (To avoid complicated notation, we always write $i_A$ and $i_S$ for the algebra and semigroup components of the universal covariant representation.) This crossed product carries a dual action $\widehat{\alpha}$ of $\mathbb{T}^2$ which leaves $i_A(C^*(G_{p,q}))$ invariant and satisfies $\widehat{\alpha}_{w,z}(i_S(m, n)) = w^nz^n i_S(m, n)$.

To compute the Fourier transform of the system, we need a description of the dual group $\widehat{G}_{p,q}$. Note that with $G_p := \mathbb{Z}[p^{-1}]/\mathbb{Z}$, the map $(r, s) \mapsto r + s$ is an isomorphism of $G_p \times G_q$ onto $G_{p,q}$, and, dually, we have $\widehat{G}_{p,q} \cong \widehat{G}_p \times \widehat{G}_q$. To describe $\widehat{G}_p$, note that $\mathbb{Z}[p^{-1}] = \bigcup_{l=1}^{\infty} p^{-l}\mathbb{Z}$, so $G_p = (\bigcup p^{-l}\mathbb{Z})/\mathbb{Z}$ has a natural description as a direct limit $\varinjlim p^{-l}\mathbb{Z}/\mathbb{Z}$, and $\widehat{G}_p$ is an inverse limit $\varprojlim (p^{-l}\mathbb{Z}/\mathbb{Z})^\sim$ of finite groups. The usual pairing $\langle t, n \rangle = \exp 2\pi i tn$ of $\mathbb{Z}$ with $\mathbb{R}/\mathbb{Z}$ induces an isomorphism of $\mathbb{Z}/p^l\mathbb{Z}$ onto $(p^{-l}\mathbb{Z}/\mathbb{Z})^\sim$, and it is easy to check that the dual of the inclusion $p^{-l}\mathbb{Z}/\mathbb{Z} \hookrightarrow p^{-(l+1)}\mathbb{Z}/\mathbb{Z}$ is the map
of \( \mathbb{Z}/p^{l+1}\mathbb{Z} \) onto \( \mathbb{Z}/p^l\mathbb{Z} \) given by reduction mod \( p^l \). Thus \( \hat{G}_p \) is naturally identified as a compact group with the inverse limit \( \limproj \mathbb{Z}/p^l\mathbb{Z} \).

Each \( \mathbb{Z}/p^l\mathbb{Z} \) is a ring, and the reduction maps are ring homomorphisms, so \( \limproj \mathbb{Z}/p^l\mathbb{Z} \) is a compact topological ring \( \mathbb{Z}_p \), which is called the ring of \( p \)-adic integers; in the previous paragraph, we identified \( \hat{G}_p \) with the additive group of \( \mathbb{Z}_p \). However, the multiplicative structure of \( \mathbb{Z}_p \) plays a crucial role in our analysis, for two reasons. First, we can use it to describe the action \( \alpha \): the reduction maps \( \mathbb{Z} \to \mathbb{Z}/p^l\mathbb{Z} \) induce an embedding of \( \mathbb{Z} \) in \( \mathbb{Z}_p \), and \( \alpha_{m,n} \) is, loosely speaking, division by \( p^m q^n \) (see Lemma 1.1 below). Second, the group \( \mathcal{U}(\mathbb{Z}_p) \) of units in \( \mathbb{Z}_p \) (the multiplicatively invertible elements) appears in our theorems. We need to know that there is a natural identification of \( \mathcal{U}(\mathbb{Z}_p) \) with \( \limproj \mathbb{U}(\mathbb{Z}/p^l\mathbb{Z}) \), and that an integer \( m \) is a unit in \( \mathbb{Z}_p \) precisely when \( m \) is coprime to \( p \). For these and other properties of \( \mathbb{Z}_p \), we refer to [16, Chapter II].

We are now ready to describe the Fourier-transform system. The dual of \( G_{p,q} \) is \( \mathbb{Z}_p \times \mathbb{Z}_q \); if \( \pi_l \) denotes the canonical map of \( \mathbb{Z}_p \) onto \( \mathbb{Z}/p^l\mathbb{Z} \), then the pairing is given by

\[
\langle r + s, (x, y) \rangle = \exp 2\pi i (r \pi_l(x) + s \pi_l(y)) \quad \text{for} \quad r \in \mathbb{Z}[p^{-1}], \ s \in \mathbb{Z}[q^{-1}] \quad \text{and} \quad l \text{ large.}
\]

**Lemma 1.1.** The Fourier transform \( C^*(G_{p,q}) \cong C(\mathbb{Z}_p \times \mathbb{Z}_q) \) carries the action defined by (1.1) into the action given by

\[
\alpha_{m,n}(f)(x, y) = \begin{cases} 
 f(p^{-m}q^{-n}x, p^{-m}q^{-n}y) & \text{if } x \in p^m q^n \mathbb{Z}_p \text{ and } y \in p^m q^n \mathbb{Z}_q, \\
 0 & \text{otherwise.}
\end{cases}
\]

**Proof.** We aim to apply [13, Proposition 4.5]. To do this, note that \( \alpha_{m,n} \) is defined by averaging over the solutions \( s \) of \( \beta_{m,n}(s) = r \), where \( \beta_{m,n} \) is the endomorphism of \( G_{p,q} \) defined by \( \beta_{m,n}(s) = p^m q^n s \). From the pairing (1.4), we see that the endomorphism \( \hat{\beta}_{m,n} \) of \( \mathbb{Z}_p \times \mathbb{Z}_q \) is given in terms of the ring structure by

\[
\hat{\beta}_{m,n}(x, y) = (p^m q^n x, p^m q^n y).
\]

Thus the Lemma follows directly from [13, Proposition 4.5]. \( \square \)

# 2. The structure theorem

Our main theorem describes the structure of \( C^*(G_{p,q}) \rtimes_{\alpha} \mathbb{N}^2 \) — or, equivalently, of the crossed product \( C(\mathbb{Z}_p \times \mathbb{Z}_q) \rtimes_{\alpha} \mathbb{N}^2 \) of the Fourier-transform system described in Lemma 1.1. To state it, we need a number-theoretic lemma. If \( k \) and \( m \) are coprime integers, so that \( m \) is a unit in \( \mathbb{Z}/k\mathbb{Z} \), we write \( \omega_k(m) \) for the order of \( m \) in \( \mathcal{U}(\mathbb{Z}/k\mathbb{Z}) \).

**Lemma 2.1.** Let \( p \) and \( q \) be distinct odd primes. Then there is a positive integer \( L = L_{p,q} \) such that

\[
\omega_{p^l}(q) = \begin{cases} 
 \omega_p(q) & \text{if } 1 \leq l \leq L \\
 p^{L-l} \omega_p(q) & \text{if } l > L.
\end{cases}
\]
This lemma is presumably well-known; certainly some of its immediate consequences are (see Remark 3.8). We are not going to prove it now, because we shall prove a slightly more general result in Theorem 3.1. However, we want to use the integers \( L_p(q) \) from this lemma in the statement of our main theorem.

