Ramsey Games: Avoiding Triangles in Graphs

Jane Butterfield

University of Minnesota - Twin Cities

MathPath 2013

Ramsey theory

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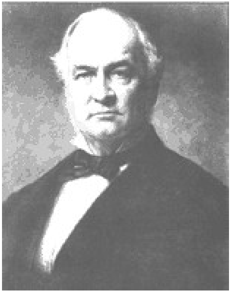
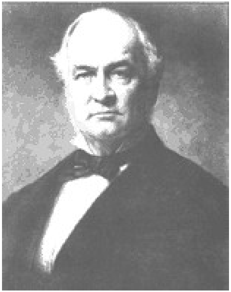


Alexander Ramsey,
first Governor of the Minnesota Territory

Ramsey theory

The Wrong Ramsey



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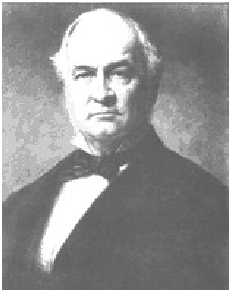

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Not-much-fun fact: Ramsey died at the age of 26 from complications after an abdominal operation to treat his jaundice.

The Right Ramsey

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What's the Pattern?

$1, 2, 3$

$1, 4, 7, 10, 13$

$2, 6, 10, 14$

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Just to be formal:

Definition

An Arithmetic Progression of length k (called k-AP for short) is a sequence of k numbers such that the difference between any consecutive numbers is constant.

Avoiding Arithmetic Progressions

... to the document camera!

Bartel Leendert van der Waerden

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In 1927, proved "Van der Waerden's Theorem"

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Van der Waerden's Theorem

What has he got to do with Arithmetic Progressions?

Theorem

For any natural number k there exists some number n large enough that any r-coloring of the integers $1, 2, \ldots, n$ must contain either a red k-AP or a blue k-AP.

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For $k = 3$ and $r = 2$, it turns out that 9 is big enough.

Van der Waerden Numbers

Just for fun...

	3	4	5	6	7
2 colors	9	35	178	1,132	
3 colors	27	293			
4 colors	76				
5 colors					
6 colors					

Van der Waerden Numbers

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	3	4	5	6	7
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3 colors	27	293	$> 2,173$	$> 11,191$	$> 48,811$
4 colors	76	$> 1,048$	$> 17,705$	$> 91,331$	$> 420,217$
5 colors	> 170	$> 2,254$	$> 98,740$	$> 540,035$	
6 colors	> 223	$> 9,778$	$> 98,748$	$> 816,981$	

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Lunchtime challenge: can you 3-color the numbers $1, 2, \ldots, 26$ without making a monochromatic 3-AP?

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However, we haven't found them. Some examples:
$1, 2, 3$
$1, 3, 5, 7$
Dinnertime challenge: what is the longest arithmetic progression of prime numbers you can find?
Here is the longest one currently known:

$$43,142,746,595,714,191 + 23,681,770 \cdot 223,092,870 \cdot n,$$

for $n = 0$ to 25.

Where are the graph games?

So much for Arithmetic Progressions.

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What happens when a mathematician reads chemistry papers?

A crash course in Graph Theory

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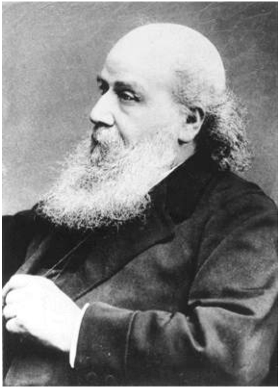
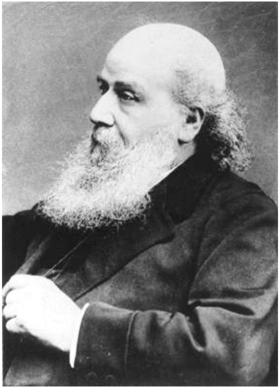


