$C^*$-algebras and Tilings, Aperiodic Order, CIRM, Luminy

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1. $C^*$-algebra basics

2. $C^*$-algebras from dynamics

3. Morita equivalence

4. $C^*$-algebras from projection tilings

5. K-theory for $C^*$-algebras
Part 1: $C^*$-algebra basics

Definition 1. A $C^*$-algebra is a set $A$:

- $A$ is an algebra over $\mathbb{C}$, the complex numbers (Not nec. commutative or unital)
- there is an involution $a \rightarrow a^*$, $a \in A$
- $A$ has a norm, $\| \|$,

such that

- $(a + \lambda b)^* = a^* + \overline{\lambda}b^*$, $a, b \in A$,
- $(ab)^* = b^*a^*$, $a, b \in A$,
- $A$ is complete in $\| \|$,
- $\| a^*a \| = \| a \|^2$, $a \in A$. 

Examples:

- $\mathbb{C}$, the complex numbers,

- For $n \geq 1$, $M_n(\mathbb{C})$, $n \times n$ complex matrices. $\ast = \text{conjugate transpose.}$

- For $\mathcal{H}$ a complex Hilbert space, $\mathcal{B}(\mathcal{H})$, the bounded linear operators on $\mathcal{H}$. $\ast = \text{adjoint.}$

- Any $A \subset \mathcal{B}(\mathcal{H})$ which is an algebra, closed under $\ast$, closed in the norm topology.
Let $X$ be a compact, Hausdorff space.

$$C(X) = \{ f : X \rightarrow \mathbb{C} \mid f \text{ continuous } \}.$$  

It is a $C^*$-algebra with pointwise algebraic operations, $\ast =$ pointwise complex conjugation, $\|\|\|$ is the supremum norm.

We can generalize: if the space $X$ is locally compact, replace $C(X)$ with $C_0(X)$, the continuous complex functions which vanish at infinity. This is unital if and only if $X$ is compact.

These are both commutative.

**Gelfand-Naimark Theorem:** Every commutative $C^*$-algebra arises in this way. $C_0(X)$ and $C_0(Y)$ are isomorphic if and only if $X$ and $Y$ are homeomorphic.

**Theorem 2.** The functor $X \rightarrow C_0(X)$ is an equivalence of categories between locally compact, Hausdorff spaces and commutative $C^*$-algebras.
• Can we extend standard topological notions to $C^*$-algebras?

• Are the some geometric constructions of non-commutative $C^*$-algebras?
Gelfand-Naimark dictionary:

<table>
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<tr>
<th>Topology</th>
<th>Commutative $C^*$-alg’s</th>
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<tbody>
<tr>
<td>closed set</td>
<td>closed ideal</td>
</tr>
<tr>
<td>$Y \subset X$</td>
<td>$I = { f \in C(X) \mid f</td>
</tr>
<tr>
<td>Borel measure</td>
<td>functional</td>
</tr>
<tr>
<td>$\mu$</td>
<td>$\varphi_\mu(f) = \int_X f , d\mu$</td>
</tr>
<tr>
<td></td>
<td>$\varphi_\mu : C_0(X) \to \mathbb{C}$</td>
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<tr>
<td>K-theory</td>
<td>K-theory</td>
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An application: Hilbert space $L^2[0, 1]$.

Let $Cut = \{p2^{-k} \mid p, k \in \mathbb{Z}\} \cap [0, 1]$.

For each $a < b$ in $Cut$, let $\chi_{[a,b)}$ denote the characteristic function of $[a, b)$, which we regard as an operator on $L^2[0, 1]$ by pointwise multiplication.

Let $A$ be the closed linear span of $\{\chi_{[a,b)} \mid a < b, a, b \in Cut\}$ in $B(L^2[0, 1])$.

This is a commutative, unital $C^*$-algebra. Hence, $A \cong C(X)$, for some $X$. What is $X$?