**Theorem 2.2.** Let \( p \) and \( q \) be distinct odd primes. Then there are \( \hat{\alpha} \)-invariant ideals \( I_1 \) and \( I_2 \) in \( C^*(G_{p,q}) \times_\alpha \mathbb{N}^2 \) such that \( I_1 \subset I_2 \),

\[
\begin{align*}
(2.2) \quad I_1 &\cong \mathcal{K}(l^2(\mathbb{N}^2)) \otimes C(U(\mathbb{Z}_p) \times U(\mathbb{Z}_q)), \\
(2.3) \quad I_2/I_1 &\cong ((\mathcal{K}(l^2(\mathbb{N})) \otimes C) \oplus (\mathcal{K}(l^2(\mathbb{N})) \otimes D), \text{ and} \\
(2.4) \quad (C^*(G_{p,q}) \times_\alpha \mathbb{N}^2)/I_2 &\cong C(T^2),
\end{align*}
\]

where \( C \) is the direct sum of \( (p-1)p^{l_p(q)-1}/o_p(q) \) Bunce-Deddens algebras with supernatural number \( o_p(q)p^\infty \) and \( D \) is the direct sum of \( (q-1)q^{l_q(p)-1}/o_q(p) \) Bunce-Deddens algebras with supernatural number \( o_q(p)q^\infty \).

The algebra \( C^*(G_{p,q}) \cong C(\mathbb{Z}_p \times \mathbb{Z}_q) \) decomposes as a tensor product \( C(\mathbb{Z}_p) \otimes C(\mathbb{Z}_q) \), and the action \( \alpha \) given by (1.5) decomposes as a tensor product of two actions of \( \mathbb{N}^2 \). At this point, we cannot separate the actions of the two copies of \( \mathbb{N} \) (as Bost and Connes say, the two primes interact), but there is a large invariant ideal \( C_0(\mathbb{Z}_p \setminus \{0\}) \) in \( C(\mathbb{Z}_p) \) where the action does split as a tensor product of two actions of \( \mathbb{N} \). The ideals \( I_1 \) and \( I_2 \) will be crossed products of different invariant ideals in \( C(\mathbb{Z}_p) \otimes C(\mathbb{Z}_q) \) built from \( C_0(\mathbb{Z}_p \setminus \{0\}) \) and its twin.

For ordinary crossed products \( A \times G \) by group actions, invariant ideals in \( A \) give rise to short exact sequences

\[
0 \longrightarrow I \times G \longrightarrow A \times G \longrightarrow (A/I) \times G \longrightarrow 0.
\]

For semigroup crossed products \( A \rtimes_\alpha S \), one has to know that the ideal \( I \) is extendibly invariant, in the sense that each endomorphism \( \alpha_s \) extends to endomorphisms of \( M(I) \) and \( M(A) \) in such a way that \( \alpha_s(1_M(I)) = \alpha_s(1_M(A)) \) as multipliers of \( I \) (see [1, 13]). Since the endomorphism \( x \mapsto p^m q^n x \) of \( \mathbb{Z}_p \) leaves both \( \mathbb{Z}_p \setminus \{0\} \) and \( \{0\} \) invariant, it follows from Lemma 1.1 and [13, Theorem 4.3] that \( I := C_0(\mathbb{Z}_p \setminus \{0\}) \) and \( J := C_0(\mathbb{Z}_q \setminus \{0\}) \) are extendibly invariant ideals in \( A := C(\mathbb{Z}_p) \) and \( B := C(\mathbb{Z}_q) \).

We can therefore apply [13, Theorem 3.1] to deduce that the ideals \( I_1 := (I \otimes J) \times \mathbb{N}^2 \) and \( I_2 := (I \otimes B + A \otimes J) \times \mathbb{N}^2 \) form a composition series in which

\[
\begin{align*}
(2.5) \quad I_1 &\cong (I \otimes J) \times_\alpha \mathbb{N}^2, \\
(2.6) \quad I_2/I_1 &\cong ((A/I) \otimes J) \times \mathbb{N}^2 \oplus (I \otimes (B/J)) \times \mathbb{N}^2, \text{ and} \\
(2.7) \quad (A \otimes B) \times_\alpha \mathbb{N}^2/I_2 &\cong ((A/I) \otimes (B/J)) \times \mathbb{N}^2.
\end{align*}
\]

Notice that because the ideals are crossed products, they are \( \hat{\alpha} \)-invariant. To prove Theorem 2.2, therefore, we have to identify the subquotients.

We begin by noting that the maps \( f \mapsto f(0) \) induce isomorphisms \( A/I \cong \mathbb{C} \) and \( B/J \cong \mathbb{C} \), so \( (A/I) \otimes (B/J) \cong \mathbb{C} \otimes \mathbb{C} \cong \mathbb{C} \). Thus (2.7) is \( \mathbb{C} \times_{id} \mathbb{N}^2 \). When the action is
unital, as the identity action \( \text{id} \) certainly is, the covariance relation (1.3) implies that the isometries are all unitary; thus \( \mathbb{C} \rtimes \text{id} \mathbb{N}^2 \) is the universal \( \ast \)-algebra generated by a unitary representation of \( \mathbb{Z}^2 \). In other words, \( \mathbb{C} \rtimes \text{id} \mathbb{N}^2 = \ast(\mathbb{Z}^2) \cong \ast(\mathbb{T}^2) \), and we have proved (2.4).

For the other two parts, we need the promised decomposition of the action of \( \mathbb{N}^2 \) on \( I = C_0(\mathbb{Z}_p \setminus \{0\}) \).

**Lemma 2.3.** The map \( (n, x) \mapsto p^n x \) is a homeomorphism of \( \mathbb{N} \times \mathcal{U}(\mathbb{Z}_p) \) onto \( \mathbb{Z}_p \setminus \{0\} \).

**Proof.** Since every nonzero \( p \)-adic number can be uniquely written as a power of \( p \) times a unit (by Proposition 2 of [16, Chapter II], for example), the map is a bijection. It is a homeomorphism because it carries the basic open sets \( \{n\} \times V \) for the topology on \( \mathbb{N} \times \mathcal{U}(\mathbb{Z}_p) \) into the basic open sets \( p^n V \) for the topology on \( \mathbb{Z}_p \setminus \{0\} \).

The lemma implies that \( I = C_0(\mathbb{Z}_p \setminus \{0\}) \cong c_0(\mathbb{N}) \otimes \mathcal{U}(\mathbb{Z}_p) \). To describe what happens to the action \( \alpha \) under this isomorphism, we need some notation. We let \( \tau \) denote the action of \( \mathbb{N} \) on \( c_0(\mathbb{N}) \) by forward shifts; if we think of elements of \( c_0(\mathbb{N}) \) as functions on \( \mathbb{N} \), then

\[
\tau_m(f)(k) = \begin{cases} 
    f(k - m) & \text{if } k \geq m \\
    0 & \text{if } k < m.
\end{cases}
\]

Since \( (q, p) = 1 \), \( q \) is a unit in \( \mathbb{Z}_p \), and division by powers of \( q \) defines an action \( \sigma = \sigma^{p, q} \) of \( \mathbb{Z} \) by automorphisms of \( \mathcal{U}(\mathbb{Z}_p) \): \( \sigma_n(f)(x) = f(q^{-n}x) \). We now have the following immediate corollary of Lemma 2.3:

**Corollary 2.4.** The isomorphism \( C_0(\mathbb{Z}_p \setminus \{0\}) \cong c_0(\mathbb{N}) \otimes \mathcal{U}(\mathbb{Z}_p) \) induced by the homeomorphism of Lemma 2.3 carries \( \alpha \) into the tensor product action \( \tau \otimes \sigma : (m, n) \mapsto \tau_m \otimes \sigma_n \).

