THE UNIVERSITY OF VICTORIA MATHEMATICS COMPETITION
October 6, 2015

- No calculators, books or notes are allowed.
- Write solutions in the booklets provided. Clearly separate rough workings from solutions.
- All the necessary work to justify an answer and all the necessary steps of a proof must be shown clearly to obtain full credit.
- Partial credit will be given only for substantial progress toward a solution.
- Questions are of equal value.

Duration: 2 hours

Question 1. Consider the following two-player game. There is a single pile of matches, and players take turns adjusting the pile. A player on his/her turn may either add a match (from an unlimited supply), or, if the size of the pile is evenly divisible by 3, that player can remove two-thirds of the matches. The first player to be given a pile with 1 match loses.

Ann and Bob are playing. Ann goes first. The pile initially consists of 10 matches. Can she guarantee a win against any response from Bob? Can Bob guarantee a win? Or neither?

Question 2. Determine whether or not there are real solutions x to the equation

$$2\sin x + 3\cos x = \tan x + 4\cot x.$$

Question 3. Let a and b be positive real numbers. Prove that $\text{round}(a \cdot \text{round}(bn)) = n$ for infinitely many natural numbers n if and only if $ab = 1$.

Here, for a positive real number x, we let $\text{round}(x) = \lfloor x + \frac{1}{2} \rfloor$ denote the usual rounding of x to the nearest integer. For instance, $\text{round}(\pi) = 3$, $\text{round}(7.77) = 8$, and $\text{round}(\frac{1}{2}) = 1$.

Question 4. An ellipse E which is not a circle is reflected in a line l passing through its centre. The resulting ellipse is E^l. Describe, with proof, the lines l which minimize the area inside both of E and E^l.