Weak-$L^1$ Estimates and Ergodic Theorems

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Abstract. We prove that for any dynamical system $(X, \Sigma, m, T)$, the maximal operator defined by

$$N^* f(x) = \sup_n \frac{1}{n} \# \{ 1 \leq i : \frac{f(T^i x)}{i} \geq \frac{1}{n} \}$$

is almost everywhere finite for $f$ in the Orlicz class $L \log \log L(X)$, extending a result of Assani [2]. As an application, a weighted return times theorem is also proved.

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1. Introduction

Let $T$ be a measure preserving transformation of a probability space $(X, \Sigma, m)$. We call $(X, \Sigma, m, T)$ a dynamical system. The following return times theorem was proved in [4]:

Theorem 1 (Bourgain). Let $1 \leq p \leq \infty$ and let $1/p + 1/q = 1$. For each dynamical system $(X, \Sigma, m, T)$ and $f \in L^p(X)$, there is a set $X_0 \subset X$ of full measure, such that for any other dynamical system $(Y, \mathcal{F}, \mu, S)$, $g \in L^q(Y)$ and $x \in X_0$, the limit,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n f(T^k x) g(S^k y),$$

exists for $\mu$ a.e. $y$.

One of the most interesting unanswered questions that emerges from this result is whether or not the fact that $f$ and $g$ lie in dual spaces is in general necessary in order to have a positive result. Neither of the existing proofs of Theorem 1 gives any indication on this, since each of them relies on Hölder’s inequality.

On the other hand, if $(gS^k)$ is replaced with a sequence $(\xi_k)$ of independent identically distributed random variables such that $\mathbb{E}(|\xi_1|) < \infty$, then the following

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**Theorem 2** (Jamison, Orey and Pruitt). Let \((a_k)\) be a sequence of positive real numbers and let \(N^* = \sup_n \frac{1}{n} \# \{k : a_k/\sum_{i=1}^{k} a_i \geq 1/n\}\), then the following are equivalent:

1. \(N^* < \infty\);
2. For any i.i.d. sequence of random variables \((\xi_k)\) such that \(\mathbb{E}(|\xi_1|) < \infty\), defining a new sequence \((\Xi_n)\) of random variables by

   \[
   \Xi_n(\omega) = \frac{\sum_{k=1}^{n} a_k \xi_k(\omega)}{\sum_{k=1}^{n} a_k},
   \]

   the sequence \((\Xi_n)\) converges pointwise almost surely.

Motivated by this criterion, Assani [1] introduced the following maximal function: given \(f \in L^1(X)\), consider

\[
N^* f(x) = \sup_n \frac{1}{n} \# \{1 \leq i : \frac{f(T^i x)}{i} \geq \frac{1}{n}\}.
\]

He proved in [2] for \(f \in L \log L(X)\), \(N^* f \in L^1\) and in particular \(N^* f(x) < \infty\) for a.e. \(x\). Based on this and Theorem 2, the following “duality-breaking” version of Theorem 1 follows almost immediately:

**Corollary 3** (Assani). For each dynamical system \((X, \Sigma, m, T)\) and function \(f\) such that \(\int |f| \log^+ |f| dm < \infty\) (i.e. \(f \in L \log L(X)\)), there is a set \(X_0 \subset X\) of full measure, such that for any sequence \((\xi_k)\) of i.i.d. random variables on the probability space \((\Omega, \mathcal{F}, \mu)\) with \(\xi_1 \in L^1(\Omega)\) and any \(x \in X_0\)

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f(T^k x) \xi_k(\omega)
\]

exists for \(\mu\) a.e. \(\omega\).

Moreover in [1] it is proved that if Theorem 1 is true for \(p = q = 1\), then \(N^* f(x)\) must be finite almost everywhere for all \(f \in L^1(X)\). This connection sheds more light on the importance of the operator \(N^*\) and motivates its further study.

In the next section we will prove the finiteness of \(N^*\) for functions in the larger class \(L \log \log L\). Note that while Assani shows that \(N^* f \in L^1\) for \(f \in L \log L\), our result establishes that \(N^* f \in L^{1, \infty}\) for \(f \in L \log \log L\) (i.e. that \(\sup_t \mathbb{E}[x : N^* f(x) > t] < \infty\)) so that while our hypothesis is weaker, so is our conclusion. Note however that since our conclusion implies that \(N^* f(x) < \infty\) for almost every \(x\), it is sufficient to imply a corollary like Corollary 3 in the case where \(f \in L \log \log L\).

