A Survey of Permutation Codes

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Permutation codes
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Classical codes and error-correction
Permutation codes
Motivation
Some bounds
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Analysis of a challenging case with algebraic combinatorics
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**Example.** \( \{000000, 000111, 111000, 111111\} \) is a binary code of length \( n = 6 \) and minimum distance \( d = 3 \).

Some nice algebraic constructions exist; for instance, the ideal

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\langle x^3 + x + 1 \rangle \subset \mathbb{F}_2[x]/\langle x^7 - 1 \rangle
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leads to a code with \( n = 7 \), \( d = 3 \), and \( |C| = 16 \).
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Hamming distance

The function $d_H : \{0, 1\}^n \times \{0, 1\}^n \rightarrow \mathbb{Z}_{\geq 0}$ defined by

$$d_H(x, y) = |\{i : x_i \neq y_i\}|$$

is a metric on $\{0, 1\}^n$ called Hamming distance.

Binary codes are sets in $\{0, 1\}^n$ which are well-separated under $d_H$.

Applications: data compression, error-correction, and the “prisoner’s hat problem”.
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Consider the code $C_7$ containing 0000000, 1101000, and closed under cyclic shifts and complements. This is the code coming from the ideal $\langle x^3 + x + 1 \rangle$ mentioned earlier.

The minimum Hamming distance of $C_7$ is 3. We have

$$|C_7| = 16 = \frac{2^7}{\binom{7}{0} + \binom{7}{1}}.$$ 

So every binary word is either in $C_7$, or within one bit of a unique word in $C_7$. That is, if we send words chosen from $C_7$, the transmission is robust against one error.
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Distance between permutations

Hamming distance $d_H$ makes sense in $S_n$ if we write permutations in “one line notation” as rearrangements of the alphabet $\{1, 2, \ldots, n\}$.

Example. 

35412 and 32415 are at distance 2.

This distance is still a metric; however, observe that $d_H = 1$ is never achieved for permutations.
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Alternatively, for $\sigma, \tau \in S_n$, their Hamming distance is the number of non-fixed points of $\sigma \tau^{-1}$.

**Example.**

35412 $\rightarrow$ (134)(25)
32415 $\rightarrow$ (134)

have quotient (25), with $d_H = 2$ non-fixed points.

With this, it is clear that $d_H$ is translation-invariant:

$d_H(\sigma, \tau) = d_H(\sigma \alpha, \tau \alpha) = d_H(\alpha \sigma, \alpha \tau)$. 
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A **permutation code** of length $n$ and distance $d$ is a subset $\Gamma \subseteq S_n$ such that the distance between distinct members of $\Gamma$ is at least $d$.

**Example.**

\[\{1234, 2143, 3412\}\]

is a permutation code of length 4 and distance 4. So is

\[\{1234, 2143, 3412, 4321\}\].

Including any additional permutation will decrease the minimum distance.
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A toy application

Suppose we wish to transmit information using amplitude modulation on electrical power lines.

Using an ordinary binary code has the disadvantage of introducing possibly long stretches of low (or high) voltage.

A permutation code enjoys the property that the sum of amplitudes on each codeword is a constant. So over a relatively short block of time, the average deviation from ambient voltage is zero.

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The question

Given $n$ and $d$, how large can a permutation code be with these parameters?

The maximum is denoted $M(n, d)$; this is increasing in $n$ and decreasing in $d$.

- finding a nice code gives a lower bound
- (linear) algebraic arguments offer upper bounds

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**Theorem**

*For* $n \geq 3$, $M(n, 3) = n!/2$.

**Proof sketch.**

In the group $A_n$, the quotient of any two permutations is even, so can’t be a transposition. Therefore, $M(n, 3) \geq |A_n| = n!/2$. Conversely, if $|\Gamma| > n!/2$, there must exist two elements in $\Gamma$ belonging to the same pigeonhole $\{\sigma, (12)\sigma\}$. This contradicts $d = 3$. 


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**Theorem**

\[ M(n, d) \leq n \ M(n - 1, d). \]

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Take all codewords which begin with a common symbol, and delete that symbol. After relabelling, this is a permutation code of length \( n - 1 \) and minimum distance \( d \).

**Corollary (Johnson bound)**

\[ M(n, d) \leq n(n - 1) \cdots (d + 1)d = n!/(d - 1)!. \]
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The denominator counts the ball \( B \) of radius \( r = \frac{d-1}{2} \) in \( S_n \).

If \( \Gamma \subseteq S_n \) is a permutation code realizing \( M(n, d) \), then the balls of radius \( r \) centred at the codewords must be disjoint. That is,

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Constructions and lower bounds

- MOLS($n$) lead to permutation codes of distance $n - 1$.  
  (Colbourn, Kløve, Ling)

- Sharply $k$-transitive permutation groups lead to maximum permutation codes of distance $n - k + 1$.  
  (Deza, Vanstone)

- Permutation polynomials of degree $t$ can be used for distance $n - t$.  (Chu)

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Other explicit constructions

- **Computer search**
  - Greedy selection of codewords, with modifications
  - Clique search, often assuming automorphisms
- **Partitioning and gluing**
- **Isometric embeddings** of some structure into $S_n$
MOLS construction

Record the list of row indices for each symbol in each square:

A: 1234, J: 2143, Q: 3412, K: 4321,
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\[ \spadesuit : 1423, \quad \heartsuit : 4132, \quad \diamondsuit : 3241, \quad \clubsuit : 2314. \]
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A challenging case: $d = 4$

Consider upper bounds on $M(n, 4)$.

