Splitting factor maps into u- and s-bijective maps.

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1st example-Shifts of finite type

Let G be a finite directed graph which consists of a vertex set G^0, an edge set G^1, and two maps $r, s : G^1 \to G^0$. The source vertex of edge e is given by $s(e)$ and the range vertex is given by $r(e)$.

Definition

We define

$$\Sigma_G = \{(x_n)_{n\in\mathbb{Z}} \mid x_n \in G^1,\ r(x_n) = s(x_{n+1}) \ \text{ for all } n \text{ in } \mathbb{Z}\}$$

With the left shift map $\sigma : \Sigma_G \to \Sigma_G$,

$$\sigma(x)_n = x_{n+1}.$$

2nd example: Hyperbolic toral automorphism

Let $\hat{A} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

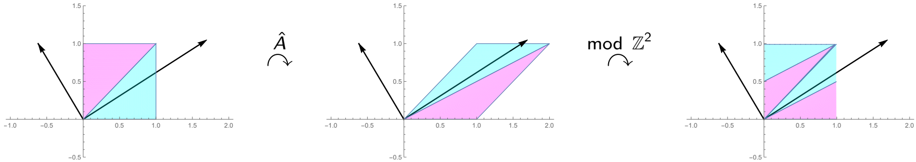
Define $A : \mathbb{T}^2 \to \mathbb{T}^2$ by

$$A([x]) = [\hat{A}x]$$

where x is in $\mathbb{R}^2$ and $[x]$ denotes its equivalence class in $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. By the integer components and the determinant, A is an invertible map.

Eigenvalues : γ and $-\gamma^{-1}$, where $\gamma = \frac{1+\sqrt{5}}{2} > 1$.

Eigenvectors: $v_u = \begin{bmatrix} \gamma \\ 1 \end{bmatrix}$ and $v_s = \begin{bmatrix} -\gamma^{-1} \\ 1 \end{bmatrix}$.



Notice $\mathbb{R}^2 = \{tv_u \mid t \in \mathbb{R}\} \oplus \{tv_s \mid t \in \mathbb{R}\} = E^u \oplus E^s$

For general $\hat{A}$ in $GL_d(\mathbb{R})$ we define,

$$E^s = \{x \in \mathbb{R}^d \mid \|\hat{A}^n x\| \to 0,\ n \to +\infty\}$$
$$E^u = \{x \in \mathbb{R}^d \mid \|\hat{A}^n x\| \to 0,\ n \to -\infty\}$$

Definition

We say a matrix $\hat{A}$ is hyperbolic if $\hat{A}$ is in $GL_d(\mathbb{R})$ and,

$$\mathbb{R}^d = E^s \oplus E^u.$$

With these in mind, the A from our example, is a hyperbolic toral automorphism.

Globally: Let f be a homeomorphism.

Definition

We say two points x, y in X are stably equivalent and write $x \overset{s}{\sim} y$ if

$$\lim_{n \to +\infty} d(f^n(x), f^n(y)) = 0$$

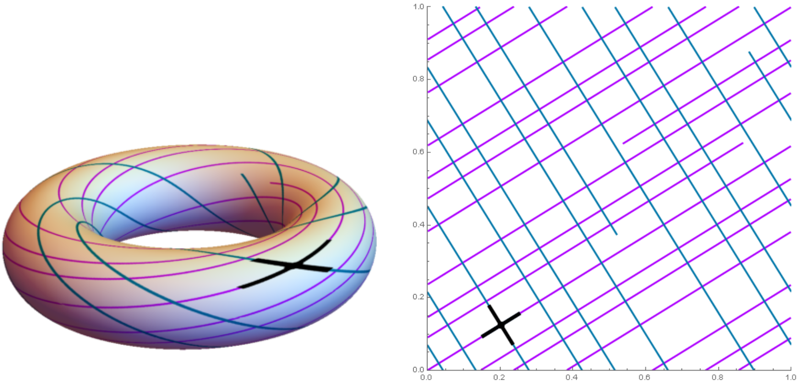
We let $X^s(x)$, the set of y with $x \overset{s}{\sim} y$.

We say that x, y are unstably equivalent and write $x \overset{u}{\sim} y$ if

$$\lim_{n \to -\infty} d(f^n(x), f^n(y)) = 0$$

We let $X^u(x)$ be the set of y with $x \overset{u}{\sim} y$.

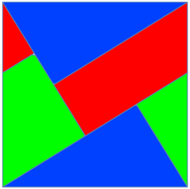
On a hyperbolic toral automorphism the global unstable and stable sets wrap around densely.

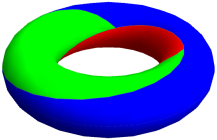


The local stable and unstable sets are given by moving a little bit along the eigendirections. Locally, $\mathbb{T}^2$ can be viewed as $\mathbb{R} \times \mathbb{R}$.

