SEMIGROUPS GENERATED BY SIMILARITY ORBITS

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Abstract. We investigate the semigroups in $M_n(F)$ generated by the similarity orbit of single matrices.

0. Introduction

Question. What is the semigroup in $M_n(F)$ generated by the similarity orbit of a single matrix of rank $k$?

In section 1 and 2 we consider the semigroup $S$ in $M_n(F)$ generated by the similarity orbit of an invertible matrix $A$. In this case $S$ is of course a semigroup in $GL_n(F)$, and it is a normal subgroup if and only if $\det A$ is a root of unity in $F^*$. For a non-scalar $A$, except when $n = 2$ and $|F| = 3$, these normal subgroups are isomorphic to semi-direct products $S \cong SL_n(F) \rtimes U$, where $U$ is the cyclic subgroup of $F^*$ generated by $\det A$.

Some bounds for the number of similarity factors required are found in section 2. Let $(A)_m = \{A_1 A_2 \ldots A_m | A_j \sim A \text{ for } j = 1, 2, \ldots, m\}$. If $A = \lambda I$ is scalar, then of course $(A)_m$ is the singleton $\{\lambda^m I\}$. If $A$ is not scalar, an obvious necessary condition for $T$ to be in $(A)_m$ is that $\det(T) = (\det(A))^m$. We prove that this condition is sufficient if $m$ is large enough. We find a bound on $m$ in terms of the number of linear invariant factors of $A$; this bound never exceeds $4n$.

In section 3 we find that the semigroup in $M_n(F)$ generated by the similarity orbit of a singular matrix $A$ with rank $A = r < n$ consists of all matrices of rank less than or equal to $r$.

1. Semigroups generated by the similarity orbit of an invertible matrix

The semigroup $S$ in $GL_n(F)$ generated by the similarity orbit of a matrix $A$ of finite multiplicative order is automatically a normal subgroup of $GL_n(F)$. It is therefore useful to characterize the normal subgroups of $SL_n(F)$ and of $GL_n(F)$ first. Recall that $SL_n(F)$ is perfect, i.e. $SL_n(F)_{ab} = SL_n(F)/[SL_n(F), SL_n(F)]$ is trivial, and

Research supported in part by the NSERC of Canada and by the Ministry of Science and Technology of Slovenia.

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PSL\(_n(F)\) is simple when \(n \neq 2\) and \(|F| \neq 2, 3\). Moreover, \(Z(\mathrm{SL}_n(F)) = Z(\mathrm{GL}_n(F)) \cap \mathrm{SL}_n(F)\) for every field \(F\).

**Lemma 1.1.** Let \(H\) be a normal subgroup of \(\mathrm{SL}_n(F)\), where \(n \neq 2\) and \(|F| \neq 2, 3\). Then either

1. \(H\) consists of scalar matrices and is therefore a cyclic subgroup generated by an \(n\)-th root of unity, or
2. \(H\) contains a non-scalar matrix and is equal to \(\mathrm{SL}_n(F)\).

**Proof.** If \(Z = Z(\mathrm{SL}_n(F)) = \mathrm{SL}_n(F) \cap Z(\mathrm{GL}_n(F))\) is the center of \(\mathrm{SL}_n(F)\), i.e. the cyclic subgroup of \(n\)-th roots of unity, then the obvious commutative diagram

\[
\begin{array}{cccccc}
1 & 1 & 1 \\
\downarrow & \downarrow & \downarrow \\
1 & \longrightarrow & H \cap Z & \longrightarrow & Z & \longrightarrow & Z/H \cap Z & \longrightarrow & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & \longrightarrow & H & \longrightarrow & \mathrm{SL}_n(F) & \longrightarrow & \mathrm{SL}_n(F)/H & \longrightarrow & 1 \\
\downarrow & \eta & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & \longrightarrow & \eta(H) & \longrightarrow & \mathrm{PSL}_n(F) & \longrightarrow & \mathrm{PSL}_n(F)/\eta(H) & \longrightarrow & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & 1 & 1 & 1
\end{array}
\]

has exact rows and columns. Now, either \(\eta(H) = 1\) or \(\eta(H) = \mathrm{PSL}_n(F)\), since \(\mathrm{PSL}_n(F)\) is simple. If \(\eta(H) = 1\) then \(H \cap Z = H\), so that \(H \subset Z\). If \(\eta(H) = \mathrm{PSL}_n(F)\) then \(Z/H \cap Z \cong \mathrm{SL}_n(F)/H\) is abelian, hence trivial, since \(\mathrm{SL}_n(F)\) is perfect (i.e. \(\mathrm{SL}_n(F)_{ab}\) is trivial), so that \(H = \mathrm{SL}_n(F)\). □

Observe that the exact sequence of groups

\[
1 \rightarrow \mathrm{SL}_n(F) \rightarrow \mathrm{GL}_n(F) \xrightarrow{\det} F^* \rightarrow 1
\]

splits; for example the homomorphism \(s : F^* \rightarrow \mathrm{GL}_n(F)\) defined by \(s(x) = x \oplus I_{n-1}\) is a section. Thus \(\mathrm{GL}_n(F) \cong \mathrm{SL}_n(F) \rtimes F^*\), the semidirect product, where the action \(\alpha : F^* \times \mathrm{SL}_n(F) \rightarrow \mathrm{SL}_n(F)\) is given by \(\alpha(x, A) = s(x)As(x)^{-1}\).

