SCATTERING FOR THE TWO-DIMENSIONAL NLS WITH EXPONENTIAL NONLINEARITY

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Abstract. We investigate existence and asymptotic completeness of the wave operators for nonlinear Schrödinger equations with a defocusing exponential nonlinearity in two space dimensions. A certain threshold is defined based on the value of the conserved Hamiltonian, below which the exponential potential energy is dominated by the kinetic energy via a Trudinger-Moser type inequality. We prove that if the Hamiltonian is below to the critical value, then the solution approaches a free Schrödinger solution at the time infinity.

1. Introduction

We study the scattering theory in the energy space for nonlinear Schrödinger equation (NLS):

$$\begin{cases} i\dot{u} + \Delta u = f(u), \quad u : \mathbb{R}^{1+2} \to \mathbb{C}, \\ u(0,x) = u_0(x) \in H^1(\mathbb{R}^2), \end{cases} \tag{1.1}$$

where the nonlinearity $f : \mathbb{C} \to \mathbb{C}$ is defined by

$$f(u) = \left(e^{4\pi|u|^2} - 1 - 4\pi|u|^2\right) u. \tag{1.2}$$

Solutions of (1.1) satisfy the conservation of mass and Hamiltonian

$$M(u,t) := \int_{\mathbb{R}^2} |u|^2 dx, \tag{1.3}$$

$$H(u,t) := \int_{\mathbb{R}^2} \left(|\nabla u|^2 + 2F(u)\right) dx. \tag{1.4}$$

Also we define

$$E(u,t) := H(u,t) + M(u,t)\ . \tag{1.5}$$

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The exponential type nonlinearities appear in several applications, as for example the self trapped beams in plasma. (See [13]). From the mathematical point of view, Cazenave in [4] considered the Schrödinger equation with decreasing exponential and showed the global well-posedness and scattering. With increasing exponentials, the situation is much more complicated (since there is no a priori L^∞ control of the nonlinear term). The two dimensional case is particularly interesting because of its relation to the critical Sobolev (or Trudinger-Moser) embedding. On the other hand, we have subtracted the cubic part from our nonlinearity f in order to avoid another critical exponent related to the decay property of solutions. To explain these issues, we start with a brief review of the more familiar power case.

1.1. **The energy critical NLS.** Recall the monomial defocusing semilinear Schrödinger equation in space dimension $d \geq 1$

$$i\,\dot{u} + \Delta u = |u|^{p-1}u, \quad u : \mathbb{R}^{1+d} \longmapsto \mathbb{C}, \tag{1.6}$$

which has the critical exponents $p^* = \frac{d+2}{d-2}$ (only for $d \geq 3$) and $p_* = 1 + \frac{4}{d}$.

For the *energy subcritical* case ($p < p^*$), an iteration of the local-in-time well-posedness result using the *a priori* upper bound on $\|u(t)\|_{H^1}$ implied by the conservation laws establishes global well-posedness for (1.6) in H^1. Those solutions scatter when $p > p_*$ [10, 15].

The *energy critical* case ($p = p^*$) was actually harder than the nonlinear Klein-Gordon equation, for which the finite propagation property was crucial to exclude possible concentration of energy, whereas there is no upper bound on the propagation speed for the Schrödinger. Nevertheless, based on new ideas such as induction on the energy size and frequency split propagation estimates, Bourgain [3] proved the global well-posedness and the scattering for radially symmetric data, and it was extended to the general case by [8] using a new interaction Morawetz inequality.

For the exponential nonlinearity in two spatial dimensions, small data global well-posedness together with the scattering was worked out by Nakamura-Ozawa in [14]. Later on, the size of the initial data for which one has local existence was quantified for (1.1) in [9], and a notion of criticality was proposed:

Definition 1.1. *The Cauchy problem* (1.1) *is said to be subcritical if* $H(u_0) < 1$, *critical if* $H(u_0) = 1$ *and supercritical if* $H(u_0) > 1$.

Indeed, one can construct a unique local solution if $\|\nabla u_0\|_{L^2} < 1$, and the time of existence depends only on $\eta := 1 - \|\nabla u_0\|_{L^2}$ and $\|u_0\|_{L^2}$. Hence the maximal local solutions are indeed global in the subcritical case. The critical case is more delicate due to the possible concentration of the Hamiltonian. The following result is proved in [9].