The HTA can be modeled using symbolic dynamics by way of Markov partitions, where $\pi : (\Sigma_G, \sigma) \to (\mathbb{T}^n, A)$ is a finite-to-one factor map.

- Adler and Weiss 1967 for the case of dimension $d = 2$.
- Sinai 1968 any finite dimension d.
- Bowen 1970, for basic sets of Axiom A diffeomorphisms.

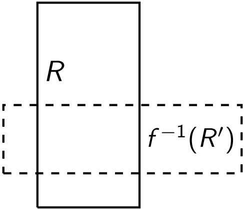




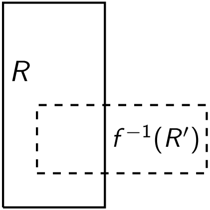
Markov Property

When $int(R) \cap f^{-1}(int(R'))$ is non-empty, then for all x in R and y in $f^{-1}(R')$, $[y, x]$ is defined and we have,

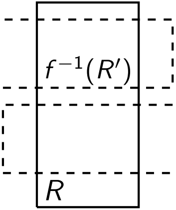
$$[f^{-1}(R'), R] = f^{-1}(R') \cap R.$$



Good



Bad covering



Bad covering

Definition

A factor map π, has a splitting, if it is a composition of a u and s-bijective map.

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$$\begin{tikzcd}
 & (\Sigma_G, \sigma) \arrow[dl, dashed, "\mu_s"'] \arrow[dd, "\pi"] \\
(\Omega, \omega) \arrow[dr, dashed, "\rho_u"'] & \\
 & (\mathbb{T}^d, A)
\end{tikzcd}$$

Definition

We say that $\pi : (X, f) \to (Y, g)$ is s-bijective if, for any x in X, its restriction to $X^s(x)$ is a bijection to $Y^s(\pi(x))$.

Theorem

Let $\pi : (X, f) \to (Y, g)$ be an s-bijective map. Then for every x in X, the map $\pi : X^s(x, \epsilon) \to Y^s(\pi(x), \epsilon')$ is a local homeomorphism.

A u-bijective map is defined and characterized analogously.

Given $(\mathbb{T}^d, A)$, we can find a factor map π.

$$\begin{tikzcd}
(\Sigma_G, \sigma) \arrow[d, "\pi"] \\
(\mathbb{T}^d, A)
\end{tikzcd}$$

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$$\begin{tikzcd} (\Sigma_G, \sigma) \arrow[d, "\pi"] \\ (\mathbb{T}^d, A) \end{tikzcd} \quad \text{locally represented as,} \quad \begin{tikzcd} \text{Cantor} \times \text{Cantor} \arrow[d, "\pi"] \\ \mathbb{R}^m \times \mathbb{R}^n \cong E^s \times E^u \end{tikzcd}$$

where $m + n = d$.

Note: This map cannot be s-bijective nor u-bijective.

Suppose we also have μ_s, an s-bijective map and ρ_u, a u-bijective map such that,

$$\begin{tikzcd} & (\Sigma_G, \sigma) \arrow[dl, "\mu_s"'] \arrow[dd, "\pi"] \\ (\Omega, \omega) \arrow[dr, "\rho_u"'] & \\ & (\mathbb{T}^d, A) \end{tikzcd}$$

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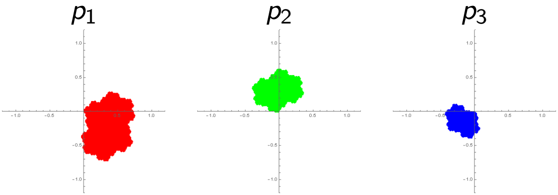
\qquad
\begin{tikzcd}
 & \text{Cantor} \times \text{Cantor} \arrow[dl, "\mu_s"'] \arrow[dd, "\pi"] \\
??? \arrow[dr, "\rho_u"'] & \\
 & \mathbb{R}^m \times \mathbb{R}^n \cong E^s \times E^u
\end{tikzcd}$$

What must ??? look like locally?
What is a candidate space for ??? ?

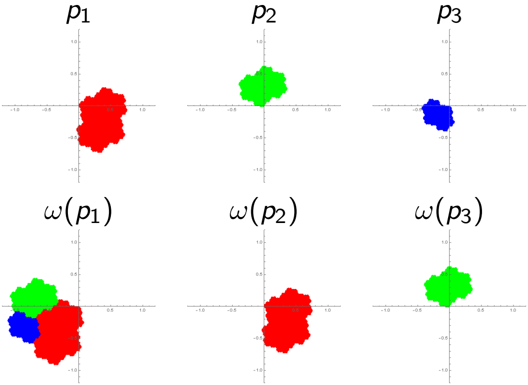
3rd Example: Substitution tiling systems, $(\Omega, \mathcal{P}, \omega)$

Prototiles, $\mathcal{P} = \{p_1, p_2 \ldots, p_n\}$. Each $p_i \subseteq \mathbb{R}^d$ is the closure of its interior.