**Proposition 1.2.** Let \(G\) be a normal subgroup of \(\mathrm{GL}_n(F)\), where \(n \neq 2\) and \(|F| \neq 2, 3\). Then, either

1. \(G\) consists of scalar matrices and therefore \(G \subset Z(\mathrm{GL}_n(F)) \cong F^*\), or
2. \(G\) contains a non-scalar matrix and is a semidirect product \(G \cong \mathrm{SL}_n(F) \rtimes U\), where \(U = \det(G) \subset F^*\).
Proof. The commutative diagram

\[
\begin{array}{cccc}
1 & 1 & 1 \\
\downarrow & \downarrow & \downarrow \\
G \cap \text{SL}_n(F) & \longrightarrow & G & \longrightarrow & U & \longrightarrow & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\text{SL}_n(F) & \longrightarrow & \text{GL}_n(F) & \longrightarrow & F^* & \longrightarrow & 1 \\
\end{array}
\]

has exact rows and columns. The bottom sequence is split by the homomorphism \( s : F^* \to \text{GL}_n(F) \) defined by \( s(x) = x \oplus I_{n-1} \), so that \( \text{GL}_n(F) \cong \text{SL}_n(F) \rtimes F^* \). The action \( \alpha : F^* \times \text{SL}_n(F) \to \text{SL}_n(F) \) is given by \( \alpha(x, A) = s(x)As(x)^{-1} \).

If \( G \) consists of non-scalar matrices then the assertion is obvious. If \( G \) contains a non-scalar matrix \( A \) then for some \( S \in \text{SL}_n(F) \) the element \( [S, A] = SAS^{-1}A^{-1} \) of \( G \cap \text{SL}_n(F) \) is not scalar. For, suppose to the contrary that \( [S, A] = SAS^{-1}A^{-1} = \lambda_S I \), i.e. \( SAS^{-1} = \lambda_S A \), for all \( S \in \text{SL}_n(F) \). Then \( \lambda : \text{SL}_n(F) \to F^* \) is a homomorphism of groups and in particular \( \lambda_{[S,T]} = 1 \) for all \( S, T \in \text{SL}_n(F) \). Since \( \text{SL}_n(F) \) is perfect, i.e. \( [\text{SL}_n(F), \text{SL}_n(F)] = \text{SL}_n(F) \), it follows that \( \lambda_S = 1 \) and hence \( [S, A] = I \) for all \( S \in \text{SL}_n(F) \), which means that \( A \) is scalar. Thus, if \( G \) contains a non-scalar matrix then so does \( G \cap \text{SL}_n(F) \), and \( G \cap \text{SL}_n(F) = \text{SL}_n(F) \) by Lemma 1.1. Then \( \det^{-1}(U) = G \), hence the top exact sequence of the diagram splits, and \( G \cong \text{SL}_n(F) \times U \). □

**Corollary 1.3.** If \( n \neq 2 \) and \( |F| \neq 2, 3 \) then the subgroup \( G \) of \( \text{GL}_n(F) \) generated by the similarity orbit of a non-scalar invertible matrix \( A \) is of the form \( G \cong \text{SL}_n(F) \rtimes U \), where \( U \) is the cyclic subgroup of \( F^* \) generated by \( \det A \).

To determine the semigroup \( S \) (as opposed to the group) generated by the similarity orbit of an invertible matrix is more complicated. Since every square matrix has a rational canonical form it is useful to start with the companion matrix of a polynomial, i.e. a cyclic matrix.

**Lemma 1.4.** The semigroup \( S \) in \( \text{GL}_n(F) \) generated by the similarity orbit of the companion matrix \( A \) of the polynomial \( p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \) with \( \det A = a_0 \neq 0 \) contains the diagonal matrix \( I_{n-1} \oplus a_0^2 \) and the scalar matrix \( a_0^2 I \).

**Proof.** If \( Q \) is the involution obtained from the identity \( I \) by reversing the order of the rows then \( B = QAQ \) is the matrix obtained from \( A \) by first reversing the order of the rows of \( A \) to get a matrix \( C \) and then reversing the order of the columns of \( C \) to get \( B \). Then

\[
BA = \begin{pmatrix} I_{n-1} & X \\ 0 & a_0^2 \end{pmatrix}
\]

for some \( X \), where \( I_{n-1} \) is the identity matrix of size \( n - 1 \). If \( a_0^2 \neq 1 \) then \( BA \) is similar to \( I_{n-1} \oplus a_0^2 \), as can be seen by replacing the last vector in the standard
ordered basis \( \{e_i | 1 \leq i \leq n \} \) of \( \mathbb{F}^n \) by \( e_n + (1/(a_0^2 - 1)) \sum_{i=1}^{n-1} x_i e_i \). Thus we are done in this case, since by a cyclic permutation similarity argument \( a_0^2 I \) is in \( S \). If \( a_0^2 = 1 \) then \( BA = I + N \) with \( N^2 = 0 \). Since \( I + N \) is similar to \( I - N \), which is easily seen by replacing \( e_n \) by \(-e_n \) in the standard ordered basis of \( \mathbb{F}^n \), it follows that \( (I + N)(I - N) = I \) is in \( S \). \( \square \)

**Proposition 1.5.** The semigroup \( S \) in \( \text{GL}_n(\mathbb{F}) \) generated by the similarity orbit of an invertible matrix \( A \) contains an upper-triangular matrix \( U \) with \( \det U = \det A^2 \), a diagonal matrix \( D \) with \( \det D = \det A^4 \) and a non-zero scalar matrix \( \lambda I \) with \( \lambda = \det A^{4n} \).