In a preprint that appeared at around the time this paper was submitted, Assani, Buczolich, and Mauldin [3] show that there exists an \(f \in L^1(X)\) such that \(N^* f(x) = \infty\) almost everywhere.
2. Main results

Throughout this section we will denote the natural logarithm of $x$ by $\log x$ and the weak-$L^1$ norm of $f$ by

$$\|f\|_{1,\infty} = \sup_{\lambda > 0} \lambda m\{x : |f(x)| > \lambda \}.$$ 

Also, as usual, we denote the ergodic maximal function by $f^*$, where $f^*(x) = \sup_n \frac{1}{n} \sum_{k=1}^{n} f(T^k x)$. The maximal ergodic theorem asserts that $\|f^*\|_{1,\infty} \leq \|f\|_{1}$ for all $f \in L^1(X)$. The following inequality from [7] turns out to be extremely useful to our investigation:

**Lemma 4.** Suppose that for $i = 1, 2, \ldots$, $g_i(x)$ is a nonnegative function on a measure space with $\sum_i \|g_i\|_{1,\infty} \leq 1$. Set $K = \sum_{i=1}^{\infty} \|g_i\|_{1,\infty} \log(1/\|g_i\|_{1,\infty})$, the entropy of the sequence $(\|g_i\|_{1,\infty})$. Then $\sum_{i=1}^{\infty} \|g_i\|_{1,\infty} \leq 2(K + 2)$.

We can now prove our main result.

**Theorem 5.** For each dynamical system $(X, \Sigma, m, T)$ and each $f \in L\log\log L(X)$ (that is $f$ satisfying $\int |f| \log^+ |f| \, dm < \infty$), $N^* f(x) < \infty$ for a.e. $x$.

**Proof.** It is enough to consider $f$ positive. Making use of the fact that $f(x) \leq \sum_{i=1}^{\infty} 2^i \chi_{A_i}(x)$, where $A_i = \{x : 2^{i-1} < f(x) \leq 2^i\}$ for $i \geq 2$ and $A_1 = \{x : f(x) \leq 2\}$, it easily follows that for each $n$,

$$\frac{1}{n} \sum_{1 \leq k \leq n} \frac{1}{k} \leq \frac{1}{n} \sum_{i=1}^{\infty} \sum_{k=1}^{n} \chi_{A_i}(T^k x) \leq \sum_{i=1}^{\infty} 2^i (\chi_{A_i})^*(x).$$

We will show that the last term in the above inequality is finite a.e. by proving that its $L_{1,\infty}$ norm is finite. Let $M = \sum_{i=1}^{\infty} 2^i m(A_i)$ and let $g_i(x) = 2^i (\chi_{A_i})^*(x)/M$. By the maximal ergodic theorem, we see

$$\|g_i\|_{1,\infty} \leq \frac{2^i m(A_i)}{M} = \frac{2^i m(A_i)}{\sum_{i=1}^{\infty} 2^i m(A_i)}$$

so that $\sum_i \|g_i\|_{1,\infty} \leq 1$. Based on Lemma 4, it will be sufficient to prove that $\sum_{i=1}^{\infty} \|g_i\|_{1,\infty} \log(1/\|g_i\|_{1,\infty}) < \infty$. It is quickly seen that this condition is equivalent to establishing that $\sum_{i=1}^{\infty} 2^i m(A_i) \log(1/(2^i m(A_i))) < \infty$.

Here and in the future the summation only runs over the indices $i$ for which $m(A_i) > 0$. Consider now $S_1 = \{i : 2^i m(A_i) \leq 1/i^2\}$ and $S_2 = \{i : 2^i m(A_i) > 1/i^2\}$. Now

$$\sum_{i \in S_1} 2^i \|\chi_{A_i}\|_1 \log \left( \frac{1}{2^i \|\chi_{A_i}\|_1} \right) \leq \sum_{j=2}^{\infty} \log(j^2)/j^2 < \infty$$

since $\psi(t) = t \log (1/t)$ is increasing on $[0, 1/e]$. On the other hand

$$\sum_{i \in S_2} 2^i \|\chi_{A_i}\|_1 \log \left( \frac{1}{2^i \|\chi_{A_i}\|_1} \right) < 2 \sum_{i \in S_2} 2^i \|\chi_{A_i}\|_1 \log i < \infty$$

since $f \in L\log\log L(X)$. This ends the proof. \qed
Corollary 6. For each dynamical system \((X, \Sigma, m, T)\) and non-negative function \(f \in L \log \log L(X)\), there is a set \(X_0 \subset X\) of full measure, such that for any sequence \((\xi_k)\) of i.i.d. random variables on the probability space \((\Omega, \mathcal{F}, \mu)\) with \(\xi_1 \in L^1(\Omega)\) and any \(x \in X_0\)

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f(T^k x) \xi_k(\omega)
\]

exists for \(\mu\) a.e. \(\omega\).