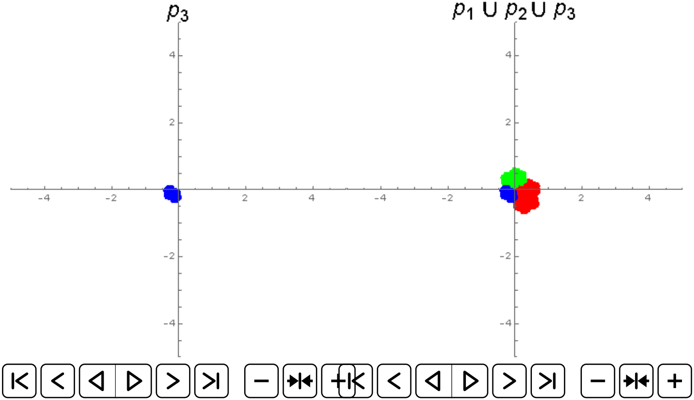


A tile t is a translation of some prototile.

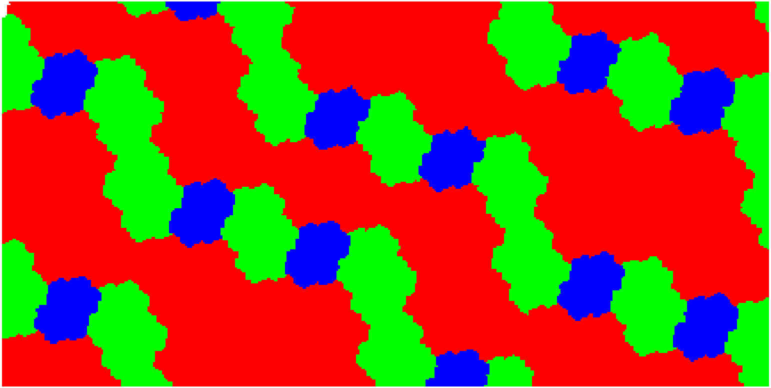
A substitution rule $\omega(p_i)$ that inflates, possibly rotates and subdivides with translates of prototiles.



A partial tiling is a collection of tiles whose interiors are pairwise disjoint. A tiling is a partial tiling whose union is $\mathbb{R}^d$.
The substitution can be iterated and extended to all tilings.



We define Ω to be the set of tilings T such that if $P \subseteq T$ then $P \subseteq \omega^k(t)$ for some tile t.

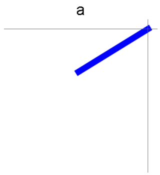
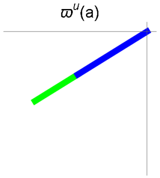
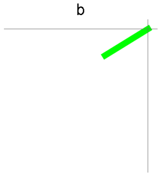
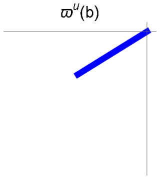


Forcing the border

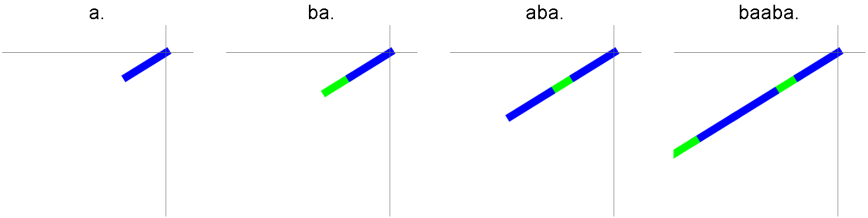
Definition

A tiling system $(\Omega, \mathcal{P}, \omega)$ forces its border if there is a $k \geq 1$ such that, if T and T' are two tilings containing a tile t, then the patches in $\omega^k(T)$ and $\omega^k(T')$ consisting of all tiles which meet $\omega^k(t)$ are identical.