**Proof.** We may assume without loss of generality that \( A \) is in rational canonical form. Apply Lemma 1.4 to each companion matrix in the rational decomposition of \( A \) to get an upper-triangular matrix \( BA \simeq (I + N) \oplus D \), where \( B \) is similar to \( A \), \( D \) is diagonal, \( \det D = \det A^2 \) and \( N^2 = 0 \). Again, since \( (I + N) \oplus D \) is similar to \( (I - N) \oplus D \), it follows that \( (I + N) \oplus D)((I - N) \oplus D) = I \oplus D^2 \). Cyclically permuting the diagonal entries of \( I \oplus D^2 \) yields \( n \) mutually similar diagonal matrices. The product of these diagonal matrices is the scalar matrix \( \lambda I \in S \), where \( \lambda = \det A^{4n} \). \( \square \)

**Corollary 1.6.** Let \( S \) be the semigroup in \( \text{GL}_n(\mathbb{F}) \) generated by the similarity orbit of an invertible matrix \( A \). Then \( S \) is a normal subgroup of \( \text{GL}_n(\mathbb{F}) \) if and only if \( \det A \) is a root of unity. If \( d = \det A \) is a root of unity and \( A \) is not scalar then \( S \cong \text{SL}_n(\mathbb{F}) \ltimes \ltimes d > \), except when \( n = 2 \) and \( |\mathbb{F}| = 2, 3 \). In particular, if \( d = 1 \) then \( S = \text{SL}_n(\mathbb{F}) \), except when \( n = 2 \) and \( |\mathbb{F}| = 2, 3 \).

**Proof.** If \( d = \det A \) is not a root of unity then \( \det S \neq 1 \) for all \( S \in S \) and the semigroup \( S \) is not a subgroup of \( \text{GL}_n(\mathbb{F}) \). If \( d^m = 1 \) then \( I = D^m = XSA \text{A}^{-1} \) in \( \text{SL}_n(\mathbb{F}) \) for some \( X \in \text{SL}_n(\mathbb{F}) \) and some \( S \in \text{GL}_n(\mathbb{F}) \), where \( D \) is the diagonal matrix of Proposition 1.5. Thus, \( A^{-1} = S \text{A}^{-1} X \in S \) and \( S \) is a subgroup of \( \text{GL}_n(\mathbb{F}) \). Now apply Proposition 1.2. \( \square \)

In the two exceptional cases \( n = 2 \) and \( |\mathbb{F}| = 2, 3 \) the group \( \text{PSL}_n(\mathbb{F}) \) is not simple and \( \text{SL}_n(\mathbb{F}) \) is not perfect. These cases have to be considered separately.

The group \( \text{GL}_2(\mathbb{Z}_2) \) is not abelian and \( |\text{GL}_2(\mathbb{Z}_2)| = 6 \), so that \( \text{PSL}_2(\mathbb{Z}_2) \cong \text{SL}_2(\mathbb{Z}_2) \cong \text{GL}_2(\mathbb{Z}_2) \cong S_3 \), the symmetric group on three symbols. The only proper normal subgroup of \( \text{GL}_2(\mathbb{Z}_2) \) is therefore the cyclic subgroup \( C_3 \) of order 3 generated by

\[
\begin{pmatrix}
1 & 1 \\
1 & 0
\end{pmatrix}
\text{ or its inverse }
\begin{pmatrix}
0 & 1 \\
1 & 1
\end{pmatrix}.
\]

**Proposition 1.7.** If \( I \neq \lambda \in \text{GL}_2(\mathbb{Z}_2) \) then \( S \cong C_3 \) if \( A \) has order 3 and \( S = \text{GL}_2(\mathbb{Z}_2) \) otherwise. \( \square \)

In the case of \( \text{GL}_2(\mathbb{Z}_3) \) we have \( |\text{GL}_2(\mathbb{Z}_3)| = 48 \) and \( Z(\text{GL}_2(\mathbb{Z}_3)) \cong C_2 \) is the cyclic subgroup of order 2 generated by \( 2I \). In the commutative diagram with exact rows
and columns

\[
\begin{array}{ccc}
Z & \longrightarrow & Z \\
\downarrow & & \downarrow \\
\text{SL}_2(\mathbb{Z}_3) & \longrightarrow & \text{GL}_2(\mathbb{Z}_3) \\
\eta & \downarrow & \eta \\
\text{PSL}_2(\mathbb{Z}_3) & \longrightarrow & \text{PGL}_2(\mathbb{Z}_3) \quad \overset{\text{det}}{\longrightarrow} \quad \mathbb{Z}_3^* \\
\end{array}
\]