**Lemma 2.5.** There is an isomorphism

\[
I_2/I_1 \cong \mathcal{K}(l^2(\mathbb{N})) \otimes (\mathcal{U}(\mathbb{Z}_p)) \rtimes (\sigma^{p, q} \mathbb{Z}) \oplus \mathcal{K}(l^2(\mathbb{N})) \otimes (\mathcal{U}(\mathbb{Z}_q)) \rtimes (\sigma^{q, p} \mathbb{Z}).
\]

**Proof.** First, recall that \( A/I \cong \mathbb{C} \) and \( B/J \cong \mathbb{C} \), so from (2.6) we have

\[
I_2/I_1 \cong (I \rtimes_\alpha \mathbb{N}^2) \oplus (J \rtimes_\alpha \mathbb{N}^2^2).
\]

Next, we use the decomposition of Corollary 2.4 and [13, Theorem 2.6] (which applies because our action satisfies (1.2)), to see that

\[
I \rtimes_\alpha \mathbb{N}^2 \cong (c_0(\mathbb{N}) \rtimes_\tau \mathbb{N}) \otimes (\mathcal{U}(\mathbb{Z}_p)) \rtimes (\sigma^{p, q} \mathbb{N}).
\]

Because \( \sigma^{p, q} \) consists of automorphisms, the isometries in any covariant representation of \( (\mathcal{U}(\mathbb{Z}_p), \mathbb{N}, \sigma) \) are unitary, and \( \mathcal{U}(\mathbb{Z}_p) \rtimes_\sigma \mathbb{N} \) is the usual crossed product \( \mathcal{U}(\mathbb{Z}_p) \rtimes_\sigma \mathbb{Z} \).

To handle the other factor in (2.10), recall that \( c \rtimes_\tau \mathbb{N} = B_{\mathbb{N}} \rtimes_\tau \mathbb{N} \) is the Toeplitz algebra, and \( c_0(\mathbb{N}) \rtimes_\tau \mathbb{N} \) is the ideal of compact operators. More precisely, let \( M \) denote the representation of \( c \) by multiplication operators on \( l^2(\mathbb{N}) \), and let \( S \) be the unilateral shift on \( l^2(\mathbb{N}) \). Then \( (M, S) \) is a covariant representation of \( (c, \mathbb{N}, \tau) \).
such that $M \times S$ is an isomorphism of $c \rtimes \tau \mathbb{N}$ onto the $C^*$-algebra generated by $S$. (This formulation of Coburn’s Theorem is described in [2], for example.) It is easy to check that $M \times S$ carries the ideal $c_0 \rtimes \tau \mathbb{N}$ onto $\mathcal{K}(l^2(\mathbb{N}))$. Thus (2.10) implies that $I \rtimes \mathbb{N}^2 \cong \mathcal{K} \otimes (C(\mathbb{U}(\mathbb{Z}_p)) \rtimes_{\sigma} \mathbb{Z})$. Swapping $p$ and $q$ gives an analogous description of $J \rtimes_\alpha \mathbb{N}^2$, and the Lemma follows from (2.9) \hfill \Box.

The description of $I_2/I_1$ in (2.3) will follow from this lemma and Theorem 3.1.

To describe $I_1 := (I \otimes J) \rtimes_\alpha \mathbb{N}^2$, we use two applications of Corollary 2.4 to get an isomorphism

$$I \otimes J = C_0(\mathbb{Z}_p \setminus \{0\}) \otimes C_0(\mathbb{Z}_q \setminus \{0\}) \cong C_0(\mathbb{N} \times \mathbb{N} \times \mathbb{U}(\mathbb{Z}_p) \times \mathbb{U}(\mathbb{Z}_q))$$

which carries the endomorphism $\alpha_{m,n}$ into $\tau_m \otimes \tau_n \otimes \sigma_n^{p,q} \otimes \sigma_m^{q,p}$. We now borrow another idea from the theory of ordinary crossed products: recall that $(C_0(G) \otimes A) \rtimes_{\tau \otimes \beta} G \cong (C_0(G) \rtimes \sigma) \otimes A$ for any action $\beta$. Because $q \in \mathbb{U}(\mathbb{Z}_p)$ and $p \in \mathbb{U}(\mathbb{Z}_q)$, the endomorphism $\phi$ of $C_0(\mathbb{N} \times \mathbb{N} \times \mathbb{U}(\mathbb{Z}_p) \times \mathbb{U}(\mathbb{Z}_q))$ defined by

$$\phi(f)(k,l,x,y) = f(k,l,q^l x, p^k y)$$

is an automorphism. A quick calculation shows that

$$\phi \circ (\tau_m \otimes \tau_n \otimes \sigma_n^{p,q} \otimes \sigma_m^{q,p}) = \tau_m \otimes \tau_n \otimes \text{id} \otimes \text{id},$$

so $\phi$ induces an isomorphism

$$(I \otimes J) \rtimes_\alpha \mathbb{N}^2 \cong (c_0(\mathbb{N} \times \mathbb{N}) \rtimes_{\tau \otimes \tau} (\mathbb{N} \times \mathbb{N})) \otimes C(\mathbb{U}(\mathbb{Z}_p) \times \mathbb{U}(\mathbb{Z}_q)).$$

To finish off the proof of (2.2), either note that

$$c_0(\mathbb{N}^2) \rtimes_{\tau \otimes \tau} \mathbb{N}^2 \cong (c_0 \rtimes \tau \mathbb{N}) \otimes (c_0 \rtimes \tau \mathbb{N}) \cong \mathcal{K}(l^2(\mathbb{N})) \otimes \mathcal{K}(l^2(\mathbb{N})) = \mathcal{K}(l^2(\mathbb{N}^2)),$$

or check directly that the natural covariant representation of $B_{\mathbb{N}^2} \rtimes_{\tau} \mathbb{N}^2$ on $l^2(\mathbb{N}^2)$ restricts to an isomorphism of $c_0(\mathbb{N}^2) \rtimes \mathbb{N}^2$ onto $\mathcal{K}(l^2(\mathbb{N}^2))$.

To prove Theorem 2.2, therefore, it remains to prove Lemma 2.1 and to identify $C(\mathbb{U}(\mathbb{Z}_p)) \rtimes_{\sigma} \mathbb{Z}$ with the appropriate number of Bunce-Deddens algebras. We do this in Theorem 3.1.

