SATANIC COCIRCULARITIES OF KNOTS
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INTRODUCTION
The problem of classifying and distinguishing various knots lies at the heart of knot theory. Even determining if a given knot is non-trivial (i.e., isotopic to the standard embedding of the circle in the 3-sphere S^3) is known to be an NP-hard problem [1]. Thus, we rely on knot invariants to aid us in deciding if two knots are equivalent (isotopic). A numerical knot invariant assigns to each knot a number in such a manner so that two equivalent knots are assigned the same number.

In the 1990s, Victor Vassiliev developed a series of numerical knot invariants, now known as Vassiliev invariants. These invariants are quite powerful in their ability to distinguish knots, and have generated a great deal of interest in the community of knot theorists. The computation of the invariants is largely algebraic, and until recently, the resulting numerical values were not known to describe any geometric properties of the knots. By examining occurrences of five points on a knot that lie on a common circle, we have established a geometric interpretation of the second Vassiliev invariant τ_2. Our knots will be interchangeably referred to as an embedding f : S^1 → S^3, or as the image K of such an embedding.

CONFIGURATION SPACES
Consider the space consisting of n distinct points on a manifold. Such a space is known as a configuration space. For our purposes, we consider only the configuration spaces of S^1 and S^3.

C_n(S^1) := {(x_1, ..., x_n) ∈ (S^1)^n : x_i ≠ x_j (for all i ≠ j)}

C_n(S^3) := {(t_1, ..., t_n) ∈ (S^3)^n : t_i ≠ t_j and t_1, ..., t_n are in counterclockwise order}

Note the additional requirement that the coordinates of a point in C_n(S^3) must be in counterclockwise order.

Given a knot f : S^1 → S^3, we extend f to the evaluation map C_n(f) : C_n(S^1) → C_n(S^3) that sends (t_1, ..., t_n) to (f(t_1), ..., f(t_n)). The image of this map is the space of n distinct points on the knot K, appearing in the order prescribed by the map f, and is denoted C_n(K).

SATANIC CIRCLES
A point (x_1, ..., x_n) ∈ C_n(S^3) is considered satanic if the following two conditions are met:
1. x_1, ..., x_n lie on a common circle in S^3.
2. The points are arranged in a ‘star-shaped’ pattern. That is, x_i is not adjacent to x_{i+1} on their common circle, with subscripts working modulo n.

Denote the space of all satanic points in C_n(S^3) by S_n. This space is a submanifold of dimension 11 in the manifold C_n(S^3). The space C_n(S^3) is 15-dimensional, and C_n(K) is 5-dimensional. Therefore, we would expect the intersection of C_n(K) with S_n in C_n(S^3) to be a 11 + 5 − 15 = 1-dimensional manifold. Indeed, this is the case for general knots, and in fact, the 1-manifold is a collection of closed curves in C_n(S^3).

Let S_n := S ∩ C_n(K). So S_n consists of points (x_1, ..., x_n) ∈ C_n(S^3) such that each x_i lies on the knot K (in order), and (x_1, ..., x_n) is satanic.

WINDING NUMBERS
Consider the curves f^−1(S) for the knot f that has been used in our previous examples (this knot is referred to as the 5_3 knot). We may project C_5(S^3) down onto C_5(S^1) by simply ‘forgetting’ the last three coordinates. The space C_5(S^1) is diffeomorphic to S^3/(0,1). Below is a projection of the curve f^−1(S) ∈ C_5(S^3) onto C_5(S^1).

BIBLIOGRAPHY