It should be a space where our functions $\chi_{[a,b)}$ are continuous: from $[0, 1]$, remove each point $a$ in $Cut$ and replace it with two points $a^-, a^+$. Topologically, imagine $a^-$ as a left endpoint of $[0, a]$ and $a^+$ as a right endpoint for $[a, 1]$, separated by a gap. This is $X$ and it is a Cantor set.
Part 2: $C^*$-algebras from dynamics

Situation 1: Topological equivalence relations

Let $X$ be a compact, Hausdorff space.

$R$ an equivalence relation on $X$.

$r, s : R \to X$ are the projections:

$$r(x, y) = x, s(x, y) = y, (x, y) \in R.$$ 

Assume $R$ has an étale topology: $r, s$ are open and local homeomorphisms.

Idea: if $(x, y)$ is in $R$, there are open sets $x \in U$, $y \in V$ and a (unique) homeomorphism $\rho : U \to V$ such that

$$\rho(x) = y,$$

$$\{(u, \rho(u)) \mid u \in U\} \subseteq R.$$
$C^*(R)$:

First look at $C_c(R)$, the continuous, complex-valued functions of compact support on $R$. It is a linear space in an obvious way. Define a product and involution:

$$(f \cdot g)(x, y) = \sum_{(x, z) \in R} f(x, z) g(z, y),$$

$$f^*(x, y) = \overline{f(y, x)}.$$

Complete in a norm to get a $C^*$-algebra, $C^*(R)$.

Example: $X = \{1, 2, \ldots, N\}, R = X \times X$.

$C^*(R) = M_N(\mathbb{C})$.

Start with $C(X) = \mathbb{C}^N = span\{\chi_1, \ldots, \chi_N\}$ and add $e_{i,j}$ such that

$$e_{i,j}^* e_{i,j} = \chi_j,$$

$$e_{i,j} e_{i,j}^* = \chi_i,$$
The last example illustrates a general property:

\[ f \in C(X) \rightarrow \delta(f)(x, y) = \begin{cases} f(x) & x = y \\ 0 & x \neq y \end{cases} \]

embeds \( C(X) \) as a unital subalgebra of \( C^*(R) \).

Assume \( U, V \subset X \) are clopen, \( \rho : U \rightarrow V \) as before, let \( w(x, y) = 1, x \in U, y = \rho(x), w(x, y) = 0, \) otherwise.

\[
\begin{align*}
w^*w &= \delta(\chi_U), \\
ww^* &= \delta(\chi_V), \\
w\delta(f)w^* &= \delta(f \circ \rho)
\end{align*}
\]

if \( f \) is supported in \( U \).

Example: \( X \) locally compact, \( R = = \) (equality).

\[
C^*(R) = C_0(X).
\]
Example (Kellendonk): $\mathcal{P} = \{p_1, \ldots, p_N\}$, a finite set of prototiles in $\mathbb{R}^d$. Each has a distinguished interior point $x(p_i)$ called a puncture.

Translate: $x(p_i + y) = x(p_i) + y, y \in \mathbb{R}^d$

Suppose $\Omega$ a compact, translation invariant collection of tilings which are made from translates of $\mathcal{P}$.

$$\Omega_{punc} = \{T \in \Omega \mid x(t) = 0, \text{ for some } t \in T\}.$$ 

$$R_{punc} = \{(T, T + x) \mid T, T + x \in \Omega_{punc}, x \in \mathbb{R}^d\}$$ is an étale groupoid.

Let $T \in \Omega$, $t_1, t_2 \in T$:

$$U = \{T' \mid t_1 - x(t_1), t_2 - x(t_1) \in T'\}$$

$$V = \{T' \mid t_1 - x(t_2), t_2 - x(t_2) \in T'\}$$

$$\rho(T') = T' + x(t_1) - x(t_2).$$
Situation 2: Actions of countable groups

$G$ a countable abelian (for notation) group, $X$ a loc. cmpct Hausdorff space, $\varphi$ an action of $G$ on $X$:

$$s \in G, \varphi^s : X \to X,$$

is a homeomorphism.