3. The crossed products $C(\mathbb{U}(\mathbb{Z}_p)) \rtimes_{\sigma} \mathbb{Z}$

Our analysis of $C(\mathbb{U}(\mathbb{Z}_p)) \rtimes_{\sigma} \mathbb{Z}$ does not require that $q$ is prime, only that it is coprime to $p$. We therefore fix an odd prime $p$ and an integer $m$ coprime to $p$, and consider the action $\sigma = \sigma^{p,m}$ of $\mathbb{Z}$ on $C(\mathbb{U}(\mathbb{Z}_p))$ defined by

$$\sigma^{p,m}(f)(x) = f(m^{-n} x).$$

Theorem 3.1. Suppose that $p$ is an odd prime and $(m,p) = 1$, and denote by $a_p(m)$ the order of $m$ in $\mathbb{U}(\mathbb{Z}/p^l\mathbb{Z})$. Then there is a positive integer $L$ such that

$$a_p(m) = \begin{cases} a_p(m) & \text{if } 1 \leq l \leq L \\ p^l \cdot L \cdot a_p(m) & \text{if } l > L, \end{cases}$$

(3.2)
and \( C(U(\mathbb{Z}_p)) \times_{\sigma, p, m} \mathbb{Z} \) is the direct sum of \( p^{L-1}(p-1)/o_p(m) \) Bunce-Deddens algebras with supernatural number \( o_p(m)p^m \).

We begin by establishing the number-theoretic statements. Because \( U(\mathbb{Z}/p^l\mathbb{Z}) \) is cyclic of order \((p - 1)p^{l-1}\) (see Theorem 2 of [8, Chapter 4], for example), we can apply the following elementary lemma about cyclic groups.

**Lemma 3.2.** Suppose that \((n, p) = 1\) and \( G, H \) are cyclic groups of orders \( p^l n, p^{l-1} n \), respectively. If \( \pi : G \to H \) is a surjective homomorphism and \( g \) is a generator of \( G \), then the order of \( \pi(g^r) \) is given by

\[
o_\pi(g^r) = \begin{cases} |G|/(r, |G|) & \text{if } p^l \text{ divides } r \\
|G|/(r, |G|) & \text{if } p^l \text{ does not divide } r.
\end{cases}
\]

**Proof.** Since \( \pi(g) \) is a generator of \( H \), we have

\[
o_\pi(g^r) = o(\pi(g^r)) = \frac{|H|}{(r, |H|)} = \frac{|G|}{p(r, |H|)}.
\]

If \( p^l \) divides \( r \), say \( r = sp^l \), then

\[p(r, |H|) = p(p^l s, p^{l-1} n) = p^l (p s, n) = p^l (s, n) = (r, p^l n) = (r, |G|),\]

as claimed. If \( p^l \) does not divide \( r \), then \((r, |G|) = (r, p^l n) = (r, p^{l-1} n) = (r, |H|)\). \( \square \)

**Corollary 3.3.** Suppose \( p \) is prime and \((p, m) = 1\). Then

\[o_{p^l}(m) = \begin{cases} o_{p^{l+1}}(m) & \text{if } p \text{ does not divide } o_{p^{l+1}}(m) \\
o_{p^{l+1}}(m)/p & \text{if } p \text{ does divide } o_{p^{l+1}}(m).
\end{cases}
\]

**Proof.** Since a number is coprime to \( p^l \) iff it is coprime to \( p^{l+1} \), the reduction map \( \pi \) is a homomorphism of \( U(\mathbb{Z}/p^l\mathbb{Z}) \) onto \( U(\mathbb{Z}/p^{l+1}\mathbb{Z}) \), and Lemma 3.2 applies. Indeed, there is a generator \( g \) such that \( m = g^r \) where \( r := (p - 1)p^l / o_{p^{l+1}}(m) \). Then

\[o_{p^l}(m) = o(\pi(g^r)) = \begin{cases} o_{p^{l+1}}(m) & \text{if } p^l \text{ divides } (p - 1)p^l / o_{p^{l+1}}(m) \\
o_{p^{l+1}}(m)/p & \text{if } p^l \text{ does not divide } (p - 1)p^l / o_{p^{l+1}}(m),
\end{cases}
\]

which translates into what we want. \( \square \)

**Corollary 3.4.** There is a positive integer \( L \) such that (3.2) holds.

**Proof.** We first note that the sequence \( \{o_{p^l}(m) : l \in \mathbb{N}\} \) must be unbounded: for fixed \( N, m^N \) is eventually less than \( p^l \), and then \( o_{p^l}(m) > N \). In particular, \( \{o_{p^l}(m)\} \) is certainly not constant. Let \( L \) be the first integer such that \( o_{p^l}(m) < o_{p^{L+1}}(m) \). Then \( o_{p^l}(m) = o_p(m) \) for \( 1 \leq l \leq L \), and by Corollary 3.3, we have \( o_{p^{L+1}}(m) = p o_p(m) \), and \( p \) divides \( o_{p^{L+1}}(m) \). Since \( o_{p^{L+1}}(m) \) divides \( o_{p^l}(m) \) for all \( l > L \), it follows that \( p \) divides \( o_{p^l}(m) \) for all \( l > L \), and \( L - L \) applications of Corollary 3.3 show that \( o_{p^l}(m) = p^{L-L} o_{p^L}(m) = p^{L-L} o_p(m) \). \( \square \)
Remark 3.5. The referee has pointed out that one can also deduce Corollary 3.4 from the isomorphism of $\mathcal{U}(\mathbb{Z}/p\mathbb{Z}) \times p\mathbb{Z}_p^+$ onto $\mathcal{U}(\mathbb{Z}_p)$ provided by sending elements of $\mathcal{U}(\mathbb{Z}/p\mathbb{Z})$ to their Teichmüller representatives and the exponential isomorphism of the additive group $p\mathbb{Z}_p^+$ onto $1+p\mathbb{Z}_p$ (see [7, Corollary 4.5.10], for example). This isomorphism is compatible with the inverse limit decompositions of $\mathcal{U}(\mathbb{Z}_p)$ and $p\mathbb{Z}_p^+$, and hence it suffices to prove the analogous properties of additive orders in $p\mathbb{Z}_p^+$.

Let $H$ be the closed subgroup of $\mathcal{U}(\mathbb{Z}_p)$ generated by $m$. Then $H$ is invariant under multiplication by powers of $m$, and the formula (3.1) also defines an action $\sigma$ of $\mathbb{Z}$ on $C(H)$. This is where the Bunce-Deddens algebras come from:

**Proposition 3.6.** The crossed product $C(H) \rtimes_\sigma \mathbb{Z}$ is a Bunce-Deddens algebra with supernatural number $o_p(m)p^\infty$.

The Bunce-Deddens algebras were originally defined to be the $C^*$-algebras generated by certain weighted shifts on $l^2$ [5, §V.3], but we shall recognise them as crossed products associated to odometer actions. Let $\{n_k\}$ be a sequence of integers each of which is at least 2, and let $X_k = \{0, 1, \ldots, n_k - 1\}$. The odometer action $\tau$ of $\mathbb{Z}$ on $\prod_{k \geq 1} X_k$ is given by addition with carry over: let $N_1 = 1$, $N_k := \prod_{i < k} n_i$ for $k > 1$, and then

$$\tau_n(\{a_k\}) = \{b_k\} \text{ where } \sum_{k \geq 1} b_k N_k \equiv n + \sum_{k \geq 1} a_k N_k \pmod{N_{k+1}}.$$ 

The crossed product $C(\prod_{k \geq 1} X_k) \rtimes_\tau \mathbb{Z}$ is then a Bunce Deddens algebra with supernatural number $\prod_{k \geq 1} n_k$ [5, Theorem VIII.4.1]. In general, Bunce-Deddens algebras are simple [5, Theorem V.3.3], and are determined up to isomorphism by their supernatural number [5, Theorem V.3.5].