Tiling example 2-Fibonacci

$\omega^u : a. \mapsto ba.$		
$\omega^u : b. \mapsto a.$		

ω^u: Does not force border ($a \mapsto ba$ and $b \mapsto a$)



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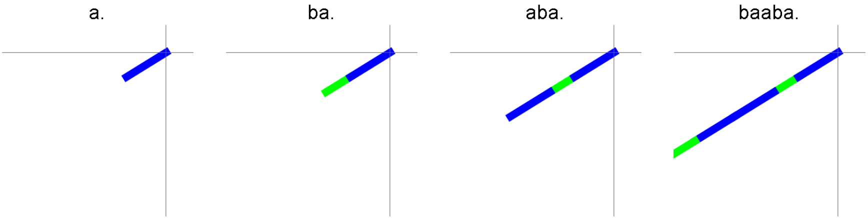

Extend on the right by a
$\dots baba a\boxed{\text{b}}a.\boxed{\text{a}} \dots$

Extend on the right by b
$\dots baba a\boxed{\text{b}}a.\boxed{\text{b}} \dots$

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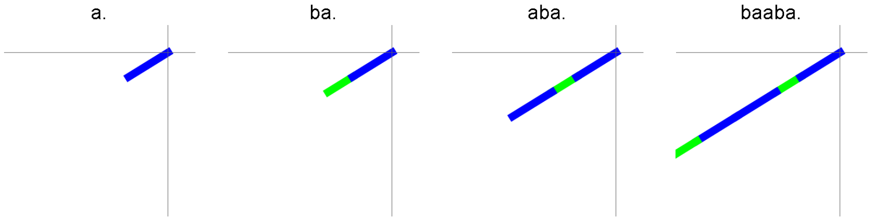

	Extend on the right by a	Extend on the right by b
	$\dots babaa\boxed{\mathrm{b}}a.\boxed{\mathrm{a}}\ \dots$	$\dots babaa\boxed{\mathrm{b}}a.\boxed{\mathrm{b}}\dots$
ω^u	$\dots baba\boxed{\mathrm{a}}ba.\boxed{\mathrm{b}}a\dots$	$\dots baba\boxed{\mathrm{a}}ba.\boxed{\mathrm{a}}\dots$

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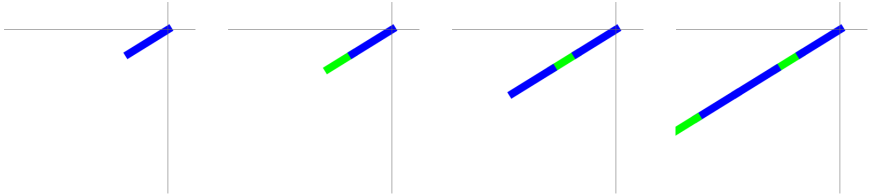

	Extend on the right by a	Extend on the right by b
	$\dots babaa\boxed{\mathrm{b}}a.\boxed{\mathrm{a}}\ \dots$	$\dots babaa\boxed{\mathrm{b}}a.\boxed{\mathrm{b}}\dots$
ω^u	$\dots baba\boxed{\mathrm{a}}ba.\boxed{\mathrm{b}}a\dots$	$\dots baba\boxed{\mathrm{a}}ba.\boxed{\mathrm{a}}\dots$
$(\omega^u)^2$	$\dots bab\boxed{\mathrm{a}}aba.\boxed{\mathrm{a}}ba\dots$	$\dots bab\boxed{\mathrm{a}}aba.\boxed{\mathrm{b}}a\dots$

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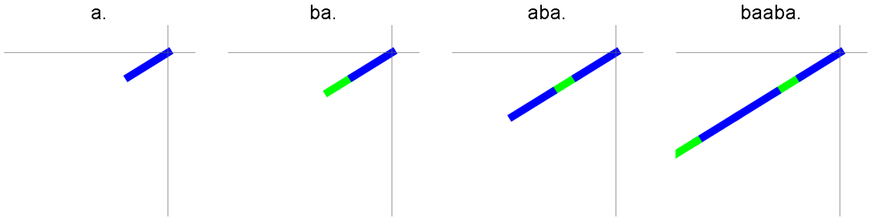

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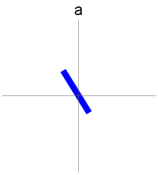
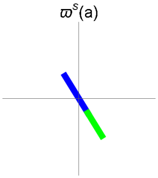
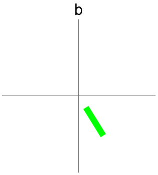
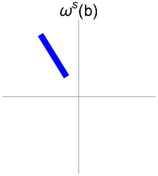
$(\omega^u)^2$	$\dots bab\boxed{\mathrm{a}}aba.\boxed{\mathrm{a}}ba\dots$	$\dots bab\boxed{\mathrm{a}}aba.\boxed{\mathrm{b}}a\dots$
$(\omega^u)^3$	$\dots b\boxed{\mathrm{a}}baaba.\boxed{\mathrm{b}}aba\dots$	$\dots b\boxed{\mathrm{a}}baaba.\boxed{\mathrm{a}}ba\dots$

Tiling example 3: Also Fibonacci

$\omega^s : a \mapsto (-1)b.a$		
$\omega^s : .b \mapsto (-1).a$		

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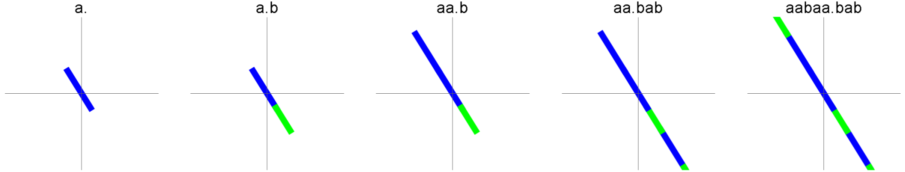

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\end{tikzcd}$$

Is there a necessary and sufficient condition for a given factor map, π, to have a splitting?