Theorem 1.2 (Global well-posedness [9])**.** *Assume that* $H(u_0) \leq 1$, *then the problem* (1.1) *has a unique global solution* u *in the class*

$$\mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^2)).$$

Moreover, $u \in L^4_{loc}(\mathbb{R},\ \mathcal{C}^{1/2}(\mathbb{R}^2))$ *and satisfies the conservation laws* (1.3) *and* (1.4).

Recall that $\mathcal{C}^{1/2}(\mathbb{R}^2)$ denotes the space of 1/2-Hölder continuous functions.

1.2. **Main result.** The main goal in this paper is to show that every global solution of (1.1) with $H(u) \leq 1$ approaches solutions to the associated free equation

$$i\dot{v} + \Delta v = 0, \tag{1.7}$$

in the energy space H^1 as $t \to \pm\infty$. Unfortunately, we have not succeeded to handle the critical case $H(u) = 1$ and we have to restrict ourselves to the subcritical one. The reason is that to trace the concentration radius, as defined in [12], the finite speed of propagation of energy is essential in our argument for NLKG, which is not available for NLS. The main ingredient for the subcritical NLS is a new interaction Morawetz estimate, proved independently by Colliander et al. and Planchon-Vega [7, 16]. This estimate gives a priori global bound of u in $L^4_t(L^8_x)$. Hence, by complex interpolation we deduce that some of the Strichartz norms used in the nonlinear estimate go to zero for large time and the scattering in the subcritical case follows. More precisely, we have

Theorem 1.3. *For any global solution u of (1.1) in H^1 satisfying $H(u) < 1$, we have $u \in L^4(\mathbb{R}, \mathcal{C}^{1/2})$ and there exist unique free solutions $u_\pm$ of (1.7) such that*

$$\|(u - u_\pm)(t)\|_{H^1} \to 0 \qquad (t \to \pm\infty).$$

Moreover, the maps

$$u(0) \longmapsto u_\pm(0)$$

are homeomorphisms between the unit balls in the nonlinear energy space and the free energy space, namely from $\{\varphi \in H^1\ ;\ H(\varphi) < 1\}$ onto $\{\varphi \in H^1\ ;\ \|\nabla\varphi\|_{L^2} < 1\}$.

This paper is organized as follows. In Section 2, we recall some useful lemmas from the literature. Section 2 is devoted to the proof of our main result (Theorem 1.3). In Section 4, we show the optimality of our nonlinear estimate with respect to the H^1 norm.

2. Background Material

In this section, we introduce some notation and recall several lemmas we use to prove the main result. First, recall the sharp Trudinger-Moser inequality on $\mathbb{R}^2$ [1, 17]. It is the limit case of the Sobolev embedding. For any $\mu > 0$ we have

$$\sup_{\|\varphi\|_{H_\mu} \leq 1} \int (e^{4\pi|\varphi|^2} - 1) dx < \infty, \tag{2.1}$$

where H_μ is defined by the norm $\|u\|^2_{H_\mu} := \|\nabla u\|^2_{L^2} + \mu^2\|u\|^2_{L^2}$. We can change $\mu > 0$ just by scaling $\varphi(x) \mapsto \varphi(x/\mu)$.

It is known that the $H^1(\mathbb{R}^2)$ functions are not generally in L^∞. The following lemma shows that we can estimate the L^∞ norm by a stronger norm but with a weaker growth (namely logarithmic).

Lemma 2.1 (Logarithmic inequality [11], Theorem 1.3)**.** *Let* $0 < \alpha < 1$*. For any real number* $\lambda > \frac{1}{2\pi\alpha}$*, a constant* C_λ *exists such that for any function* $\varphi \in H_0^1 \cap \dot{\mathcal{C}}^\alpha(|x| < 1)$*, one has*

$$\|\varphi\|_{L^\infty}^2 \le \lambda \|\nabla\varphi\|_{L^2}^2 \log\left(C_\lambda + \frac{\|\varphi\|_{\dot{\mathcal{C}}^\alpha}}{\|\nabla\varphi\|_{L^2}}\right). \tag{2.2}$$

We also recall the whole space version of the above inequality.

Lemma 2.2 ([11], Theorem 1.3)**.** *Let* $0 < \alpha < 1$*. For any* $\lambda > \frac{1}{2\pi\alpha}$ *and any* $0 < \mu \le 1$*, a constant* $C_\lambda > 0$ *exists such that, for any function* $u \in H^1(\mathbb{R}^2) \cap \mathcal{C}^\alpha(\mathbb{R}^2)$

$$\|u\|_{L^\infty}^2 \le \lambda \|u\|_{H_\mu}^2 \log\left(C_\lambda + \frac{8^\alpha \mu^{-\alpha}\|u\|_{\mathcal{C}^\alpha}}{\|u\|_{H_\mu}}\right). \tag{2.3}$$

Finally, we recall the Strichartz estimate for the free Schrödinger equation. (See [5]).