Action is free if $\varphi^s(x) = x \Rightarrow s = 0$.

$C_0(X) \times_\varphi G$: Generators: $C_0(X)$, $u_s, s \in G$, Relations:

\[
\begin{align*}
    u_0 &= 1, \\
    u_su_t &= u_{s+t}, \\
    u_s^* &= u_{-s}, \\
    u_sf u_s^* &= f \circ \varphi^{-s} \\
    u_sf &= (f \circ \varphi^{-s})u_s
\end{align*}
\]

$s, t \in G, f \in C_0(X)$. 

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Consider all formal sums

$$\sum_{s \in G} f_s u_s$$

where only finitely many $f_s \in C(X)$ are non-zero. The rules above define product and involution. We give this a norm and then complete.

Idea: Each $s$ in $G$ defines an automorphism of $C_0(X)$: $f \rightarrow f \circ \varphi^{-s}$. Here $\delta(f) = f u_0$ and $C_0(X) \subset C_0(X) \times_\varphi G$ and all these automorphisms become inner. $u_s$ is a unitary. (Caution: $u_s$ is in $C_0(X) \times_\varphi G$ only if $X$ is compact.)

Example: $X = \{1, \ldots, N\}, G = \mathbb{Z}_N$, $\varphi$ is addition, mod $N$. $C(X) \times G \cong M_N$. 

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Gelfand-Naimark dictionary (for free actions):

<table>
<thead>
<tr>
<th>Dynamics $(X, G, \varphi)$</th>
<th>$C^*$-alg. $C_0(X) \times \varphi G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>closed invariant set $Y \subset X$</td>
<td>two-sided closed ideal $I = {\sum_s f_s u_s \mid f_s</td>
</tr>
<tr>
<td>invariant measure $\mu$</td>
<td>trace $\tau_\mu(\sum_s f_s u_s) = \int_X f_0 d\mu$</td>
</tr>
</tbody>
</table>

\[
\tau_\mu(ab) = \tau_\mu(ba)
\]
Comparison of topological equivalence relations and actions of countable groups.

Start with \((X, G, \varphi)\).

Let \(R_\varphi = \{(x, \varphi^s(x)) \mid x \in X, s \in G\}\), is an equivalence relation. The classes are the orbits.

If \(G\) acts freely \((\varphi^s(x) = x \text{ only if } s = e)\), this can be given an étale topology. The local homeomorphisms are \(\varphi^s, s \in G\).

\[ C(X) \times_\varphi G \cong C^*(R_\varphi). \]
Situation 3: Continuous group actions

$G$ a locally compact abelian group, $X$ a locally compact Hausdorff space, $\varphi$ an action of $G$ on $X$:

$$s \in G, \varphi^s : X \to X,$$

is a homeomorphism.

$C_c(X \times G)$ is a linear space and is given a product and involution:

$$(f \cdot g)(x, s) = \int_G f(x, t) g(\varphi^t(x), s - t) d\lambda(t),$$

$$f^*(x, s) = f(\varphi^{-s}(x), s),$$

$f, g$ in $C_c(X \times G)$, $x$ in $X$, $s$ in $G$,

$\lambda$ is Haar measure on $G$.

$G$ discrete: 

$$u_s(x, t) = \begin{cases} 
1 & t = s \\
0 & t \neq s
\end{cases}$$
Part 4: Morita equivalence for $C^*$-algebras
(Rieffel, Muhly-Renault-Williams)

“Morita equivalence is more natural than isomorphism” - A. Connes.