Theorem

If a splitting for the factor map $\pi : \Sigma \to \mathbb{T}^d$ exists then,

for every x in $\mathbb{R}^d$ for which $q(x)$ is periodic in $\mathbb{T}^d$, there exist open sets $U \subseteq E^s$ and $V \subseteq E^u$ containing 0, with the property that for all y and $\overline{y}$ in $V \setminus (\partial^u \mathcal{R} - x)$,

$$U \cap (\partial^s \mathcal{R} - y - x) = U \cap (\partial^s \mathcal{R} - \overline{y} - x).$$

To understand the condition, let us first define $\mathcal{R}$.

Constructing $\mathcal{R}$

Let $\nu : G^1 \to \mathbb{Z}^d$ be a labelling of the edges of a finite graph G.
Let $\nu(e)^s$ be the projection onto E^s through E^u.

Define $\pi^s : \Sigma_G \to E^s$ by,

$$\pi^s(x) = \sum_{n \leq 0} \hat{A}^{-n} \nu(x_n)^s$$

Define $\pi^u : \Sigma_G \to E^u$ by,

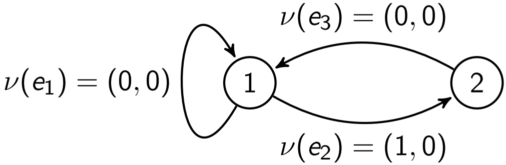
$$\pi^u(x) = -\sum_{n \geq 1} \hat{A}^{-n} \nu(x_n)^u$$

and $\pi' : \Sigma_G \to \mathbb{R}^d$ by,

$$\pi'(x) = \pi^s(x) + \pi^u(x) \in \mathbb{R}^d$$

Let G_{fib} be the following finite directed graph with labelling map,

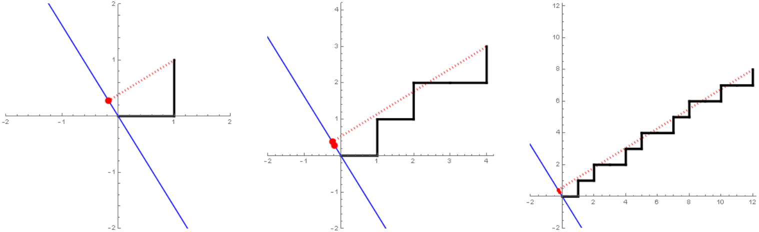
$$\nu : G^1_{\text{fib}} \to \mathbb{Z}^2.$$



Suppose we take $x = \dots e_2 e_3 e_2 e_3 e_2 e_3 . e_2 e_3 e_2 e_3 e_2 e_3 \dots$ from $\Sigma_{G_{\text{fib}}}$.

Visualizing the map π^s

$$\pi^s(x) = \sum_{n\leq 0} A^{-n}\nu(x_n)^s = \lim_{k\to\infty}\left(\sum_{n=0}^{-k} A^{-n}\nu(x_n)\right)^s$$



k=1 k=3 k=5

Markov partition

Definition

Let $\mathcal{R}_{G,\nu}$ be defined by the following sets, for $i \in G^0$,

$$R_i^s = \pi^s\{x \in \Sigma_G \mid r(x_0) = i\} \subseteq E^s$$
$$R_i^u = \pi^u\{x \in \Sigma_G \mid r(x_0) = i\} \subseteq E^u$$

and

$$\mathcal{R}_{G,\nu} = \{R_i^s + R_i^u \mid i \in G^0\}$$

$$R_1^s = \pi^s\{x \in \Sigma_{G_{\text{fib}}} \mid r(x_0) = 1\}$$
$$= \left\{ \sum_{n\geq 0} a_n(-\gamma)^{-n}(1,0)^s \;\middle|\; \begin{array}{r} a_n \in \{0,1\} \\ a_n a_{n+1} = 0 \\ a_0 = 0 \end{array} \right\}$$
$$= \text{blue set}$$

$$R_2^s = \pi^s\{x \in \Sigma_{G_{\text{fib}}} \mid r(x_0) = 2\}$$
$$= \left\{ \sum_{n\geq 0} a_n(-\gamma)^{-n}(1,0)^s \;\middle|\; \begin{array}{r} a_n \in \{0,1\} \\ a_n a_{n+1} = 0 \\ a_0 = 1 \end{array} \right\}$$
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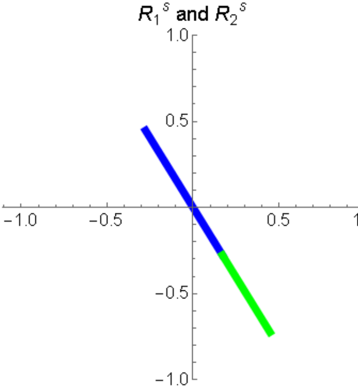

