UVic Mathematics Competition

September 21, 2010

University of Victoria

- No calculators, books or notes are allowed.
- Write solutions in the booklets provided. Clearly separate rough work from solutions.
- All the necessary work to justify an answer and all the necessary steps of a proof must be shown clearly to obtain full credit.
- Partial credit will be given only for substantial progress toward a solution.
- Questions are of equal value.

Duration: 2 hours

Question 1. For each positive integer n, let

$$f(n) = \lfloor 2\sqrt{n} \rfloor - \lfloor \sqrt{n-1} + \sqrt{n+1} \rfloor.$$

(For a real number x, the notation $\lfloor x \rfloor$ represents the greatest integer less than or equal to x.) Determine all values of n for which $f(n) = 1$.

Question 2. Given seven points in the plane, determine the maximum number of lines that each intersect at least 3 of the points.

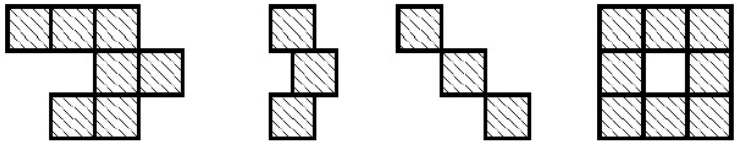
Question 3. Let α and β be real numbers satisfying

$$\begin{aligned} \alpha^3 - 3\alpha^2 + 5\alpha - 17 &= 0, \\ \beta^3 - 3\beta^2 + 5\beta + 11 &= 0. \end{aligned}$$

Find $\alpha + \beta$.

Question 4. A polyomino is a shape that can be obtained by gluing together 1×1 squares full-edge-to-full-edge in such a way that the resulting shape is connected (i.e. you can move from any square to any other through edges that are connected) and simply-connected (i.e. the shape has no holes).

For example in the following figure the first is a polyomino; the second is not (the edges do not fully line up); the third is not (same reason) and the fourth is not (it fails to be simply connected).



Prove that if $P(n)$ denotes the number of polyominoes that can be built from n unit squares (identifying two polyominoes if one can be transformed into another by a rigid motion), then $P(n)$ satisfies $A^n < P(n) < B^n$ for all $n \geq 3$ for suitable constants $1 < A < B$.