**Proof.** Write $d$ for $o_p(m)$, and let

$$O := \{0, 1, \ldots, d - 1\} \times \{0, 1, \ldots, p - 1\}^\mathbb{N}.$$ 

For $l > L$, we define $h_l : O \to \mathcal{U}(\mathbb{Z}/p^l\mathbb{Z})$ by

$$h_l(\{a_n\}) = m^{a_0 + da_1 + dp a_2 + \cdots + dp^{l-2} a_{l-2} - a_{l-1} - L} \pmod{p^l \mathbb{Z}};$$

because the order of $m$ in $\mathcal{U}(\mathbb{Z}/p^l\mathbb{Z})$ is $dp^{l-2}$, the maps $h_l$ satisfy $h_{l+1}(\{a_n\}) = h_l(\{a_n\})$ (mod $p^l \mathbb{Z}$). Since the $h_l$ are continuous by definition of the product topology, they induce a continuous map $h : O \to C(\mathbb{Z}/p^L\mathbb{Z})$, which is an injection because $h_l(\{a_n\})$ determines $a_0, \ldots, a_{l-1}$ uniquely. The range of $h$ is a compact subgroup, and contains the positive powers of $m$, which are the images of the sequences in $O$ which are eventually zero; since such sequences are dense in $O$, their images generate the range. In other words, $h$ is a continuous injection of $O$ onto $H$, and is therefore a homeomorphism. Since $h(\tau\{a_n\}) = mh(\{a_n\})$ for all $\{a_n\}$, we deduce that the Bunce-Deddens algebra $C(O) \rtimes_\tau \mathbb{Z}$ is isomorphic to $C(H) \rtimes_\sigma \mathbb{Z}$.

To finish the proof of our theorem, we need to decompose the dynamical system $(C(\mathcal{U}(\mathbb{Z}_p)), \mathbb{Z}, \sigma)$ as a sum of copies of $(C(H), \mathbb{Z}, \sigma)$. This needs a simple group-theoretic lemma.
Lemma 3.7. Suppose $G = \lim G_n$ is a compact group which is the inverse limit of finite groups $G_n$, and suppose that the canonical maps $\pi_n : G \to G_n$ are surjective. If $H$ is a closed subgroup of $G$ and there is an integer $k$ such that $|G_n/\pi_n(H)| = k$ for all $n$, then $|G/H| = k$.

Proof. Certainly $|G/H| \geq |\pi_n(G)/\pi_n(H)| = k$. Suppose $g_1 H, \ldots, g_{k+1} H$ are cosets in $G/H$; we shall prove that two must be the same. The hypothesis implies that for each $n$, two of $\pi_n(g_i H)$ coincide. Since there are only finitely many possibilities, we can assume by passing to a subsequence that the same two coincide in each $G_n/\pi_n(H)$; say $\pi_n(g_1 H) = \pi_n(g_2 H)$ for all $n$. Then $\pi_n(g_1 g_2^{-1}) \in \pi_n(H)$; say $\pi_n(g_1 g_2^{-1}) = \pi_n(h_n)$. By definition of the topology on the inverse limit, we have $h_n \to g_1 g_2^{-1}$ in $G$, so that $g_1 g_2^{-1} \in H$ and $g_1 H = g_2 H$.

End of the proof of Theorem 3.1. Since $\pi_1(H)$ is the subgroup of $U(\mathbb{Z}/p^l \mathbb{Z})$ generated by $m$, we have

$$|U(\mathbb{Z}/p^l \mathbb{Z})/\pi_1(H)| = (p - 1)p^{l-1} / o_p(m) = (p - 1)p^{l-1} / o_p(m)$$

for all $l \geq L$.

We can therefore apply Lemma 3.7 to $U(\mathbb{Z}_p) = \lim(U(\mathbb{Z}/p^l \mathbb{Z}), l \geq L)$ to deduce that $H$ has index $N := (p - 1)p^{L-1} / o_p(m)$ in $U(\mathbb{Z}_p)$.

Next, note that because $H$ is a closed subgroup of finite index, it is also open: its complement is the finite union of cosets of $H$, and hence closed. Since $H$ is by definition invariant under multiplication by powers of $m$, it follows that $U(\mathbb{Z}_p)$ is the disjoint union of $N$ open and closed invariant sets of the form $xH$, and $C(U(\mathbb{Z}_p))$ is the direct sum of $\sigma$-invariant ideals of the form $C(xH)$. The dynamical systems $(C(xH), \mathbb{Z}, \sigma)$ are all conjugate to $(C(H), \mathbb{Z}, \sigma)$. Thus the Theorem follows from Proposition 3.6.

Remark 3.8. An integer $m$ which generates $U(\mathbb{Z}/p^l \mathbb{Z})$ is called a primitive root modulo $p^l$. If $m$ is a primitive root modulo $p^l$ for one $l > 1$, then (3.2) implies that $L_p(m) = 1$ and $o_p(m) = p - 1$, and hence that $m$ is a primitive root modulo $p^k$ for all $k$. (This is known; see [6, §17, Exercise VI.4], for example.) Theorem 3.1 gives a curious $C^*$-algebraic characterisation of primitive roots: $m$ is primitive modulo $p^l$ for all $l$ if and only if $C(U(\mathbb{Z}_p)) \rtimes_\sigma \mathbb{Z}$ is simple. More generally, the cardinality of Prim $C(U(\mathbb{Z}_p)) \rtimes \mathbb{Z}$ determines the orders $o_p(m)$ of $m$ in $U(\mathbb{Z}/p^l \mathbb{Z})$.

The relations (3.2) are the only restrictions on the possible values of $o_p(m)$. Indeed, given an odd prime $p$, a divisor $d$ of $p - 1$, and an integer $L \geq 1$, there are infinitely many primes $q$ with $o_p(q) = d$ and $L_p(q) = L$. To see this, choose $k$ such that $o_{p^{L+1}}(k) = pd$. Then every integer $q$ in the arithmetic progression $k + np^{L+1}$ has $o_p(q) = d$ and $o_{p^{L+1}}(q) = pd$, and it follows from (3.2) that $o_{p^{L}}(q) = p^{L} - d$ for all $l > L$. Now our assertion follows from Dirichlet’s Theorem: every arithmetic progression $k + nr$ with $(k, r) = 1$ contains infinitely many primes [8, §16.1].
4. The primitive ideal space

