THE UNIVERSITY OF VICTORIA MATHEMATICS COMPETITION
September 29, 2016

- No calculators, books or notes are allowed.
- Write solutions in the booklets provided. Clearly separate rough workings from solutions.
- All the necessary work to justify an answer and all the necessary steps of a proof must be shown clearly to obtain full credit.
- Partial credit will be given only for substantial progress toward a solution.
- Questions are of equal value.

Duration: 2 hours

Question 1. Let p be an odd prime. Prove that p is the harmonic mean of one and only one pair of positive integers m and n with $m < n$. Recall that the harmonic mean h of two real numbers a and b is defined by $\frac{1}{h} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right)$, for example 3 is the harmonic mean of 2 and 6.

Question 2. Show that there does not exist a non-constant polynomial p with integer coefficients such that $p(n)$ is prime for each natural number n.

Question 3. Consider an arrangement of circles in the plane such that each lattice point (m, n) other than $(0, 0)$, is the centre of a circle of diameter $\frac{1}{1000}$. Is there an infinite line through the origin that doesn't hit any of the circles?

Question 4. Let $x_1 < x_2 < \cdots$ be a sequence of positive integers satisfying $x_n \leq 2(n-1)$ for every $n > 1$. Prove that the set of differences $x_j - x_k$ includes every positive integer.