Proposition 2.3. *Let* $I \subset \mathbb{R}$ *be a time slab,* $t_0 \in I$ *and* (q, r)*,* $(\tilde{q}, \tilde{r})$ *two admissible Strichartz couples,* i.e.,

$$2 \le r, \beta < \infty \quad \textit{and} \quad \frac{1}{q} + \frac{1}{r} = \frac{1}{\alpha} + \frac{1}{\beta} = \frac{1}{2}\,.$$

There exists a positive constant C *such that if* $u := u(t, x)$ *a solution in* $\mathcal{C}(I, H^1(\mathbb{R}^2))$ *of the linear problem*

$$\mathrm{i}\dot{u} + \Delta\, u = G(t, x), \qquad u(t_0) \in H^1(\mathbb{R}^2),$$

then[1]

$$\|u\|_{L^q(I,W^{1,r})} \le C\Big(\|u(t_0)\|_{H^1} + \|G\|_{L^{\tilde{q}'}(I,W^{1,\tilde{r}'})}\Big)\,. \tag{2.4}$$

In particular, note that $(q, r) = (4, 4)$ is an admissible Strichartz couple and

$$W^{1,4}(\mathbb{R}^2) \hookrightarrow \mathcal{C}^{1/2}(\mathbb{R}^2).$$

3. Proof of Theorem 1.3

For a time slab $I \subset \mathbb{R}$, we define $S^1(I)$ via

$$\|u\|_{S^1(I)} = \|u\|_{L^\infty(I,H_x^1)} + \|u\|_{L^4(I,W^{1,4})}.$$

By the Strichartz estimates we have

$$\|u\|_{S^1} \lesssim \|u(0)\|_{H^1} + \|\langle\nabla\rangle(i\,\dot{u} + \Delta u)\|_{L^{\frac{2}{1+2\eta}}(L_x^{\frac{1}{1-\eta}})}, \tag{3.1}$$

for any $0 < \eta \le 1/2$.

The scattering result Theorem 1.3 is easily proved by the following two lemmas: First we have the Strichartz-type estimate on the nonlinearity

[1] Here p' stands for the Lebesgue conjugate exponent of $1 \le p \le \infty$, that is

$$\frac{1}{p} + \frac{1}{p'} = 1\,.$$

Lemma 3.1. *For any $H \in (0,1)$, there exists $\delta \in (0,1)$, such that for any time slab I, any $T \in I$ and any H^1 solution u of (1.1) with $H(u) \le H$, we have*

$$\|u\|_{S^1(I)} \lesssim \|u(T)\|_{H^1} + \|u\|_{L^4(I,L^8)}^{4\delta} \|u\|_{S^1(I)}^{5-4\delta},$$

Next we have a global *a priori* bound. It was proved independently by Planchon-Vega [16] and Colliander et al. [7]

Lemma 3.2. *Let u be a global solution of (1.1) in H^1. Then*

$$\|u\|_{L^4(\mathbb{R},L^8)} \lesssim \|u\|_{L^\infty(\mathbb{R};L^2)}^{3/4} \|\nabla u\|_{L^\infty(\mathbb{R};L^2)}^{1/4} \lesssim M(u)^{3/8} H(u)^{1/8}.$$

Actually both of them gave a priori bound on some Sobolev norm on $|u|^2$. The above is a consequence of it via the Sobolev embedding.

By the above global bound, we can decompose $\mathbb{R}$ into a finite number of intervals on which the $\|u\|_{L^4L^8}$ norm is sufficiently small. Then the first lemma gives a uniform bound on $\|u\|_{S^1}$ on each interval, and hence by summing it up for all intervals, we obtain a priori bound

$$\|u\|_{S^1(\mathbb{R}\times\mathbb{R}^2)} \le C(E(u)) < \infty, \tag{3.2}$$

and thereby the scattering for u.