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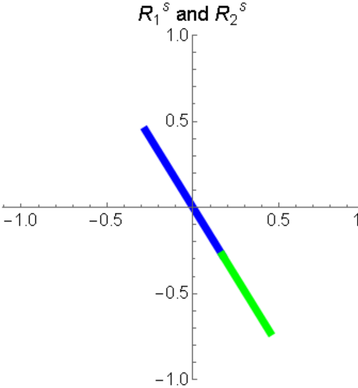

$$R_1^u = \pi^u\{x \in \Sigma_{G_{\text{fib}}} \mid r(x_0) = 1\}$$
$$= \left\{ -\sum_{n\geq 1} a_n(\gamma)^n(1,0)^u \;\middle|\; \begin{array}{r} a_n \in \{0,1\} \\ a_n a_{n+1} = 0 \end{array} \right\}$$
$$= \text{blue set}$$

$$R_2^u = \pi^u\{x \in \Sigma_{G_{\text{fib}}} \mid r(x_0) = 2\}$$
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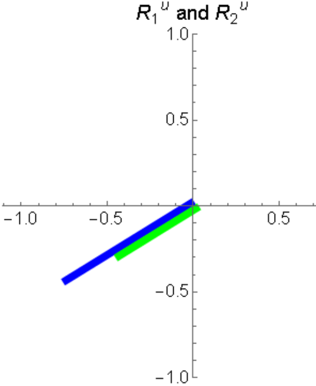
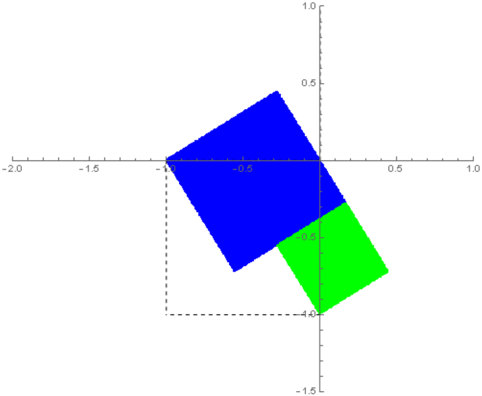
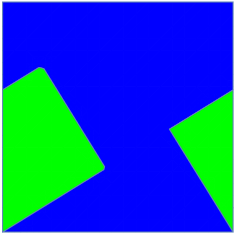


$R_1^s + R_1^u,\ R_2^s + R_2^u$

$q : \mathbb{R}^2 \to \mathbb{T}^2 \bmod \mathbb{Z}^2$ map.



$\mathcal{R}$



$q(\mathcal{R})$

Theorem

If $\mathcal{R}$ is regular, the collection $\{R_i + m \mid 1 \leq i \leq l, m \in \mathbb{Z}^d\}$ are pairwise disjoint and tile $\mathbb{R}^d$ then,

1. *The map $\pi = q \circ \pi' : \Sigma_G \to \mathbb{T}^d$ is a finite -to-one factor map.*
2. *$q(\mathcal{R})$ is a Markov partition.*
3. *There is a dense G_δ in $\mathbb{T}^d$, B, such that if x is in B then $\#\pi^{-1}\{x\} = 1$.*

Graph Iterated Function system property

Theorem

The collection of sets $\mathcal{R}_{(G,\nu)}$ satisfies the following equations,

$$AR_i^u = \bigcup_{s(e)=i} R_{r(e)}^u - \nu(e)^u,$$

$$R_j^s = \bigcup_{r(e)=j} AR_{r(e)}^s + \nu(e)^s,$$

for $1 \leq i, j \leq I$.