Since \( \text{Prim } C(X, K) \) is homeomorphic to \( X \) [15, Example A.24] and Bunce-Deddens algebras are simple [5, Theorem V.3.3], Theorem 2.2 gives us a setwise description of the primitive ideal space of the algebra \( C(Z_p \times Z_q) \). It consists of a copy \( \{I_{x,y}\} \) of \( U(Z_p) \times U(Z_q) \) embedded as an open subset, a copy \( \{L_{w,z}\} \) of \( T^2 \) embedded as a closed subset, and two finite sets \( \{J_{x,Hp}\}, \{K_{y,Hq}\} \) parametrised by the quotients \( U(Z_p)/H_p = U(Z_q)/q^2 \) and \( U(Z_q)/H_q = U(Z_q)/p^2 \) whose cardinalities determine the number of Bunce-Deddens algebras in the subquotients. The topology on \( \text{Prim } (C(Z_p \times Z_q) \times_\alpha \mathbb{N}^2) \) is then given by:

**Theorem 4.1.** The maps \( (x, y) \mapsto I_{x,y}, xH_p \mapsto J_{x,Hp}, yH_q \mapsto J_{y,Hq} \) and \( (w, z) \mapsto L_{w,z} \) combine to give a bijection of the disjoint union

\[
(U(Z_p) \times U(Z_q)) \sqcup U(Z_p)/q^2 \sqcup U(Z_q)/p^2 \sqcup T^2
\]

onto \( \text{Prim } (C(Z_p \times Z_q) \times_\alpha \mathbb{N}^2). \) Write \( \pi_p \) for the map \( U(Z_p) \times U(Z_q) \rightarrow U(Z_p) \rightarrow U(Z_p)/q^2 \). Then the hull-kernel closure of a nonempty subset \( F \) of (4.1) is

(a) the usual closure of \( F \) in \( T^2 \) if \( F \subset T^2; \)
(b) \( F \cup T^2 \) if \( F \subset U(Z_p)/q^2 \cup U(Z_q)/p^2; \)
(c) the usual closure of \( F \) in \( U(Z_p) \times U(Z_q) \) together with \( \pi_p(F) \cup \pi_q(F) \cup T^2 \) if \( F \subset U(Z_p) \times U(Z_q). \)

We shall prove this by writing down irreducible representations of \( C(Z_p \times Z_q) \times_\alpha \mathbb{N}^2 \) realising each of these primitive ideals, identifying their kernels as crossed products of invariant ideals in \( C(Z_p \times Z_q) \) using results from [13], and then reading off the topology from standard properties of the topology on \( Z_p \times Z_q. \)

The ideals \( L_{w,z} \) are lifted from the quotient \( (C(Z_p \times Z_q) \times \mathbb{N}^2)/I_2 = C \times_{id} \mathbb{N}^2, \) and are the kernels of the characters \( \gamma_{w,z} : (m,n) \mapsto w^mz^n; \) more precisely, \( L_{w,z} = \ker(\varepsilon_{w,z}) \), where \( \varepsilon_{0,0}(f) := f(0,0). \) Because \( \text{Prim}(C \times_{id} \mathbb{N}^2) \) is a closed subset of \( \text{Prim}(C(Z_p \times Z_q) \times \mathbb{N}^2), \) this also proves part (a) of Theorem 4.1.

The ideals \( J_{x,Hp} \) are lifted from the image of the surjection \( (id \otimes \varepsilon_0)^* \) of \( C(Z_p \times Z_q) \times \mathbb{N}^2 \) onto \( C(Z_p) \times \mathbb{N}^2 \) induced by \( id \otimes \varepsilon_0 : C(Z_p \times Z_q) \rightarrow C(Z_p) \), and are determined in the image by their intersections with the ideal \( C_0(Z_p \setminus \{0\}) \times \mathbb{N}^2. \) Recall that the homeomorphism \( h_p : (k, x) \mapsto p^k x \) induces an isomorphism

\[
h_p^* : C_0(Z_p \setminus \{0\}) \times \mathbb{N}^2 \cong C(U(Z_p), c_0(\mathbb{N}) \times \tau \mathbb{N}) \times_{\sigma \otimes id} \mathbb{Z}.
\]

Because \( M \times T \) is an isomorphism of \( c_0(\mathbb{N}) \times \tau \mathbb{N} \) onto \( K(l^2(\mathbb{N})) \) and \( Z \) acts freely on \( U(Z_p) = \text{Prim } C(U(Z_p), K) \), the primitive ideals of the right-hand side of (4.2) are induced from the ideals \( \ker(M \times T) \circ \varepsilon_x \). In particular, we have

\[
J_{x,Hp} \cap \left( C_0(Z_p \setminus \{0\}) \times \mathbb{N}^2 \right) = \ker \left( \left( \text{Ind}_{(0)}^{\mathbb{Z}}(M \times T) \circ \varepsilon_x \right) \circ h_p^* \circ (id \otimes \varepsilon_0)^* \right).
\]
We can now use the standard form \( \tilde{\pi} \times \lambda \) of the induced representation to see that we can realise \( J_{xH_0} \) as the kernel of the representation \( \rho_x \times (T \otimes \lambda) \) of \( C(\mathbb{Z}_p \times \mathbb{Z}_q) \times \alpha \mathbb{N}^2 \) on \( l^2(\mathbb{N} \times \mathbb{Z}) \), where

\[
(\rho_x(f)\xi)(k, l) := f(p^k q^l x, 0)\xi(k, l)
\]

Similarly, with \( \sigma_y : C(\mathbb{Z}_p \times \mathbb{Z}_q) \to B(l^2(\mathbb{Z} \times \mathbb{N})) \) defined by

\[
(\sigma_y(f)\xi)(k, l) = f(0, p^k q^l y)\xi(k, l),
\]

we have \( \ker(\sigma_y \times (\lambda \otimes T)) = K_{yH_0} \).

The ideals \( I_{x,y} \) are determined by their interaction with \( I_1 \), and \( I_{x,y} \cap I_1 \) is pulled back under the isomorphism \( (2.2) \) from the kernel of the evaluation map \( \varepsilon_{x,y} : C(\mathcal{U}(\mathbb{Z}_p) \times \mathcal{U}(\mathbb{Z}_q), \mathcal{K}) \to \mathcal{K} \). This isomorphism is induced by the homeomorphism \( h : (l, k, x, y) \mapsto p^k q^l (x, y) \) of \( \mathbb{N}^2 \times \mathcal{U}(\mathbb{Z}_p) \times \mathcal{U}(\mathbb{Z}_q) \) onto \( (\mathbb{Z}_p \setminus \{0\}) \times (\mathbb{Z}_q \setminus \{0\}) \), and the Toeplitz representation \( M \times T \) of \( C_0(\mathbb{N}^2) \times \mathbb{N}^2 \) onto \( \mathcal{K}(l^2(\mathbb{N}^2)) \). The representation

\[
(\pi_{x,y}(f)\xi)(k, l) = f(p^k q^l x, p^k q^l y)\xi(k, l)
\]

satisfies \( \pi_{x,y}|_{C_0(\mathbb{Z}_p \setminus \{0\}) \times (\mathbb{Z}_q \setminus \{0\})} = M \circ \varepsilon_{x,y} \circ (h^{-1})^* \), and it follows that \( (\pi_{x,y}, T) \) is a covariant representation of \( (C(\mathbb{Z}_p \times \mathbb{Z}_q), \mathbb{N}^2, \alpha) \) with \( I_{x,y} = \ker(\pi_{x,y} \times T) \).

