On the Chromatic Thresholds of Hypergraphs

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Let $\mathcal{F}$ be a family of r-uniform hypergraphs. The chromatic threshold of $\mathcal{F}$ is the infimum of the values $c \geq 0$ such that the subfamily of $\mathcal{F}$ consisting of hypergraphs H with minimum degree at least $c\binom{|V(H)|}{r-1}$ has bounded chromatic number.

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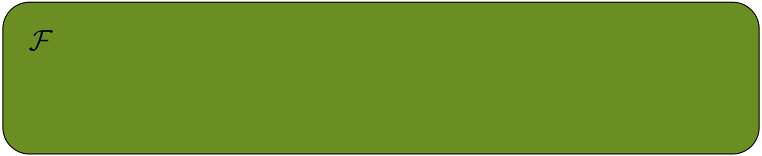

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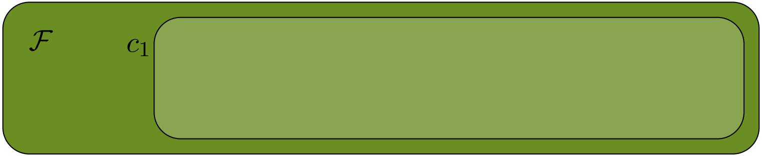

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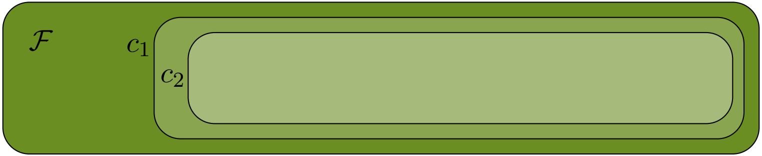

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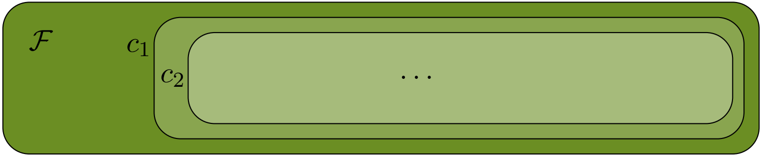

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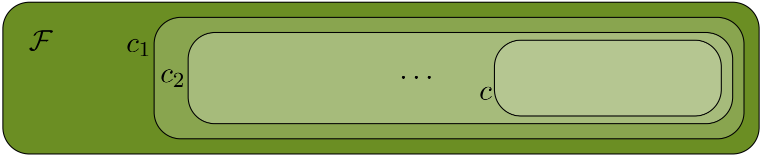

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- Andrásfai, Erdős, and Sós: the answer is $\frac{2}{5}|V(G)|$, achieved by the blowup of C_5.
- For an r-uniform hypergraph F, let $\mathcal{P}(n, F)$ be the family of r-uniform hypergraphs on n vertices that do not contain F. Let $ex(n, F) = \max\{|E(G)| : G \in \mathcal{P}(n, F)\}$.

Easy bound for graphs

- Suppose G is a graph on n vertices. If $\delta(G) > 2ex(n,F)/n$ then

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Theorem (Allen, Böttcher, Griffiths, Kohayakawa, Morris (2011+))

If G is a graph with chromatic number $r \geq 3$, then the chromatic threshold of $\mathcal{P}(n,G)$ is one of

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Similar bound holds for hypergraphs, too...

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Suppose F is an r-uniform hypergraph. If $ex(n, F)$ is known, then there is an easy upper bound for the chromatic threshold of $\mathcal{P}(n, F)$.