**Johnson bound:** $n!/6$.

**Sphere-packing bound:** $n!$. This is bad because $n$ is even.

**Theorem**

$M(n, 4) = (n - 1)!$

**Proof idea.**

Blob-packing: The blobs $A_\sigma = \{(1i)\sigma : 1 \leq i \leq n\}$, centred at codewords $\sigma \in \Gamma$, must be disjoint in any permutation code of distance 4. We have $|A_\sigma| = n$ for each $\sigma$, so $|\Gamma| \leq n!/n$. [ ]
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Association Schemes

A *k-class association scheme* on a set $X$ is a list of binary relations $R_0, \ldots, R_k$ on $X$ satisfying

- $R_0$ is the identity
- the relations partition $X^2$, and
- a strong regularity condition

given $x$ and $y$ with $(x, y) \in R_h$, the number of $z \in X$ for which both $(x, z) \in R_i$ and $(z, y) \in R_j$ is a constant depending only on $h, i, j \in \{0, \ldots, k\}$.

These values $p_{ij}^h$ are called the *structure constants*. 
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These values $p_{ij}^h$ are called the \textit{structure constants}.
Example. Let $X = \{0, 1\}^n$, $R_i$ be disagreement in $i$ places.

This is the Hamming scheme.
The Conjugacy Scheme

The symmetric group defines an association scheme, called the *conjugacy scheme*, where $X = S_n$, relations are indexed by partitions of $n$, and $(\sigma, \tau) \in R_\mu$ if and only if $\sigma \tau^{-1}$ belongs to conjugacy class $\mu$.

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p_{ij}^h = \frac{|C_i||C_j|}{n!} \sum_{\chi} \frac{\chi(\phi_i)\chi(\phi_j)\chi(\phi_h)}{\chi(id)}
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Let \((X, \{R_i\})\) be a \(k\)-class association scheme.

For \(J \subset \{1, \ldots, k\}\), a \(J\)-clique is a subset \(W\) of \(X\) such that for any \(w_1, w_2 \in W\), \((w_1, w_2) \in R_j\) for some \(j \in J\).

In the conjugacy scheme, cliques model permutation codes.
Clique

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Delsarte’s linear programming bound for $J$-cliques, specialized to permutation codes, is as follows.

\[
\text{maximize: } a_0 + a_1 + \cdots + a_{m-1} \\
\text{subject to: } \sum_{0 \leq i < m} a_i \chi_k(\phi_i) \geq 0 \quad \text{for } 0 \leq k < m, \\
a_0 = 1, \ a_i \geq 0, \quad \text{and} \\
a_i = 0 \quad \text{if } d_H(\text{id}, \phi_i) \notin D.
\]

Using this, one can obtain decent upper bounds on $M(n, d)$ for various small parameters, say up to $n = 15$. 
Sharpened bound for distance four

Theorem (joint with N. Sawchuck)

If $n$ is a square integer,

$$M(n, 4) \leq \frac{n!}{(n + 2)}.$$ 

The proof idea is to show

$$\chi(1) + 3\chi(2) + (n - 2)\chi(3) \geq 0$$

using ‘local optimization’ on integer partitions of $n$. 
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The full bound is

$$\frac{n!}{M(n, 4)} \geq 1 + \frac{(n + 1)n(n - 1)}{n(n - 1) - (n - k^2)((k + 1)^2 - n)((k + 2)(k - 1) - n)}$$

for $k^2 \leq n \leq k^2 + k - 2$, which gives the best improvement for $n \approx k^2 + k/2$. 
What can we construct with distance four

On the construction side, it is difficult to do much better than greedy for $d = 4$:

$$M(n, 4) \geq \frac{n!}{B_3} = \frac{n!}{1 + \binom{n}{2} + 2\binom{n}{3}}.$$

Can we shrink the gap between these bounds for $M(n, 4)$?
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Open problems

- Find or improve bounds on the smallest undecided cases $M(7, 4)$ and $M(7, 5)$. All we know is $343 \leq M(7, 4) \leq 535$ and $77 \leq M(7, 5) \leq 134$.

- Obtain better constructions for $M(n, n-1)$ when $n$ is not a prime power. For general $n$, we only have $M(n, n-1) \geq n^{1+1/14.8}$, coming from the lower bound on mutually orthogonal latin squares. Some improvements are possible for special values of $n$.

- Study codes constrained by prescribed distance sets, or (more generally) conjugacy classes. As a special case, one has “equidistant permutation arrays” in which any two distinct codewords have the same distance.

- Add to the growing body of work on other metrics on $S_n$, such as the Kendall-$\tau$ metric or Ulam metric.

- Further study generalizations in two directions: “constant composition codes” and “injection codes”.

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- Study codes constrained by prescribed distance sets, or (more generally) conjugacy classes. As a special case, one has “equidistant permutation arrays” in which any two distinct codewords have the same distance.

- Add to the growing body of work on other metrics on $S_n$, such as the Kendall-$\tau$ metric or Ulam metric.

- Further study generalizations in two directions: “constant composition codes” and “injection codes”.
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