To identify the kernels of these representations, we shall use the following analogue of the standard characterisations of faithful representations.

**Lemma 4.2.** Let \( (\eta, T) \) be a covariant representation of a semigroup dynamical system \( (A, \mathbb{N}^k, \alpha) \) with extendible endomorphisms. Suppose that \( \ker \eta \) is an extendibly \( \alpha \)-invariant ideal, and that there is a unital representation \( \mathcal{W} \) of \( \mathbb{T}^k \) such that \( \eta \times T, \mathcal{W} \) is a covariant representation of the dual system \( (A \rtimes \mathbb{N}^k, \mathbb{T}^k, \hat{\alpha}) \). Then

\[
\ker(\eta \times T) = (\ker \eta) \rtimes \mathbb{N}^k = \text{span}\{i_S(m)^* i_A(a) i_S(n) : m, n \in \mathbb{N}^k, a \in \ker \eta\}.
\]

**Proof.** We know from [13, Theorem 1.8] that \( (\ker \eta) \rtimes \mathbb{N}^k \) is naturally isomorphic to the ideal

\[
\text{span}\{i_S(m)^* i_A(a) i_S(n) : m, n \in \mathbb{N}^k, a \in \ker \eta\} \subset A \rtimes \mathbb{N}^k,
\]

and that the quotient map \( \pi : A \to A/(\ker \eta) \) induces a homomorphism \( \pi \times \text{id} \) of \( A \rtimes \mathbb{N}^k \) onto \( (A/\ker \eta) \rtimes \mathbb{N}^k \) with kernel \( (\ker \eta) \rtimes \mathbb{N}^k \). There is a faithful representation \( \zeta \) of \( A/\ker \eta \) such that \( \eta = \zeta \circ \pi \), and then \( (\zeta, T) \) and \( (\zeta \times T, \mathcal{W}) \) are covariant. It suffices to prove that \( \zeta \times T \) is faithful, for then \( \eta \times T = (\zeta \times T) \circ (\pi \times \text{id}) \), and

\[
\ker(\eta \times T) = \ker(\zeta \times T) \circ (\pi \times \text{id}) = \ker(\pi \times \text{id}) = (\ker \eta) \rtimes \mathbb{N}^k.
\]

To prove \( \zeta \times T \) faithful, we follow the standard procedure of [4, Lemma 2.2]. Write \( C = A/\ker \eta \), and let \( \theta : C \rtimes \mathbb{N}^k \to C \rtimes \mathbb{N}^k \) be the expectation obtained by averaging over the dual action \( \hat{\alpha} \) on \( C \rtimes \mathbb{N}^k \), which is faithful on positive elements by [10, Remark 3.6]. Because \( S = \mathbb{N}^k \) is abelian, \( C \rtimes \mathbb{N}^k \) is spanned by the elements \( i_S(m)^* i_C(c) i_S(n) \) \([13, Lemma 1.3]\), and hence \( \theta(C \rtimes \mathbb{N}^k) \) is spanned by the elements \( i_S(m)^* i_C(c) i_S(n) \); because every finite set of elements in \( \mathbb{N}^k \) has an upper bound, we can imitate the
proof of [2, Lemma 1.5] to see that \( \zeta \times T \) is faithful on \( \theta(C \times \mathbb{N}^k) \). Now we can use the covariance of \( (\zeta \times T, W) \) to get an estimate
\[
\| (\zeta \times T)(\theta(f)) \| = \left\| \int_{\mathbb{T}_h} W_z^*(\zeta \times T)(f) W_z \, dz \right\|
\leq \int_{\mathbb{T}_h} \| W_z^* \zeta \times T(f) W_z \| \, dz
= \| \zeta \times T(f) \|
\]
and follow the argument of [4, Lemma 2.2] to see that \( \zeta \times T \) is faithful.

The ideal \( \ker \pi_{x,y} \) consists of the functions which vanish on the closure of the orbit \( p^n q^n(x, y) \); to check that \( \ker \pi_{x,y} \) is extendibly invariant, we need to know exactly what this closure is.

**Lemma 4.3.** Let \( (x, y) \in \mathbb{Z}_p \times \mathbb{Z}_q \). Then \( q^n x \) has the same closure in \( \mathbb{Z}_p \) as \( q^n x \), and the closure of \( p^n q^n(x, y) \) in \( \mathbb{Z}_p \times \mathbb{Z}_q \) is
\[
(4.3) \quad p^n q^n(x, y) \cup (p^n q^n x \times \{0\}) \cup \{0\} \times p^n q^n y.
\]

**Proof.** Since \( q \in \mathcal{U}(\mathbb{Z}_p) \), multiplication by \( q \) is a homeomorphism of \( \mathcal{U}(\mathbb{Z}_p) \), and defines a free and minimal action of \( \mathbb{Z} \) on \( q^n x \). The sequence \( \{q^k x : k \in \mathbb{N}\} \) has a convergent subsequence, \( q^{kn} x \to x_0 \), say, and then \( q^n x_0 = \overline{q^n x} \) by minimality. Thus every element of \( \overline{q^n x} \) can be approximated first by \( q^n x_0 \), and then by elements \( q^{n+kn} x \) of \( q^n x \). Thus \( \overline{q^n x} = \overline{q^n x} \). This argument also shows that every element of \( \overline{q^n x} \) is the limit of a sequence \( q^{m_n} x \) in which \( m_n \to \infty \).

Since \( (0, 0) = \lim_n p^n q^n (x, y) \), it certainly belongs to the orbit closure. Suppose \( p^{kn} q^n x \to s \) and \( s \neq 0 \). Write \( s = p^l s_0 \) for \( s_0 \in \mathcal{U}(\mathbb{Z}_p) \). Then \( p^l \mathcal{U}(\mathbb{Z}_p) \) is an open neighbourhood of \( s \), so \( k_n = i \) for large \( n \), and \( q^{kn} x \to p^l s \). As observed above, we may as well suppose \( l_n \to \infty \); but then \( q^{kn} y \to 0 \), and \( p^n q^n(x, y) \to (s, 0) \). Thus \( (p^n q^n x \times \{0\}) \) is contained in the orbit closure, and, by symmetry, so is \( \{0\} \times \overline{p^n q^n y} \).

For the other inclusion, suppose \( (w, z) \in \mathbb{Z}_p \times \mathbb{Z}_q \), and \( p^l q^n(x, y) \to (w, z) \). It is obvious that \( (w, z) \) belongs to (4.3) if one of \( w \) or \( z \) is 0, so suppose \( w \) and \( z \) are both nonzero. We can write \( (w, z) = (p^iw_0, q^jz_0) \) for units \( w_0, z_0 \) and \( i, j \in \mathbb{N} \), and then \( p^i \mathcal{U}(\mathbb{Z}_p) \times q^j \mathcal{U}(\mathbb{Z}_q) \) is a neighbourhood of \( (w, z) \). Thus \( (k_n, l_n) = (i, j) \) for large \( n \), and \( (w, z) = p^n q^n(x, y) \) belongs to \( p^n q^n(x, y) \), as required.