There does not seem to be a better way of exploiting Lemma 4 in order to extend even more the class of functions for which \(N^* f\) is almost everywhere finite. Moreover, as we show in the following proposition, the inequality in Lemma 4 is sharp up to a constant. We note that a more general version of this proposition appears in work of Kalton [6]

Proposition 7. Given positive numbers \(a_1, \ldots, a_n\) which sum to 1, there exist functions \(g_1, \ldots, g_n\) with \(\|g_i\|_{1, \infty} = a_i\) such that \(\|g_1 + \cdots + g_n\|_{1, \infty} \geq \frac{1}{8} (2 + K) \sum \|g_i\|_{1, \infty}\), where \(K\) is the entropy of the sequence \((a_i)\).

Proof. For each \(i\), let \(\xi_i\) be a random variable taking the value \(1/n\) with probability \((1 - a_i)^{n-1} a_i\). Moreover, the \(\xi_i\)'s will be chosen to be independent. One can then check that \(P(\xi_i > \lambda) \leq a_i/\lambda\) while \(P(\xi_i \geq 1 - \epsilon) = a_i\) for \(\epsilon\) small enough, so that \(\|\xi_i\|_{1, \infty} = a_i\).

We see that

\[
E(\xi_i) = \sum_{n=1}^{\infty} a_i \frac{(1 - a_i)^{n-1}}{n} = -\frac{a_i}{1 - a_i} \log a_i \geq -a_i \log a_i.
\]

Similarly, we see that

\[
E(\xi_i^2) = a_i \sum_{n=1}^{\infty} \frac{1}{n^2} (1 - a_i)^{n-1} \leq a_i \sum_{n=1}^{\infty} \frac{1}{n^2} \leq 2a_i.
\]

In particular, setting \(\Xi = \xi_1 + \cdots + \xi_n\), we see that \(E(\Xi) \geq K\) but \(\text{Var}(\Xi) \leq 2\).

Using Tchebychev’s inequality, we see that

\[
P(\Xi \geq K - 2) \geq P(|\Xi - E(\Xi)| \leq 2) \geq 1 - \frac{\text{Var}(\Xi)}{2^2} \geq \frac{1}{2}.
\]

If \(K > 4\), we have \(P(\Xi \geq K/2) \geq \frac{1}{2}\) so that the weak-\(L^1\) norm exceeds \(K/4\), which in turn exceeds \((K + 2)/6\). If \(K \leq 4\), take \(f\) to be any function of weak \(L^1\) norm 1 and let \(f_n = a_n f\), so that \(\|f_n\|_{1, \infty} = a_n\). Then \(\sum f_i = f\), so that \(\|\sum f_i\|_{1, \infty} = 1 \geq \frac{1}{8} (K + 2) \sum \|f_i\|_{1, \infty}\). This completes the proof of the proposition. \(\square\)

Remark 8. Note that although \(f \in L \log \log L\) is sufficient to guarantee that \(N^* f < \infty\) almost everywhere, there are functions \(f\) outside \(L \log \log L(X)\), for which \(N^* f(x) < \infty\) for a.e. \(x\). In particular, it is easy to construct functions outside \(L \log \log L\) for which the entropy computed in Theorem 5.

Further, if we are willing to restrict the system, we see that no condition on the distribution of \(f\) can guarantee the divergence of \(N^* f(x)\). Specifically, Lemma 1 of [1] guarantees that whenever \(T^k f\) are independent random variables (take for
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example $(X, \Sigma, m, T)$ to be a Bernoulli shift and $f$ to depend only on the first coordinate), then $N^* f(x) < \infty$ for a.e. $x$.

Another consequence of Theorem 5 is the following weighted version of Theorem 3.

**Theorem 9.** For each dynamical system $(X, \Sigma, m, T)$ and $f \in L^1(X)$, there is a set $X_0 \subset X$ of full measure, such that for any sequence $(\xi_k)$ of i.i.d. random variables on the probability space $(\Omega, \mathcal{F}, \mu)$ with $\xi_1 \in L^1(\Omega)$ and any $x \in X_0$

$$\lim_{n \to \infty} \frac{1}{n \log n} \sum_{k=1}^{n} f(T^k x) \xi_k(\omega) = 0$$

for $\mu$ a.e. $\omega$.