If $A$ and $B$ are Morita equivalent ($A \sim B$), then

- $A$ and $B$ have isomorphic lattices of closed two-sided ideals
- there is a bijection between classes of representations as operators on Hilbert space
- $A$ and $B$ have isomorphic K-theory
What is not preserved:

- linear dimension
- commutativity

Example 1: $M_m(\mathbb{C}) \sim M_n(\mathbb{C})$ are Morita equivalent for all $m, n \geq 1$. 
Example 2: $\varphi$ a free, wandering action of $G$ on $X$. $q : X \rightarrow X/R\varphi$ is the quotient map. Wandering implies that the space of orbits $X/R\varphi$ is Hausdorff in the quotient topology.

$$A = C_0(X) \times \varphi G \sim B = C_0(X/R\varphi)$$ are Morita equivalent.

e.g. $C_0(\mathbb{R}) \times \mathbb{Z} \sim C(S^1)$.

Moral: if the quotient $X/R\varphi$ is a bad space (there is some recurrence in $\varphi$), then $C_0(X) \times \varphi G$ is its non-commutative replacement.
Example 3: \( X \) locally compact, Hausdorff, \( \varphi \) an action of \( G \), \( \psi \) an action of \( H \),

\[ \varphi^s \circ \psi^t = \psi^t \circ \varphi^s, s \in G, t \in H. \]

If the actions \( \varphi \) and \( \psi \) are both wandering, then

\[ A = C_0(X/R_{\varphi}) \times_{\psi} H \]
\[ B = C_0(X/R_{\psi}) \times_{\varphi} G \]
\[ C = C_0(X) \times_{\varphi \times \psi} (G \times H) \]

are all Morita equivalent.

Example 4: If \( \varphi \) is an \( \mathbb{R} \)-action on \( X \) and has a transversal \( T \), let \( \psi \) be the Poincaré first return map on \( T \). Under mild conditions,

\[ C_0(X) \times_{\varphi} \mathbb{R} \sim C_0(T) \times_{\psi} \mathbb{Z}. \]
Example 5: Let Ω be a continuous hull. It has an action of $\mathbb{R}^d$ and we consider the $C^*$-algebra $C(\Omega) \times \mathbb{R}^d$.

Recall

$$\Omega_{punc} = \{ T \in \Omega \mid x(t) = 0, \text{ some } t \in T \}$$

and

$$R_{punc} = \{ (T, T + x) \mid , T, T + x \in \Omega_{punc} \}$$

and the $C^*$-algebra $C^*(R_{punc})$.

- $\Omega_{punc}$ is a transverse to the $\mathbb{R}^d$-action,

- restricting the $\mathbb{R}^d$-orbits to $\Omega_{punc}$ gives $R_{punc}$ which is étale

- every $\mathbb{R}^d$ orbit in $\Omega$ meets $\Omega_{punc}$.

$C^*(R_{punc})$ and $C(\Omega) \times \mathbb{R}^d$ are Morita equivalent.
Part 5: $C^*$-algebras for projection method tilings (Forrest-Hunton-Kellendonk)

Data:

- $\mathbb{R}^d$, physical space (to be tiled),
- $H$, internal space, locally cpct ab. group,
- $\pi : \mathbb{R}^d \times H \to \mathbb{R}^d$, $\pi^\perp : \mathbb{R}^d \times H \to H$,
- $\mathcal{L} \subset \mathbb{R}^d \times H$, discrete, co-compact (lattice),
- $\pi|\mathcal{L}, \pi^\perp|\mathcal{L}$ one-to-one, $L = \pi^\perp(\mathcal{L})$ dense in $H$,
- $W \subset H$, compact, regular, $\lambda(\partial W) = 0$. 

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A point $x$ in $\mathbb{R}^d \times H$ is non-singular if
$$\pi^\perp(x + \mathcal{L}) \cap \partial W = \emptyset.$$ 
$\mathcal{N}$ is the set of non-singular points.

$$\Lambda_x = \pi\{y \in x + \mathcal{L} \mid \pi^\perp(y) \in W\}$$
is a Delone set, called a regular model set.