Proof of Lemma 3.1. It suffices to estimate the nonlinear term in some dual Strichartz norm as in (3.1). Choose $0 < \delta < 1$ and $\lambda > 0$ such that

$$K := \frac{H+1}{2} < 1, \quad 2\pi(1+2\delta)\lambda K^2 = 2, \quad \lambda > \frac{1}{\pi(1-\delta)}. \tag{3.3}$$

We estimate only $\nabla f(u)$, since the same estimate on $f(u)$ is easier. Note that

$$|\nabla f(u)| \lesssim |\nabla u||u|^2(e^{4\pi|u|^2} - 1).$$

In the case $\|u\|_{L_x^\infty} \ge K$, we have by the Hölder inequality,

$$\|\nabla f(u)\|_{L_x^{\frac{1}{1-\delta}}} \lesssim \|\nabla u\|_{L^{\frac{2}{1-\delta}}} \|u\|_{L^{\frac{4}{\delta}}}^2 \|e^{4\pi|u|^2} - 1\|_{L^1}^{1/2-\delta} \|e^{4\pi|u|^2} - 1\|_{L^\infty}^{1/2+\delta}. \tag{3.4}$$

The third term on the right is bounded by the Trudinger-Moser (2.1). For the last term we use the H_μ version of the logarithmic inequality (2.3) with $\mu := \min(1, \sqrt{(1-H)/M}) > 0$. Since

$$\|u\|_{H_\mu}^2 = \|\nabla u\|_{L^2}^2 + \mu^2\|u\|_{L^2}^2 \le H + \mu^2 M \le \frac{H+1}{2} < 1,$$

that term is bounded by

$$e^{4\pi(1/2+\delta)\|u\|_{L^\infty}^2} \lesssim (1 + \|u\|_{C^{1/2-\delta/2}}/\|u\|_{H_\mu})^{2\pi(1+2\delta)\|u\|_{H_\mu}^2} \lesssim \|u\|_{C^{1/2-\delta/2}}^2, \tag{3.5}$$

where we used (3.3) as well as $\|u\|_{C^{1/2-\delta/2}} \ge \|u\|_{L^\infty} \ge K$. The case $\|u\|_{L^\infty} \le K$ is easy, since then $|\nabla f(u)| \lesssim |\nabla u||u|^4$.

Now we integrate in time using the Hölder to obtain

$$\|\nabla f(u)\|_{L^{\frac{2}{1+2\delta}}(L^{\frac{1}{1-\delta}})} \lesssim \|\nabla u\|_{L^{\frac{2}{\delta}}(L^{\frac{2}{1-\delta}})} \|u\|_{L^{\frac{2}{\delta}}(L^{\frac{4}{\delta}})}^2 \|u\|_{L^{\frac{4}{1-\delta}}(C^{1/2-\delta/2})}^2.$$

Finally, the complex interpolation and the Sobolev embedding imply that

$$\begin{aligned}
&\|\nabla u\|_{L^{\frac{2}{\delta}}(L^{\frac{2}{1-\delta}})} \lesssim \|\nabla u\|_{L^\infty L^2}^{1-2\delta}\|\nabla u\|_{L^4L^4}^{2\delta} \lesssim \|u\|_{S^1},\\
&\|u\|_{L^{\frac{2}{\delta}}(L^{\frac{4}{\delta}})} \lesssim \|u\|_{L^\infty L^2}^{1-2\delta}\|u\|_{L^4L^8}^{2\delta},\\
&\|u\|_{L^{\frac{4}{1-\delta}}(C^{1/2-\delta/2})} \lesssim \|u\|_{L^{\frac{4}{1-\delta}}(H^{1,\frac{4}{1+\delta}})} \lesssim \|u\|_{L^\infty H^1}^{\delta}\|u\|_{L^4H^{1,4}}^{1-\delta} \lesssim \|u\|_{S^1}.
\end{aligned} \tag{3.6}$$

Plugging them into the above, we deduce the result as desired. □

4. Criticality of the nonlinear estimate by the Strichartz norms

We see that the linear energy and the Strichartz estimate are not sufficient to control the nonlinearity in the critical case.

Proposition 4.1. *For any $\delta > 0$, there exists a sequence of radial free Schrödinger solutions v_N ($N \to \infty$) such that*

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\nabla v_N|^2 + |v_N|^2 dx < 1, \quad H(v_N, 0) \le 1 + \delta,\\
&\|\nabla f(v_N)\|_{L_t^pL_x^q(|t|\ll N^{-2}, |x|\ll N^{-1})} \ge C_\delta(\log N)^{1/2},
\end{aligned} \tag{4.1}$$

for any $(p, q) \in [1,\infty]$ satisfying $1/p + 1/q = 3/2$.

The above norm on $f(v_N)$ is the dual Strichartz norm in H^1_x for the linear Schrödinger equation. Similar result was shown for the critical Klein-Gordon equation [12].