the determinant map is split by the homomorphism \( s : \mathbb{Z}_3^* \to \text{GL}_2(\mathbb{Z}_3) \) defined by \( s(2) = \text{diag}[2, 1] \). Moreover, the Sylow 2-subgroups of \( \text{SL}_2(\mathbb{Z}_3) \) and \( \text{PSL}_2(\mathbb{Z}_3) \) are normal, they are a copy of the quaternion group \( Q \) generated by the two matrices

\[
X = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}
\quad \text{and} \quad
Y = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},
\]

and a copy of the Klein 4-group \( V \) generated by \( \eta(X) \) and \( \eta(Y) \), respectively. We have a commutative diagram with exact rows and columns

\[
\begin{array}{ccc}
Z & \longrightarrow & Z \\
\downarrow & & \downarrow \\
Q & \longrightarrow & \text{SL}_2(\mathbb{Z}_3) \\
\eta & \downarrow & \eta \\
V & \longrightarrow & \text{PSL}_2(\mathbb{Z}_3) \quad \longrightarrow \quad \text{C}_3
\end{array}
\]

in which the canonical projection \( p : \text{SL}_2(\mathbb{Z}_3) \to \text{C}_3 \) is split by the homomorphism \( t : \text{C}_3 \to \text{SL}_2(\mathbb{Z}_3) \), where

\[
t(x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},
\]

the image of a generator of \( \text{C}_3 \), generates a Sylow 3-subgroup of order 3 in \( \text{SL}_2(\mathbb{Z}_3) \). Observe that \( s(\mathbb{Z}_3^* \text{SL}_2(\mathbb{Z}_3) \to \text{GL}_2(\mathbb{Z}_3) \) acts on \( t(\text{C}_3) \) and on \( Q \) while \( t(\text{C}_3) \) acts on \( Q \) by conjugation, so that \( \text{C}_3 \rtimes \mathbb{Z}_3^* \cong S_3 \). Thus, \( \text{PSL}_2(\mathbb{Z}_3) \cong V \rtimes \text{C}_3, \text{SL}_2(\mathbb{Z}_3) \cong Q \rtimes \text{C}_3 \) and \( \text{GL}_2(\mathbb{Z}_3) \cong \text{SL}_2(\mathbb{Z}_3) \rtimes \mathbb{Z}_3^* \cong Q \rtimes S_3 \). There are three Sylow 2-subgroups of order 16 in \( \text{GL}_2(\mathbb{Z}_3) \), namely \( Q \rtimes \mathbb{Z}_3^* \) and its conjugates. They intersect in the normal subgroup \( Q \). The proper normal subgroups of \( \text{GL}_2(\mathbb{Z}_3) \) are therefore \( Z \cong \text{C}_2, Q \) and \( \text{SL}_2(\mathbb{Z}_3) \).

**Proposition 1.8.** Let \( I \neq A \in \text{GL}_2(\mathbb{Z}_3) \).

1. If \( \text{det} A = 1 \) then \( S = Z, Q, \text{SL}_2(\mathbb{Z}_3) \) depending on whether the order of \( A \) is 2, 4 or divisible by 3.
2. If \( \text{det} A = 2 \) then \( S = \text{GL}_2(\mathbb{Z}_3). \quad \square \)
The main result of [S] will be used repeatedly in the next section. We record it here, without proof, for future reference.

**Theorem 1.9.** Let \( A \in \text{GL}_n(\mathbb{F}) \) be non-scalar and let \( \beta_j, \gamma_j \ (1 \leq j \leq n) \) be elements of \( \mathbb{F}^* \) such that \( \prod_{j=1}^{n} \beta_j \gamma_j = \det A \). Then there exist matrices \( B \) and \( C \) in \( \text{GL}_n(\mathbb{F}) \) with eigenvalues \( \beta_1, \ldots, \beta_n \) and \( \gamma_1, \ldots, \gamma_n \), respectively, such that \( A = BC \). Furthermore, \( B \) and \( C \) can be chosen so that \( B \) is lower triangularizable and \( C \) is simultaneously upper triangularizable. \( \square \)

2. **Some bounds on the number of similarity factors required**

In this section we have to assume that the field \( \mathbb{F} \) has enough elements, \( |\mathbb{F}| > 2n \) should suffice. The following result of Cater [C], which we quote here without proof, will be used in our considerations.

**Lemma 2.1.** If \( M \) is a non-scalar in \( \text{GL}_n(\mathbb{F}) \) and \( \det M = x_1x_2\ldots x_n \) then there is a factorization \( M = A_1A_2\ldots A_n \) with \( \det A_i = x_i \) and \( \text{rank}(A_i - I) = 1 \) for \( i = 1, 2, \ldots, n \). \( \square \)

Observe that the properties of the matrices \( A_i \) of Lemma 2.1 imply that \( A_i \) is similar to \( (I_2 + J_2) \oplus I_{n-2} \) if \( x_i = 1 \) and similar to \( x_i \oplus I_{n-1} \) if \( x_i \neq 1 \). Here is an immediate consequence of Cater’s result.