The hull $\Omega$ is the completion of
$$\{\Lambda_x \mid x \in \mathcal{N}\}.$$

Comments:

• $\mathcal{N}$ is invariant under the actions of $\mathbb{R}^d$ and $\mathcal{L}$,

• $\Lambda_{x+s} = \Lambda_x$, if $s \in \mathcal{L}$,

• $\Lambda_{x+u} = \Lambda_x + u$, if $u \in \mathbb{R}^d$. 

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Lemma 3. Suppose $x_n \in \mathcal{N}$ converges to $x \in \mathbb{R}^d \times H$. $\Lambda_x$ converges in $\Omega$ (i.e. is Cauchy in the tiling metric) if and only if, for every $s \in L$, the sequence $\pi^\perp(x_n)$ is eventually either in $W + s$ or in its complement.

Theorem 4. For $s \in L$, 

$$\Lambda_x \to \chi_{W+s}(x), \ x \in \mathcal{N} \cap H$$

extends to a continuous function on $\Omega$.

Definition 5. Consider $A$, the $C^*$-algebra of operators on $L^2(H, \lambda)$ generated by $C_0(H)$ and $\chi_{W+s}, s \in L$. Let $\hat{H}$ be its spectrum; i.e. $A \cong C_0(\hat{H})$.

The action of $L$ on $E$ extends to $\hat{H}$. $L \subset H$ is dense implies that $\hat{H}$ is totally disconnected.
Theorem 6. The hull $\Omega$ is homeomorphic to

$$\mathbb{R}^d \times \hat{H}/\mathcal{L}$$

The actions of $\mathbb{R}^d$ and $\mathcal{L}$ on $\mathbb{R}^d \times \hat{H}$ are commuting, free and wandering:

Theorem 7. $C_0(\mathbb{R}^d \times \hat{H}/\mathcal{L}) \times \mathbb{R}^d$ is Morita equivalent to

$$C_0(\hat{H}) \times L.$$

The actions of $G = \mathbb{R}^d$ and $\mathcal{L} \cong L$ on $\mathbb{R}^d \times \hat{H}$ are commuting and wandering:

$$\mathbb{R}^d \times \hat{H}/\mathbb{R}^d \cong \hat{H}.$$
Further reductions:

Assume $H = \mathbb{R}^N$. So $L \cong \mathcal{L} \cong \mathbb{Z}^{d+N}$, as an abstract group: $C_0(\hat{H}) \times \mathbb{Z}^{d+N}$. The action is by translation by the vectors $L$, which is a dense subgroup of $\mathbb{R}^N$.

$\hat{H}$ is $\mathbb{R}^N$ disconnected along the boundaries of $W$ and its translates by $L$. In many cases, this can be done in other ways, e.g. by lines.

Example: Fibonacci: $d = 1, N = 1, L = \mathbb{Z} + \alpha \mathbb{Z}$. $W = [a, b]$. $\hat{H}$ is $\mathbb{R}^1$ disconnected along the $\mathbb{Z} + \alpha \mathbb{Z}$-orbits of $a$ and $b$ (one orbit or two?).

Example: Penrose: $d = 2, N = 2, L$ is the subgroup of the plane generated by $\exp(2\pi ij/5), j = 0, 1, 2, 3, 4$. $\hat{H}$ is the plane disconnected along the 5 lines through the origin and $\exp(2\pi ij/5), j = 0, 1, 2, 3, 4$, and all translates of them by $L$. 
Example: TTT (Tübingen triangle tiling) Same is the Penrose, but rotate the 5 original lines by $\pi/10$.

Example: Octagonal tiling: $d = 2$, $N = 2$, $L$ is the subgroup generated by $exp(\pi ij/4), j = 0, 1, 2, 3$. $\mathring{H}$ is the plane disconnected along the 4 lines through the origin and $exp(\pi ij/4), j = 0, 1, 2, 3$, and all translates by $L$.