**Lemma 4.4.** Let \( (x, y) \in \mathcal{U}(\mathbb{Z}_p) \times \mathcal{U}(\mathbb{Z}_q) \). Then
\[
(a) \quad J_{xH_p} = \text{spans} \{i_S(i, j)^* i_A(f) i_S(m, n) : f \equiv 0 \text{ on } \overline{p^n q^n x} \times \{0\}\};

(b) \quad K_{xH_q} = \text{spans} \{i_S(i, j)^* i_A(f) i_S(m, n) : f \equiv 0 \text{ on } \{0\} \times \overline{p^n q^n y}\}; \text{ and}

(c) \quad I_{x,y} = \text{spans} \{i_S(i, j)^* i_A(f) i_S(m, n) : f \equiv 0 \text{ on } \overline{p^n q^n (x, y)}\}.
\]

**Proof.** For part (a), we want to apply Lemma 4.2 with \( \eta = \rho_x \), and we therefore need to know that \( \ker \rho_x \) is extendibly invariant. We have \( \rho_x(f) = 0 \) iff \( f \equiv 0 \) on
Lemma 4.2 with must have $z$ proves that $\ker(p^k q^l)$ is both nonzero: say $z \neq 0$, then $p^k q^l(w, z)$ is certainly not in $p^k q^l x \times \{0\}$. So we consider the case $z = 0$, and suppose $p^{k_n} q^{l_n} x \to p^k q^l w$. Since $w$ cannot be 0, we can write $w = p^i w_0$ for $w_0 \in \mathcal{U}(\mathbb{Z}_p)$. Eventually $p^{k_n} q^{l_n} x \in p^{k+i} \mathcal{U}(\mathbb{Z}_p)$, so $k_n = k + i$ for large $n$, and $q^l w = \lim p^{k_n} q^{l_n} x$ belongs to $p^i (q^n x)$. Since $q^n x = q^z x$, this implies that $w \in p^i (q^n x)$, and hence that $(w, z) \in p^i q^n x \times \{0\}$, which is a contradiction. So $p^k q^l(w, z) \notin p^k q^l x \times \{0\}$ for all $k, l \in \mathbb{N}$, and we have shown that $\ker \rho_x$ is extendibly invariant.

Next we observe that $W_{w, z} \xi(k, l) := w^k z^l \xi(k, l)$ defines a unitary representation $W$ of $\mathbb{T}^2$ on $l^2(\mathbb{N} \times \mathbb{Z})$ such that $(\rho_x \times (T \otimes \lambda), W)$ is covariant for the dual action. Thus we can deduce from Lemma 4.2 that $J_{x H_p} = \ker(\rho_x \times (T \otimes \lambda))$ has the required form. This gives (a), and of course (b) is exactly the same.

For (c), we apply the same argument to

$$\ker \pi_{x, y} = \{ f \in C(\mathbb{Z}_p \times \mathbb{Z}_q) : f \equiv 0 \text{ on } p^N q^N(x, y) \};$$

as above, the crux is to prove that if $p^k q^l(w, z)$ belongs to the closure of $p^k q^l(x, y)$, then so does $(w, z)$. Suppose $(w, z) \in \mathbb{Z}_p \times \mathbb{Z}_q$ and $p^k q^l(x, y) \to p^k q^l(w, z)$. If $w$ or $z$ is 0, we are in the situation covered by the first paragraph. So suppose $w$ and $z$ are both nonzero: say $w = p^i w_0$ and $z = q^j z_0$ for units $w_0, z_0$. By Lemma 4.3, we must have $p^k q^l(w, z) = p^m q^n(x, y)$ for some $m, n \in \mathbb{N}$. Then $p^{k+i} q^l w_0 = p^m q^n x$ and $p^k q^{l+j} z_0 = p^m q^n y$. The first of these equations implies that $k + i = m$, so $k \leq m$, and the second that $l \leq n$. Thus $(w, z) = p^m q^{n-l}(x, y)$ belongs to $p^N q^N(x, y)$. This proves that $\ker \pi_{x, y}$ is extendibly invariant. Part (c) follows from an application of Lemma 4.2 with $W$ given by the same formula as before.

Proof of Theorem 4.1. We have already observed that (a) is easy. For (b), notice that for any $x H_p \in \mathcal{U}(\mathbb{Z}_p)/H_p$, the spanning elements $i_S(i, j) A_i \Lambda(f) i_S(m, n)$ of $J_{x H_p}$ go to $f(0, 0) i_S(i, j) A_i \Lambda(f) i_S(m, n)$ in the quotient $C \times_{id} \mathbb{N}^2$, and hence $J_{x H_p} \subset L_{w, z}$ for all $(w, z) \in \mathbb{T}^2$.

For (c), we observe that $\overline{F} \cap (\mathcal{U}(\mathbb{Z}_p) \times \mathcal{U}(\mathbb{Z}_q))$ is the usual closure because $(x, y) \mapsto I_{x, y}$ is a homeomorphism of $\mathcal{U}(\mathbb{Z}_p) \times \mathcal{U}(\mathbb{Z}_q)$ onto the open set $\text{Prim} I_1$. That the closure contains the other points follows from Lemma 4.4: $f \in \ker \pi_{x, y}$ implies $f \in \ker \rho_x$, so all the generators for $I_{x, y}$ described in Lemma 4.4 belong to $J_{x H_p}$, and $(x, y) \in F$ implies

$$J_{x H_p} \in \overline{F} = \{ P \in \text{Prim}(C(\mathbb{Z}_p \times \mathbb{Z}_q) \times \mathbb{N}^2) : \bigcap_{Q \in F} Q \subset P \}.$$
\( p^n q^n (x, y) \) for every \((x, y) \in F\), and hence \( i_A(f) \) belongs to \( \bigcap \{ I_{x,y} : x, y \in F \} \) but not to \( J_{x_0 H_p} \). Thus \( \overline{F} \cap \mathcal{U}(\mathbb{Z}_p)/H_p \) is precisely \( \pi_p(F) \), and part (c) follows from (b). \( \square \)

**Remark 4.5.** It is interesting to compare our description of \( \text{Prim} C^*(G_{p,q}) \times \mathbb{N}^2 \) with that obtained for the Bost-Connes algebra \( C_Q \) in [12]. In \( \text{Prim} C_Q \), the finite sets coming from \( \text{Prim} I_2/I_1 \) do not appear; loosely speaking, we believe this happens because \( C_Q \) contains all the primes, and some of these will act minimally on any given \( \mathcal{U}(\mathbb{Z}_p) \) (see Remark 3.8). So the numbers \( o_{p'}(q) \) cannot be recovered from \( \text{Prim} C_Q \). Of course this information is still buried somewhere in \( C_Q \): it follows from Theorem 2.1 of [14] that the inclusion of \( G_{p,q} \) in \( \mathbb{Q}/\mathbb{Z} \) induces an isomorphism of \( C^*(G_{p,q}) \times \mathbb{N}^2 \) into \( C^*(\mathbb{Q}/\mathbb{Z}) \times \mathbb{N}^* = C_Q \).

**References**


The Two-Prime Hecke $C^*$-algebra

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