**Proposition 2.2.** Let \( A \) be a non-scalar element of \( \text{GL}_n(\mathbb{F}) \) such that \( \text{rank}(A - I) = 1 \). If \( \det T = \det A^n \) then \( T = A_1A_2\ldots A_n \), where \( A_i \) is similar to \( A \) for \( i = 1, 2, \ldots, n \).

**Proof.** The conditions imposed on \( A \) imply that \( A \) is similar to \( (I_2 + J_2) \oplus I_{n-2} \) if \( \det A = 1 \) and similar to \( A \oplus I_{n-1} \) if \( \det A \neq 1 \). By Cater’s Lemma 2.1 we see that \( T = A_1A_2\ldots A_n \), where \( \det A_i = \det A \) and \( \text{rank}(A_i - I) = 1 \), and hence where \( A_i \) is similar to \( A \) for \( i = 1, 2, \ldots, n \). \( \square \)

**Corollary 2.3.** If \( T \) is in \( \text{SL}_n(\mathbb{F}) \) then \( T = A_1A_2\ldots A_k \) for some \( k \) such that \( 0 \leq k \leq n \), where \( A_i \) is similar to \( A = (I_2 + J_2) \oplus I_{n-2} \) for \( i = 1, 2, \ldots, k \). \( \square \)

**Lemma 2.4.** If \( A \in \text{GL}_n(\mathbb{F}) \) is cyclic, then every \( T \in \text{GL}_n(\mathbb{F}) \) with distinct eigenvalues and \( \det T = \det A^2 \) has a factorization \( T = A_1A_2 \), where \( A_i \) is similar to \( A \) for \( i = 1, 2 \).

**Proof.** The matrix \( A \) is similar to the companion matrix of its characteristic polynomial \( p(x) = a_0 + a_1x + \ldots + a_{n-1}x^{n-1} + x^n \). Thus we may assume that

\[
A = \begin{pmatrix}
0 & a_0 \\
1 & a_1 \\
& \ddots \\
& & \ddots \\
& & & 1 & a_{n-1}
\end{pmatrix}.
\]
It is easy to see that via a suitable diagonal similarity $A$ is similar to a matrix of the form
\[
B = \begin{pmatrix}
0 & b_0 \\
x_1 & b_1 \\
& \ddots \\
& & \ddots \\
x_{n-1} & b_0 \\
\end{pmatrix}
\]
where $x_1, x_2, \ldots, x_{n-1}$ can be chosen arbitrarily in $F^*$, and where the determinant condition $b_n x_1 x_2 \ldots x_{n-1} = a_0$ holds. Then
\[
S = \begin{pmatrix}
a_{n-1} & x_{n-1} \\
& \ddots \\
& & \ddots \\
b_1 & x_1 \\
b_0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & a_0 \\
1 & a_1 \\
& \ddots \\
& & \ddots \\
& & & 1 & a_{n-1}
\end{pmatrix}
= \begin{pmatrix}
x_{n-1} & \ast \\
& \ddots \\
& & \ddots \\
x_1 & \ast \\
\end{pmatrix}
\]
is upper-triangular, and the first factor of $S$ is similar to $B$ via the similarity given by the involution obtained by reversing the order of the rows of the identity matrix. Since $x_1, x_2, \ldots, x_{n-1}$ and $b_0 a_0$ can be taken to be the distinct eigenvalues of $T$ we conclude that $T$ is similar to $S$, and thus $T$ is of the desired form. □

**Proposition 2.5.** Suppose that $|F| > 2n$. If the rational canonical form of $A$ has no scalar direct summand then there exists a $T \in \text{GL}_n(F)$ with distinct eigenvalues and $\det T = \det A^2$ such that $T = A_1 A_2$ with $A_1$ and $A_2$ similar to $A$. Furthermore, the eigenvalues of $T$ can be chosen outside a given subset $E$ of $F^*$ if $|F| \geq 2(|E| + n)$.

**Proof.** Assume without loss of generality that $A$ is in rational canonical form $A = R_1 \oplus R_2 \oplus \cdots \oplus R_m$. By hypothesis each rational cell $R_i$ has size $k_i \geq 2$. Let $n_0 = 0$ and $n_j = n_{j-1} + k_j$ for $j = 1, 2, \ldots, m$. We want to apply Lemma 2.4 in sequence to each rational cell $R_i$. First choose $n_1 - 2$ distinct elements $x_1, \ldots, x_{n_1-2}$ of $F^*$ outside $E$. Then choose distinct elements $x_{n_1-1}$ and $x_{n_1}$ outside $E' = E \cup \{x_1, \ldots, x_{n_1-2}\}$ such that $x_1 x_2 \ldots x_{n_1} = \det R_1^2$. This is possible if $|F^*| > 2(|E'| + 2 = 2(|E| + n_1 - 1)$. We have now used $n_1$ distinct elements of $F^*$. Now let $E_1 = E \cup \{x_1, \ldots, x_{n_1}\}$, and choose in the same way distinct elements $x_{n_1+1}, \ldots, x_{n_2}$ of $F^*$ outside $E_1$ such that $x_{n_1+1} \ldots x_{n_2} = \det R_2^2$. This is possible if $|F^*| > 2(|E_1| + k_2 - 1) = 2(|E| + n_2 - 1)$. Continue this process to obtain a sequence $\{x_1, x_2, \ldots, x_n\}$ of distinct elements of $F^*$ outside $E$ with $x_{n_j+1} \ldots x_{n_{j+1}} = \det R_j^2$ for $j = 0, 1, \ldots, m-1$. This is possible if $|F^*| > 2(|E| + n - 1)$. Now let $T_j = \text{diag}[x_{n_{j+1}}, \ldots, x_{n_{j+1}}]$. Applying Lemma 2.4, we get factorizations $T_j = R_j' R_j''$ with $R_j'$ and $R_j''$ each similar to $R_j$. Then $T = T_1 \oplus T_2 \oplus \ldots \oplus T_m = R' R''$, where $R' = R_1' \oplus R_2' \oplus \ldots \oplus R_m'$ and $R'' = R_1'' \oplus R_2'' \oplus \ldots \oplus R_m''$ are both similar to $A$. □