One more reduction (still with $H = \mathbb{R}^N$). List a set of generators of $L$: $s_1, \ldots, s_{d+N}$. Act on a disconnected $H = \mathbb{R}^N$. The action of the first $N$ of them is free and wandering: let $\mathring{H}_0$ denote the quotient, which is a Cantor set. It is really a disconnected $N$-torus. Our $C^*$-algebra is Morita equivalent to

$$C(\mathring{H}) \times \mathbb{Z}^{d+N} = C(\mathring{H}_0) \times \mathbb{Z}^d.$$
Part 6: K-theory for $C^*$-algebras

To a $C^*$-algebra, $A$, there are associated two abelian groups, $K_0(A)$ and $K_1(A)$. These are based on

- projections $p^2 = p = p^*$
- unitaries $u^* = u^{-1}$,

respectively, in $A$. It is a recepticle for such data and also an invariant for $A$. There is (by now) quite a lot of machinery for computing it.
$K_0(A)$: Assume $A$ with unit.

$p$ is a projection if $p^2 = p = p^*$.  

Equivalence of projections:

- **Murray-von Neumann similarity**  
  \[ p \sim_s q \quad \exists v, v^*v = p, vv^* = q, \]

- **unitary eq.**  
  \[ p \sim_u q \quad \exists v^* = v^{-1}, vpv^{-1} = q \]

- **homotopy**  
  \[ p \sim_h q \quad \exists t \rightarrow pt, p_0 = p, p_1 = q \]

Note that $v$ above must be in $A$.

Addition of projections: if $p, q$ are orthogonal ($pq = 0$), then $p + q$ is a projection.

$M_n(A)$ is the set of $n \times n$ matrices with entries from $A$. It is a $C^*$-algebra. Its unit is $1_n$. For $a \in M_n(A), b \in M_m(A)$,

\[
a \oplus b = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in M_{m+n}(A).
\]
$P_n(A)$, projections in $M_n(A)$.

$$P_1(A) \subset P_2(A) \subset P_3(A) \subset$$

by identifying $p$ and $p \oplus 0$. Let $P(A) = \cup_n P_n(A)$.

Equivalence: In $P(A)$, we have $\sim = \sim_s = \sim_u = \sim_h$.

Problem: $p + p_0 \sim q + p_0 \not\Rightarrow p \sim q$.

Define $p \approx q$ if and only if $p \oplus 1_n \sim q \oplus 1_n$, for some $n$. $[p]$ is the class modulo $\approx$.

Addition: $p, q \in P(A)$, $p = p \oplus 0$, $q \sim 0 \oplus q$, which are orthogonal, and so

$$[p] + [q] = [p \oplus q]$$

is a well-defined addition.

$P(A)/\sim$ is a semi-group with identity, $[0]$. $K_0(A)$ is its Grothendieck group, i.e. formal differences of classes of $P(A)$:

$$K_0(A) = \{[p] - [q] \mid p, q \in P(A)\}.$$  

It has a natural positive cone:

$$K_0(A)^+ = \{[p] - [0] \mid p \in P(A)\}.$$
Example: $\mathbb{C}$

Consider matrices over $\mathbb{C}$:

**Lemma 8.** Two projections $p$ and $q$ in $M_n(\mathbb{C})$ are similar if and only if $\text{rank}(p) = \text{rank}(q)$.

Rank is not going to generalize easily to other $C^*$-algebras, but recall, for a projection $\text{rank}(p) = \text{Trace}(p)$.

**Proposition 9.** The map $\text{Tr} : K_0(\mathbb{C}) \to \mathbb{Z}$

$$\text{Tr}([p] - [q]) = \text{Trace}(p) - \text{Trace}(q)$$

is an isomorphism. Under this, $K_0(\mathbb{C})^+ = \{0, 1, 2, 3, \ldots\} = \mathbb{Z}^+$. 