The proof will be based on the following relative of Theorem 5. Define $L^* f(x) = \sup_n \frac{1}{n} \# \{1 \leq i : \frac{f(T^i x)}{i \log i} \geq \frac{1}{n}\}$

**Lemma 10.** For each dynamical system $(X, \Sigma, m, T)$ and each $f \in L^1(X)$, $L^* f(x) < \infty$ for a.e. $x$.

**Proof.** As usual, we can assume $f$ is positive. Fix an $n \in \mathbb{N}$. Using the fact that $f(x) \leq \sum_{i=1}^{\infty} 2^i \chi_{A_i}(x)$ we get that

$$\frac{1}{n} \# \{1 \leq k : \frac{f(T^k x)}{k \log k} \geq 1/n\} \leq \frac{1}{n} \sum_{i=1}^{\infty} \sum_{k=1}^{p_i(T^k x)} \chi_{A_i}(T^k x) \leq \frac{1}{n} \sum_{i=1}^{\infty} p_i(\chi_{A_i})^*(x)$$

where $p_i$ is the largest integer such that $p_i \log \log p_i \leq 2^i$. Letting $\phi : (1, \infty) \to \mathbb{R}$ be the increasing function $\phi(x) = x \log x$, we see that $p_i \leq \phi^{-1}(2^i)$. We claim that there exists a $C > 0$ such that $p_i \leq C \frac{2^{i+1}}{\log(i+1)}$ for all $i, n \in \mathbb{N}$. To see this, we check the existence of a $C$ such that $\phi^{-1}(2^i) \leq C \frac{2^{i+1}}{\log(i+1)}$ or equivalently $2^i \leq \phi(\frac{2^{i+1}}{\log(i+1)})$ for all $x \geq 0$. Hence

$$\sup_n \frac{1}{n} \# \{1 \leq i : \frac{f(T^i x)}{i \log i} \geq \frac{1}{n}\} \leq C \sum_{i=1}^{\infty} \left( \frac{2^i}{\log(i+1)} \right) (\chi_{A_i})^*(x)$$

Based on Lemma 4 and on the maximal ergodic theorem, it suffices to prove that

$$\sum_{i=1}^{\infty} \left( \frac{2^i}{\log(i+1)} \right) (\chi_{A_i})^*(x) < \infty.$$ By splitting the sum in two parts depending on whether or not $2^i \leq \phi(\frac{2^{i+1}}{\log(i+1)}) \leq \frac{1}{n}$ and reasoning like in the proof Theorem 5, it easily follows that the sum from above is finite.

**Proof of Theorem 9.** It suffices to assume that both $f$ and $\xi_1$ are positive. According to the previous lemma, let $X_0$ the subset of full measure of $X$ containing all the points $x$ for which $L^* f(x) < \infty$. For a fixed $x \in X_0$ denote $w_k := f(T^k x)$ and also $W_k := k \log k$. The argument of Jamison, Orey and Pruitt from [5] can be extended with really no essential changes to this case, to conclude that since

$$\sup_n \frac{1}{n} \# \{1 \leq i : \frac{w_i}{W_i} \geq \frac{1}{n}\} < \infty,$$

$$\lim_{n \to \infty} \frac{1}{W_n} \sum_{k=1}^{n} w_k \xi_k(\omega) = 0$$
for $\mu$ a.e. $\omega$. □

Remark 11. It is not known whether in Theorem 9 the weight $n \log \log n$ can be replaced with a smaller one, like $n \log \log \log n$, much less with $n$. Any improvement on this weight will necessarily have behind it an extension of the result of Theorem 5 to a larger Orlicz class.

Remark 12. It would be interesting to find the largest Orlicz class that would guarantee that $N^* f(x) < \infty$ almost everywhere. The above establishes that such an Orlicz class would contain $L \log \log L$ and the recent preprint

A careful examination of the proof of [3] demonstrates that in any Orlicz class with an essentially smaller weight than the class $L \log \log \log L$, there exists a function $f$ such that $N^* f(x) = \infty$ almost everywhere.

In particular, these two results demonstrate that the largest Orlicz class that would guarantee that $N^* f(x) < \infty$ almost everywhere lies between $L \log \log L$ and $L \log \log \log L$.

References


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