**Theorem 2.6.** If the rational canonical form of $A$ has no scalar direct summand then every matrix $B$ with $\det B = \det A^4$ is of the form $B = A_1 A_2 A_3 A_4$, where $A_i$ is similar to $A$ for $i = 1, 2, 3, 4$.

**Proof.** Use Theorem 1.9 to write $B = LU$, where $L$ is lower-triangular and $U$ is upper-triangular, each with the same spectrum as the operator $T$ of Proposition 2.5.
Thus $L$ and $U$ are both similar to $T$. It then follows from Proposition 2.5 that $B = LU = A_1A_2A_3A_4$, where $A_i$ is similar to $A$ for $i = 1, 2, 3, 4$. □

**Corollary 2.7.** Let $A \in \text{GL}_n(F)$ be such that its rational canonical form has no scalar direct summand, and let $k$ be any natural number. Then every matrix $B \in \text{GL}_n(F)$ with $\det B = \det A^{4k}$ is of the form $B = A_1A_2\ldots A_{4k}$, where $A_i$ is similar to $A$ for $i = 1, 2, \ldots, 4k$. □

**Corollary 2.8.** If the rational canonical form of $A \in \text{SL}_n(F)$ has no scalar direct summand then every matrix $B \in \text{SL}_n(F)$ is of the form $B = A_1A_2A_3A_4$ where $A_i$ is similar to $A$ for $i = 1, 2, 3, 4$.

For a matrix $A \in \text{GL}_n(F)$ whose rational canonical form has a scalar direct summand of size one the bound on the similarity factors depends on the multiplicity of this summand. The ‘worst’ case occurs when that scalar direct summand has multiplicity $n-2$, i.e. when $A$ is diagonalizable with an eigenvalue of multiplicity $n-1$.

**Theorem 2.9.** If the rational canonical form of $A \in \text{GL}_n(F)$ has a scalar direct summand of multiplicity $r - 1 \leq n - 2$ then every non-scalar $T \in \text{GL}_n(F)$ with $\det T = \det A^{4r}$ is of the form $T = A_1A_2\ldots A_{4r}$, where $A_i$ is similar to $A$ for $i = 1, 2, \ldots, 4r$.

*Proof.* Without loss of generality we may assume that the matrix $A$ is in rational canonical form $A = cI_{r-1} \oplus R_1 \oplus \ldots \oplus R_m$, where each rational cell $R_j$ has size at least 2. Apply Proposition 2.5 with $E = \{c^2\}$ to $R_1 \oplus R_2 \oplus \ldots \oplus R_m$ to get a matrix $B = A_1A_2 = c^2I_{r-1} \oplus \text{diag}[d_0, d_1, \ldots, d_{n-r}] = D_0 \oplus \text{diag}[d_1, d_2, \ldots, d_{n-r}] = D_0 \oplus D_1$ so that the entries $c^2, d_0, d_1, \ldots, d_{n-r}$ are all distinct, with $A_1$ and $A_2$ similar to $A$. This is possible if $|F^*| > 2(n - r)$. Then $D_0 = c^2I_{r-1} \oplus d_0$ and $\text{rank}(\frac{1}{c^2}D_0 - I_r) = 1$. Setting $\alpha = (-1)^{r-1}(d_0/c^2)^r$ and applying Lemma 2.1 we conclude that

$$
\begin{pmatrix}
1 & \alpha \\
& \ddots & \ddots \\
& & 1 & 0
\end{pmatrix} = M_1M_2\ldots M_r
$$

with $\det M_i = d_0/c^2 \neq 1$ and $\text{rank}(M_i - I_r) = 1$. Thus $M_i$ is similar to $I_{r-1} \oplus \frac{d_0}{c^2} = \frac{1}{c^2}D_0$. Multiplying by $c^{2r}$ we get the matrix

$$
P = 
\begin{pmatrix}
c^{2r} & c^{2r}\alpha \\
& \ddots & \ddots \\
& & c^{2r} & 0
\end{pmatrix} = P_1P_2\ldots P_r
$$
with $P_i = c^2 M_i$ similar to $D_0$ for $i = 1, 2, \ldots, r$. Moreover, by repeated applications of Theorem 1.7 we can find a diagonal matrix $Q = \text{diag}[q_1, q_2, \ldots, q_{n-r}]$ with distinct diagonal entries, distinct from the eigenvalues of $P$, such that $\det Q = \det D_i^t$ and $Q = Q_1 Q_2 \ldots Q_r$, where $Q_i$ is similar to $D_i$ for $i = 1, 2, \ldots, r$. Thus, $C = P \oplus Q$ is cyclic, $\det C = \det(P) \det(Q) = \det B^r$ and $C = B_1 B_2 \ldots B_r = A_1 A_2 \ldots A_{2r}$, where $B_i = P_i \oplus Q_i$ is similar to $B = D_0 \oplus D_1$ for $i = 1, 2, \ldots, r$ and $A_j$ is similar to $A$ for $j = 1, 2, \ldots, 2r$.