James Joseph Sylvester

A crash course in Graph Theory



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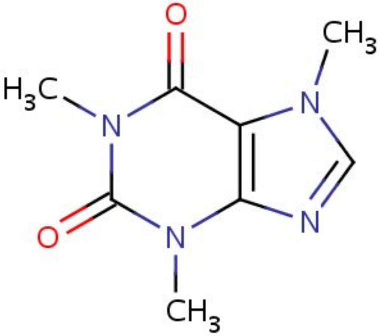

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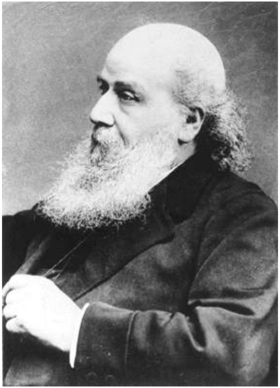

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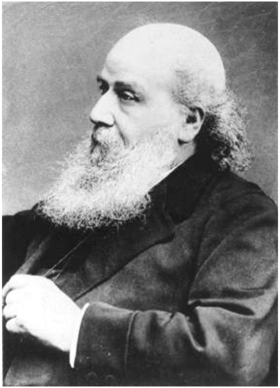

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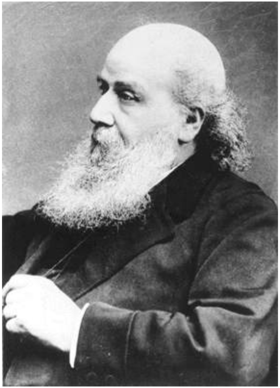

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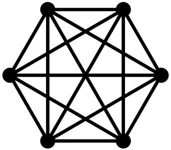
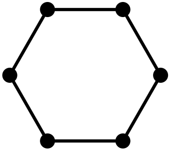
- The max degree of G, $\Delta(G)$, is $\max\{d(v) : v \in V(G)\}$.

Some important graphs



K_6



C_6

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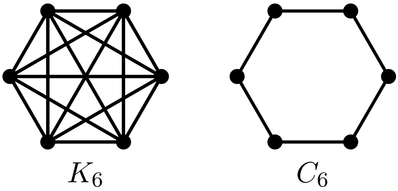

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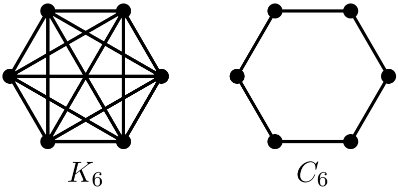

- The complete graph, K_ℓ, has ℓ vertices, and every two vertices form an edge.
- The cycle, C_ℓ, has vertices $v_1, \ldots, v_\ell$, and $v_i v_j$ is an edge iff $|i - j| = 1 (\text{mod } \ell)$.

Ramsey + Sylvester = . . .

Graph Ramsey Theory.

Challenges: (back to the document camera)

Graph Ramsey Games

Let's color edges instead of vertices this time.

- Game 1: I will give you a graph and you have to color the edges to avoid a triangle.

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Which game would you rather play? Why?

Game 1

You should not play this game! Ramsey says:

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Theorem

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In fact, a similar statement is true no matter what graph we are trying to avoid and how many colors we get to use.

Game 2 vs Game 3

- Game 2: Same as Game 1, but the graph's maximum degree will be at most 4.

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- Game 3: Same as Game 2, but I will give you the edges one at a time.

I have a lot more power in Game 3! Let's play...
(to the document camera!)

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I can win Game 1, but how? The truth is, we aren't sure!

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$R(6)$ is somewhere between 102 and 165 We would have better luck trying to destroy the aliens!

Many variations

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- What if I promise to stay in some other family of graphs (instead of max-degree-4 graphs)?

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- What if we want to avoid one graph in red and a different graph in blue?
- What if I promise to stay in some other family of graphs (instead of max-degree-4 graphs)?
- What if I am replaced by a random opponent?

Do you want to know more?

Breakout: Graph Ramsey Games, Week 4