Theorem

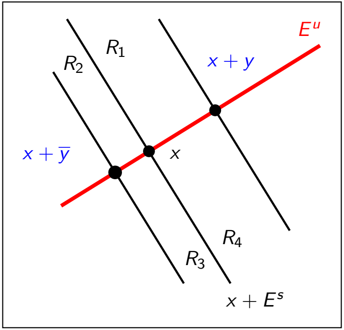
If a splitting for the factor map $\pi : \Sigma \to \mathbb{T}^d$ exists then,

for every x in $\mathbb{R}^d$ for which $q(x)$ is periodic in $\mathbb{T}^d$, there exist open sets $U \subseteq E^s$ and $V \subseteq E^u$ containing 0, with the property that for all y and $\overline{y}$ in $V \setminus (\partial^u \mathcal{R} - x)$,

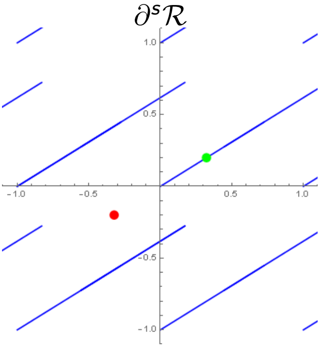
$$U \cap (\partial^s \mathcal{R} - y - x) = U \cap (\partial^s \mathcal{R} - \overline{y} - x).$$

The stable boundaries around a periodic point (for the map A) should look the same in the E^u direction.

The condition is satisfied if the boundary around a periodic point looks something like this...

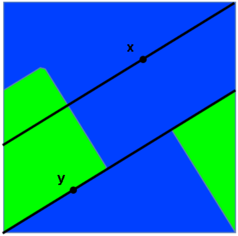


Our G_{fib} example does not satisfy the condition



Idea for proof

Suppose the condition fails.



Choose x and y unstably equivalent and stably equivalent to a periodic point, where x has one preimage under π while y has two preimages under π. Contradicts properties of u and s-bijective maps. No splitting for the map $\pi : \Sigma_{G_{\text{fib}}} \to \mathbb{T}^2$ exists.

Does there exist another SFT for which the factor map splits?

Theorem (Putnam,2005)

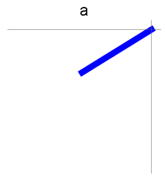
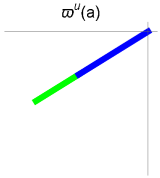
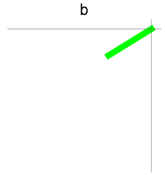
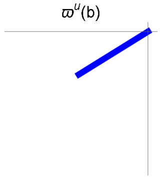
Let (Y, g) be an irreducible Smale space. Then there exists a shift of finite type (Σ, σ), another irreducible Smale space (Ω, ω), and

$$\mu : (\Sigma, \sigma) \to (\Omega, \omega)$$
$$\rho : (\Omega, \omega) \to (Y, g)$$

factor maps, such that μ is s-bijective and ρ is u-bijective.

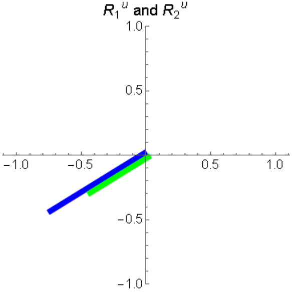
How do we find it, explicitely?

The GIFS for our G_{fib} Markov partition gives a tiling substitution.

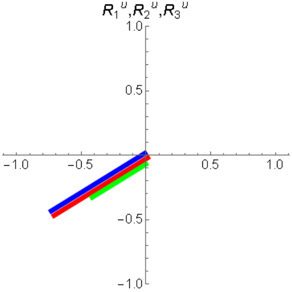
		$AR_1^u = R_1^u \cup R_2^u - (1,0)^u$
		$AR_2^u = R_1^u$

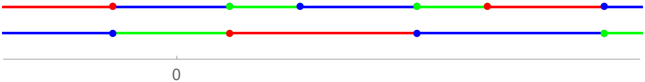
The collared tiling system $(\Omega_1, \mathcal{P}_1, \omega_1)$ forces its border.

Original prototiles



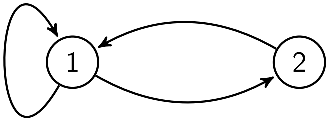
Collared prototiles



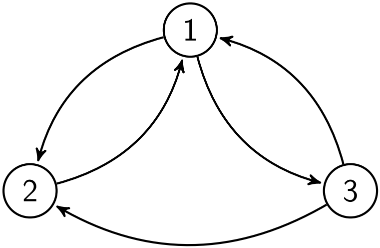


Non-conjugate shifts of finite type

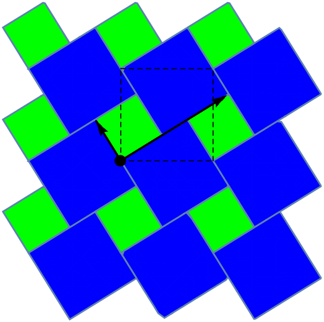
$(\Sigma_{G_{\text{fib}}}, \sigma)$ (old)



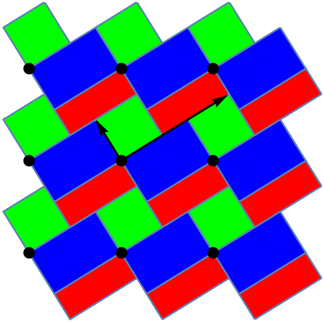
$(\Sigma_{G_{\text{fib1}}}, \sigma)$ (new)



New Markov partition.



