UVIC MATHEMATICS COMPETITION 2014

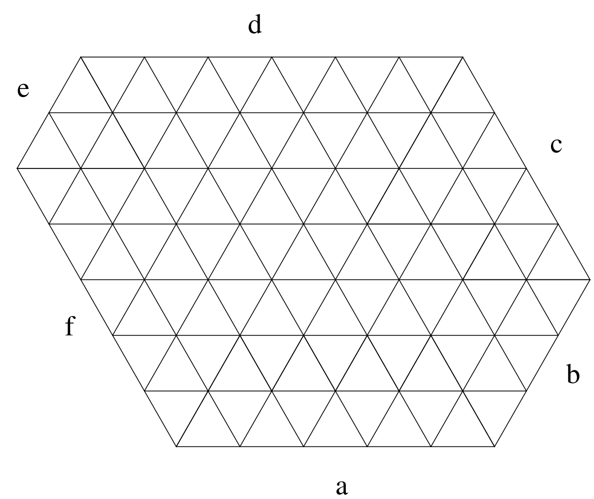
- All necessary work to justify an answer and all the steps of a proof must be clearly shown to obtain full credit.
- Partial credit will only be awarded for substantial progress towards a solution.
- All questions are worth equal marks.

NO CALCULATORS, NOTES OR BOOKS ALLOWED

1) Find all ordered triples of real numbers $(a-b, b-c, c-a)$ such that

$$a(a-1)+2bc = b(b-1)+2ac = c(c-1)+2ab.$$

2) Consider a convex hexagon made out of unit equilateral triangles as shown in the figure.



Let the number of triangles making up each side be a, b, c, d, e and f as shown. Show that $b-e = d-a = f-c$.

Supposing that the quantity above is non-negative, deduce that there are non-negative integers k, l, m, n such that $a = k$, $b = l+n$, $c = m$, $d = k+n$, $e = l$ and $f = m+n$.

Prove that the number of unit equilateral triangles in the figure is $n^2 + 2n(k+l+m) + 2(kl+km+lm)$.

3) What is the largest integer K for which it is possible to satisfy the following condition?

> There exists $x \in \mathbb{R}$ such that $2^k < x^k + x^{k+1} < 2^{k+1}$ for all $1 \leq k \leq K$.

4) Let C denote the cylinder $\{(x, y, z)\colon x^2 + y^2 \leq 1; |z| \leq 1\}$. Let P_a denote the plane $\{(x, y, z)\colon x = az\}$.

Find an expression (involving standard functions only) for the area of the intersection of C with P_a.