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Example: \( C(S^2) \)

If \( p \in M_n(C(S^2)) \), then \( \text{Trace}(p(x)) \) is continuous in \( x \). If \( p \) is also a projection, its value is integral.

\[ [p] - [q] \in K_0(C(S^2)) \rightarrow \text{Trace}(p(x)) - \text{Trace}(q(x)) \]

is a homomorphism, but is not injective. There is a projection \( p \in M_2(C(S^2)) \) such that at every point \( p(x) \) is similar to \( 1 \oplus 0 \), but this similarity cannot be made continuous over \( S^2 \).

**Proposition 10.** If \( X \) is totally disconnected, let \( C(X, \mathbb{Z}) \) be the group of continuous integer-valued functions on \( X \). The function \( Tr : K_0(C(X)) \rightarrow C(X, \mathbb{Z}) \) defined by

\[ Tr([p] - [q])(x) = \text{Trace}(p(x)) - \text{Trace}(q(x)) \]

is an isomorphism. Under this, \( K_0(C(X))^+ = C(X, \mathbb{Z}^+) \).

\( U \subset X \) clopen, \( \chi_U \) is a projection in \( C(X) \) and also in \( C(X, \mathbb{Z}) \). The map takes \([\chi_U] - [0]\) to \( \chi_U \).

**Proposition 11.** For a minimal action of $\mathbb{Z}$ on a Cantor set $X$, $K_0(C(X) \times \varphi \mathbb{Z})$ is isomorphic to

$$C(X, \mathbb{Z})/\{f - f \circ \varphi \mid f \in C(X, \mathbb{Z})\}$$

and $K_0(C(X) \times \varphi \mathbb{Z})^+$ is the image of $C(X, \mathbb{Z}^+)$. 

Inclusion $C(X) \subset C(X) \times \mathbb{Z}$ gives

$$K_0(C(X)) \cong C(X, \mathbb{Z}) \rightarrow K_0(C(X) \times \mathbb{Z}).$$

Surjectivity: every projection in $C(X) \times \mathbb{Z}$ is similar to one in $C(X)$.

Let $U \subset X$ be clopen. $\chi_U$ is a projection in $C(X)$, but

$$\chi_U \sim_u u_1 X U u_1^* = \chi_U \circ \varphi^{-1} = \chi_{\varphi(U)}.$$

If one replaces $\mathbb{Z}$ by $\mathbb{Z}^d$, $d > 1$, more sophisticated methods (spectral sequences) are needed.
Recall, every \( \varphi \)-invariant measure \( \mu \) gives a trace \( \tau_\mu \) on \( C(X) \times \mathbb{Z} \). This yields a map

\[
\hat{\tau}_\mu : K_0(C(X) \times \mathbb{Z}) \to \mathbb{R}.
\]

If \( U \) is clopen, \( \hat{\tau}_\mu[\chi_U] = \mu(U) \).

**Theorem 12.** \( a \) in \( K_0(C(X) \times \mathbb{Z}) \) is in \( K_0(C(X) \times \mathbb{Z})^+ \) if and only if \( a = 0 \) or \( \hat{\tau}_\mu(a) > 0 \), for all \( \mu \).

For \( d > 1 \), the inclusion \( C(X) \subset C(X) \times \mathbb{Z}^d \) induces \( C(X, \mathbb{Z}) \to K_0(C(X) \times \mathbb{Z}^d) \) which is not onto.

**Theorem 13** (Gap labelling: B-B-G, B-OO, K-P).

\[
\hat{\tau}_\mu(K_0(C(X) \times \mathbb{Z}^d)) = \hat{\tau}_\mu(C(X, \mathbb{Z})) = \{ \mu(U) \mid U \text{ clopen} \} + \mathbb{Z}.
\]
There are some very sophisticated machinery for computing this.

Connes’ analogue of the Thom isomorphism:

\[ K_i(C(X) \times \mathbb{R}^d) \cong K_{i+d}(C(X)). \]

Can be used in the case \( X = \Omega \), the continuous hull. \( K_i(C(X)) \) is closely related (especially in low dimensions) to the cohomology of \( X \).

However, this isomorphism does not respect the order structure on \( K_0 \).