Thus, by Theorem 1.9, every matrix $T \in \text{GL}_n(F)$ with $\det T = \det C^2 = \det B^{2r} = \det A^{4r}$ is of the form

$$T = C_1 C_2 = B_1 B_2 \ldots B_{2r} = A_1 A_2 \ldots A_{4r}$$

with $C_i$ is similar to $C$, $B_j$ is similar to $B$ and $A_k$ is similar to $A$. \qed

**Corollary 2.10.** If $A \in \text{GL}_n(F)$ is not scalar and $s = \text{lcm}(1, 2, \ldots, n - 1)$, then every $T \in \text{GL}_n(F)$ with $\det T = \det A^{4s}$ is of the form $T = A_1 A_2 \ldots A_{4s}$. \qed

3. **Semigroups generated by the similarity orbit of a singular matrix**

We first prove a preliminary result for the similarity semigroup when rank $A = n - 1$ and then apply it to to show that in the general when rank $A < n$ the similarity semigroup of $A$ consists of all matrices of rank less than or equal to rank $A$.

**Proposition 3.1.** The semigroup in $M_n(F)$ generated by the similarity orbit of a matrix $A$ with rank $A = n - 1$ consists of all matrices of rank less than or equal to $n - 1$.

**Proof.** Let $S$ be the semigroup generated by the similarity orbit of the matrix $A$ of rank $n - 1$ in $M_n(F)$. The proof will be in four steps.

Step 1) We first show that $S$ contains a matrix $C = X \oplus 0$ for some invertible $X$ of size $n - 1$. By Fitting’s Lemma, see for example [B], we have $F^n = \text{im} A^m \oplus \ker A^m$ for some natural number $m$, so that we may assume that $A = Y \oplus N$, where $Y$ is invertible and $N$ is nilpotent in Jordan canonical form. Then $B = Y \oplus N^T$ is similar to $A$ and $AB = Y^2 \oplus I \oplus 0 = X \oplus 0$, where $X$ is invertible of size $n - 1$.

Step 2) Next we can prove that $S$ contains a matrix $Y = \lambda I_{n-1} \oplus 0$, where $\lambda \neq 0$ and $I_{n-1}$ is the identity matrix of rank $n - 1$. In the matrix $C = X \oplus 0$ of step 1) the matrix $X$ is invertible and we can get the result by applying Proposition 1.5.

Step 3) Now we show that $S$ contains for each $r = 0, 1, \ldots, n - 1$ a matrix of the form $\lambda I_r \oplus N$, where $N$ is nilpotent of maximal rank $n - r - 1$. This is certainly true for $r = n - 1$ by step 2). If $r = n - 2$ and $Y = \lambda I_{n-1} \oplus 0$ is the matrix obtained in step 2) then

$$\left(\begin{array}{ccc}
\lambda & & \\
& \ddots & \\
& & \lambda \\
\end{array}\right) \left(\begin{array}{ccc}
\lambda & & \\
& \ddots & \\
& & \lambda \\
\end{array}\right) = \left(\begin{array}{ccc}
\lambda^2 & & \\
& \ddots & \\
& & \lambda^2 \\
\lambda^2 & & \\
& \ddots & \\
& & \lambda^2 \\
\end{array}\right),$$
that is
\[ Q^{-1} Y Q S^{-1} Y S = \lambda^2 I_{n-2} \oplus \lambda^2 \begin{pmatrix} \frac{1}{-1} & \frac{1}{-1} \\ \frac{1}{-1} & \frac{1}{-1} \end{pmatrix}, \]
which is similar to
\[ \begin{pmatrix} \lambda^2 I_{n-2} & \lambda^2 \\ 0 & \lambda^2 \\ 0 & 0 \end{pmatrix} = \lambda^2 I_{n-2} \oplus \lambda^2 J_2. \]
Here we used the similarities
\[ Q^{-1} Y Q = Q^{-1} Y = \lambda I_{n-2} \oplus \begin{pmatrix} \lambda & 0 \\ -\lambda & 0 \end{pmatrix} \quad \text{and} \quad S^{-1} Y S = Y S = \lambda I_{n-2} \oplus \begin{pmatrix} \lambda & \lambda \\ 0 & 0 \end{pmatrix}, \]
where the elementary matrix \( Q = E_{n,n-1} \) is obtained from \( I_n \) by adding the \((n-1)\)-th row to the \(n\)-th row and \( S = Q^T \) is the transpose.