Proof. We take the same initial data as in ([12], Proposition 7.1):

$$v_N(0) = \sqrt{\frac{2\pi}{\log N}}\frac{1}{(2\pi)^2}\int_{1<|\xi|<e^{-a}N} |\xi|^{-2}e^{i\xi x}d\xi. \tag{4.2}$$

By the Plancherel theorem,

$$\begin{aligned}
&\|\nabla v_N(0)\|_{L^2}^2 = \frac{2\pi}{\log N}\frac{1}{(2\pi)^2}\int_{1<|\xi|<e^{-a}N} |\xi|^{-2}d\xi = \frac{\log N - a}{\log N},\\
&\|v_N(0)\|_{L^2}^2 = \frac{2\pi}{\log N}\frac{1}{(2\pi)^2}\int_{1<|\xi|<e^{-a}N} |\xi|^{-4}d\xi < \frac{1}{2\log N}.
\end{aligned} \tag{4.3}$$

By the sharp Trudinger-Moser inequality (2.1), there exists $M > 0$ such that for any $\mu > 0$

$$\sup_{\|\nabla\psi\|_{L^2}^2+\mu\|\psi\|_{L^2}^2\le 1} \int F(\psi)dx \le M/\mu, \tag{4.4}$$

where μ can be removed or inserted by rescaling. Then (4.3) implies that

$$\int 2F(v_N(0))dx \le \frac{M}{a}, \quad \int_{\mathbb{R}^2} |\nabla v_N|^2 + |v_N|^2 dx < 1, \quad H(v_N, 0) < 1 + \frac{M}{a}, \tag{4.5}$$

so that we get the desired nonlinear energy bound on v_N by choosing $a \ge M/\delta$.

The free solution is given by

$$v_N(t,x) = \sqrt{\frac{2\pi}{\log N}}\frac{1}{(2\pi)^2}\int_{1<|\xi|<e^{-a}N} |\xi|^{-2}e^{-it|\xi|^2+i\xi x}d\xi, \tag{4.6}$$

we have in the region where $t \sim \varepsilon^2 N^{-2}$ and $|x| \sim \varepsilon N^{-1}$ for some small fixed $\varepsilon > 0$ and large $N \in \mathbb{N}$,

$$\Re v_N(t,x) \geq \sqrt{\frac{\log N}{2\pi}} - \frac{a+\varepsilon^2}{\sqrt{2\pi \log N}}, \quad e^{4\pi|v|^2}|v|^2 \gtrsim N^2 \log N. \tag{4.7}$$

We have to estimate ∇v. By the radial symmetry, it suffices to consider the case $x = (x_1, 0)$ and $\nabla v_N = (\partial_1 v_N, 0)$. Then

$$\begin{aligned} \partial_1 v_N = \partial_r v_N &\sim \frac{1}{\sqrt{\log N}}\int_{1<|\xi|<e^{-a}N} \frac{\xi_1}{|\xi|^2}e^{-it|\xi|^2+i\xi x}d\xi \\ &= \frac{i}{\sqrt{\log N}}\int_1^N e^{-it\rho^2}\int_{-\pi}^{\pi}\cos\theta\sin(r\rho\cos\theta)d\theta d\rho, \end{aligned} \tag{4.8}$$

since $0 < t\rho^2 < \varepsilon^2 \ll 1$ and $0 < r\rho\cos\theta < \varepsilon \ll 1$, we get

$$|\partial_r v_N| \sim \frac{1}{\sqrt{\log N}}\int_1^N r\rho d\rho \sim \frac{rN^2}{\sqrt{\log N}} \sim \frac{\varepsilon N}{\sqrt{\log N}}. \tag{4.9}$$

Thus we conclude

$$\begin{aligned} &\inf_{t\sim\varepsilon^2N^{-2}} \|e^{4\pi|v_N|^2}|v_N|^2\nabla v_N\|_{L^q_x(|x|\sim\varepsilon N^{-1})} \gtrsim N^2\log N\frac{\varepsilon N}{\sqrt{\log N}}(\varepsilon N^{-1})^{2/q}, \\ &\|\nabla f(v_N)\|_{L^p_tL^q_x(t\sim\varepsilon^2N^{-2},|x|\sim\varepsilon N^{-1})} \gtrsim \varepsilon^{2/p+2/q}N^{3-2/q-2/p}\sqrt{\log N} \sim \varepsilon^3\sqrt{\log N}. \end{aligned} \tag{4.10}$$

□

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