$\Longrightarrow$



From Anderson and Putnam 1998 and Wieler 2005.

$$\begin{tikzcd}
 & (\Sigma_{G_{\text{fib1}}}, \sigma) \arrow[dl, "\mu_s"'] \arrow[dd, "\pi"] \arrow[dr, "\mu_u"] & \\
(\Omega_1, \omega_1) \arrow[dr, "\rho_u"] & & (\Omega', \omega') \arrow[dl, "\rho_s"] \\
 & (\mathbb{T}^2, A) &
\end{tikzcd}$$

(Ω_1, ω_1)	collared fibonacci (tiling example 2)
(Ω', ω')	collared fibonacci substitution (tiling example 3)

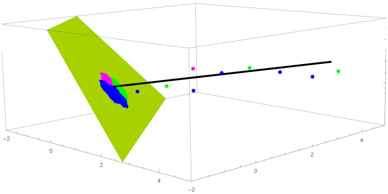
A three dimensional example

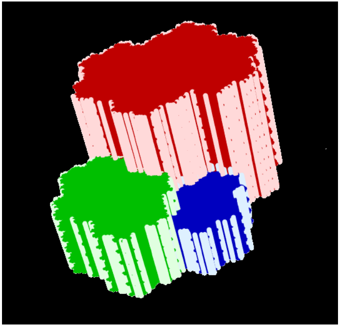
Let $\hat{B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. The induced map B defines an HTA of $\mathbb{T}^3$.

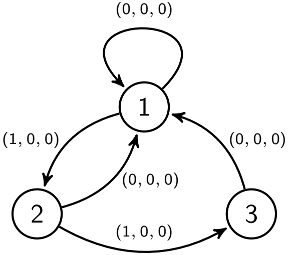
Eigenvalues: $\beta > 1$, $\alpha, \overline{\alpha}$, where $\beta^3 - \beta^2 - \beta - 1 = 0$.

Expanding line and contracting plane.

The Markov partition is given by the following (viewed in $\mathbb{R}^3$).







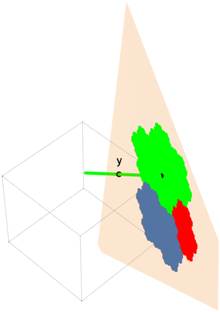
Unstable (Tribonacci)

$$AR_1^u = R_1^u \cup R_2^u - (1,0,0)^s$$
$$AR_2^u = R_1^u \cup R_3^u - (1,0,0)^s$$
$$AR_3^u = R_1^u$$

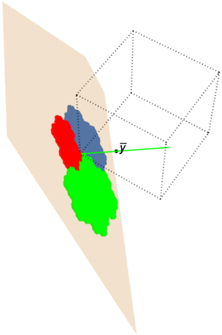
Stable (Rauzy)

$$R_1^s = AR_1^s \cup AR_2^s \cup AR_3^s$$
$$R_2^s = AR_1^s + (1,0,0)^s$$
$$R_3^s = AR_2^s + (1,0,0)^s$$

Interior point



Boundary point



No splitting for the factor map exists.

$$\begin{tikzcd}
 & (\Sigma_{G_{\text{new}}}, \sigma) \arrow[dl, "\mu_s"'] \arrow[dd, "\pi_{\text{new}}"] \arrow[dr, "\mu_u"] & \\
(\Omega_1, \omega_1) \arrow[dr, "\rho_u"] & & (\Omega', \omega') \arrow[dl, "\rho_s"] \\
 & (\mathbb{T}^2, A) &
\end{tikzcd}$$

(Ω_1, ω_1)	collared tribonacci substitution
(Ω', ω')	collared Rauzy substitution (tiling example 1)

We know:

- Existence of splitting for $\pi \implies$ condition on boundaries of MP.
- Forcing the border of $(\Omega, \omega) \implies \exists$ a map π that splits.
- We have an example of a factor map that splits, but the correspdonding tiling system (Ω, ω) does not force its border.

Questions:

- Does the condition being satisfied imply the existence of a splitting?
- If we randomly label a graph of the SFT what sort of sets in $\mathbb{R}^d$ are possible? Under which conditions?
- What does all of this have to do with Ian's homology theory for Smale spaces?

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Sinai, Ya. G. (1968). "Construction of Markov partitions". In: *Functional Analysis and Its Applications* 2.3, pp. 245–253. ISSN: 1573-8485. DOI: 10.1007/BF01076126. URL: https://doi.org/10.1007/BF01076126.

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Thank you for your attention!