Now proceed by backward induction on \( r \) using
\[ \lambda^{2(n-r-1)} \begin{pmatrix} I_r & J_{n-r} \\ J_{n-r} & I_{n-r} \end{pmatrix} \lambda^2 \begin{pmatrix} I_{r-1} & J_2 \\ J_2 & I_{n-r-1} \end{pmatrix} = \lambda^{2(n-r)} \begin{pmatrix} I_{r-1} & 0 \\ 0 & J_{n-r+1} \end{pmatrix} \]
which is the same as
\[ \lambda^{2(n-r-1)} (I_r \oplus J_{n-r}) \lambda^2 (I_{r-1} \oplus J_2 \oplus I_{n-r-1}) \simeq \lambda^{2(n-r)} (I_{r-1} \oplus J_{n-r+1}), \]
or the same as
\[ (I_r \oplus J_{n-r}) (I_{r-1} \oplus J_2 \oplus I_{n-r-1}) = I_{r-1} \oplus J_{n-r+1}, \]
where \( J_s \) is the nilpotent Jordan cell of size \( s \) and rank \( s - 1 \).

Sep 4) Finally we prove that \( \mathcal{S} \) contains every matrix of the form \( Z \oplus 0 \) for every invertible matrix \( Z \) of size \( n - 1 \). By step 3) the big Jordan cell \( J_n \) is in \( \mathcal{S} \) and so are its transpose \( J_n^t \) and all their powers. Moreover \( J_n J_n^t = I_{n-1} \oplus 0 \) is idempotent of rank \( n - 1 \) and \( J_n^k (J_n^T)^k = I_{n-k} \oplus O_k \) is idempotent of rank \( n - k \). Thus \( \mathcal{S} \) contains all idempotents of rank less than or equal to \( n - 1 \). Then
\[ \begin{pmatrix} I_{n-1} & x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I_{n-1} & 0 \\ y^t & 0 \end{pmatrix} = \begin{pmatrix} I_{n-1} + xy^t & 0 \\ 0 & 0 \end{pmatrix} \]
yields the result. This is all we need to proceed with the general case when rank \( A < n \) and the final argument is done in the proof of the next theorem. \( \Box \)
Theorem 3.2. The semigroup $S$ in $M_n(F)$ generated by the similarity orbit of a matrix $A$ of rank $r < n$ consists of all matrices of rank $\leq r$.

Proof. Let rank $A = r = n - u$. The argument used in step 1) of Proposition 3.1 shows that $S$ contains a matrix of the form $X \oplus O_u$ for some invertible matrix $X$ of size $r$. That $S$ contains a matrix $Y = \lambda I_r \oplus O_u$ for some scalar $\lambda \neq 0$ again follows from Proposition 1.5 as in step 2) of Proposition 3.1. As in step 3) of Proposition 3.1 with $n = r + 1$ we show that for each $s = 0, 1, \ldots, r$ the semigroup $S$ contains a matrix of the form $\lambda I_s \oplus N \oplus O_{u-1}$, where $N \simeq J_{r-s+1}$ is nilpotent of maximal rank $r - s$. As in step 4) of Proposition 3.1 it now follows that $S$ contains all matrices of the form $Z \oplus O_u$ for every invertible matrix $Z$ of size $r$.

This shows in particular that $K = J_{r+1} \oplus O_{u-1}$, all its powers and their transposes are in $S$. But then $K^l(K^l)^T = I_{r-l} \oplus O_{n-r-l}$ is in $S$ for $l = 1, 2, \ldots, r$, and hence $S$ contains all idempotents of rank $\leq r$, and hence all matrices of the form $C \oplus O_w$ for invertible $C$ and $u \leq w \leq n$.

Now we want to prove that if $B \in M_n(F)$ and rank($B$) = $v \leq r$ then $B \in S$. By Fitting’s Lemma $B \simeq B_0 \oplus N$, where $B_0$ is invertible of size $s \geq 0$ and $N$ is nilpotent of rank $v - s$. More precisely,

$$B \simeq B_0 \oplus N \simeq B_0 \oplus J_{s_1} \oplus J_{s_2} \oplus \ldots \oplus J_{s_t} \oplus O_w =$$

$$(B_0 \oplus (I_{s_1-1} \oplus 0) \oplus \ldots \oplus (I_{s_t-1} \oplus 0) \oplus O_w)(I_s \oplus J_{s_1} \oplus \ldots \oplus J_{s_t} \oplus O_w)$$

when $N$ is in Jordan form. Since $n = s + s_1 + s_2 + \ldots + s_t + w = v + t + w$ it follows that the number of Jordan cells is $t = n - v - w \leq n - v$. The first factor on the right is similar to $B_0 \oplus I_{v-s} \oplus O_{w+t}$, hence belongs to $S$. The second factor is in the semigroup generated by the similarity orbit of

$$I_v \oplus O_{w+t} \simeq I_s \oplus (I_{s_1-1} \oplus 0) \oplus (I_{s_2-1} \oplus 0) \oplus \ldots \oplus (I_{s_t-1} \oplus 0) \oplus O_w \in S,$$

since $J_{s_j}$ is in the semigroup generated by the similarity orbit of $I_{s_j-1} \oplus 0$ in $M_{s_j}(F)$ for $j = 1, 2, \ldots, t$ by step 3) in the proof of Proposition 3.1. This proves that $B$ is in $S$. \qed

References


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