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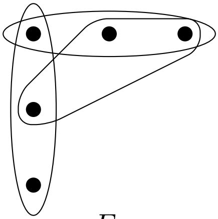

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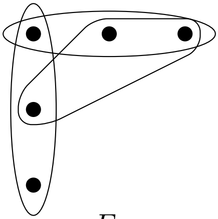

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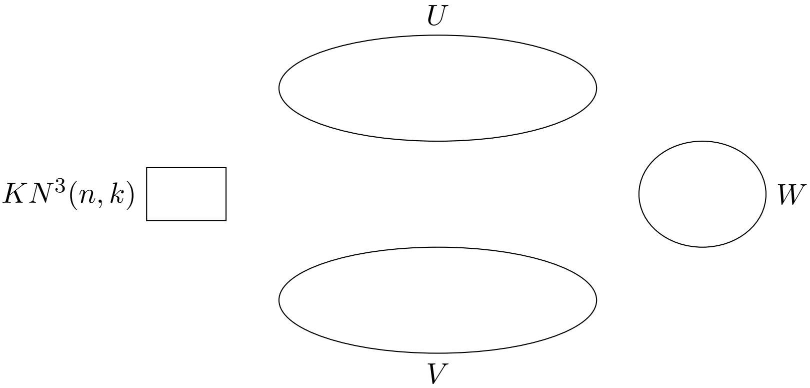

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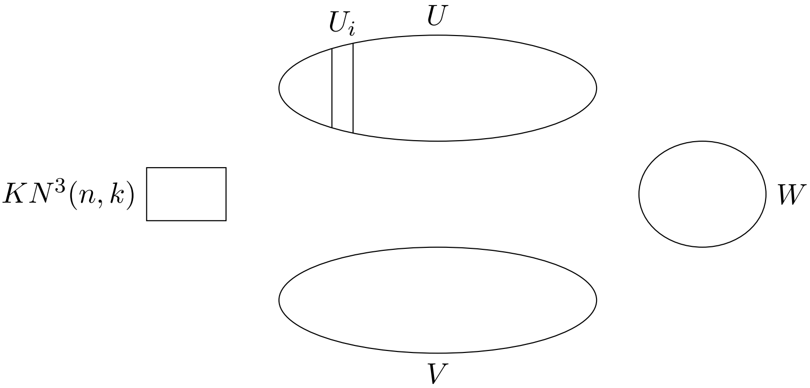

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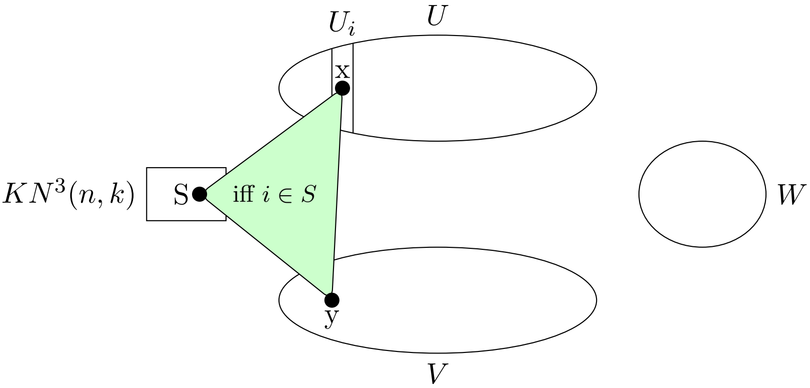

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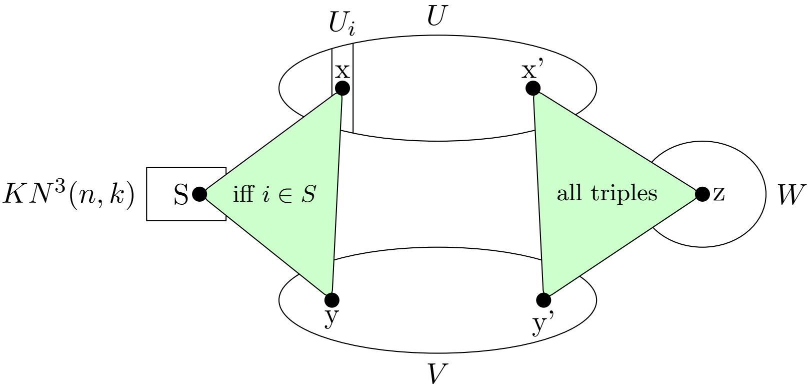

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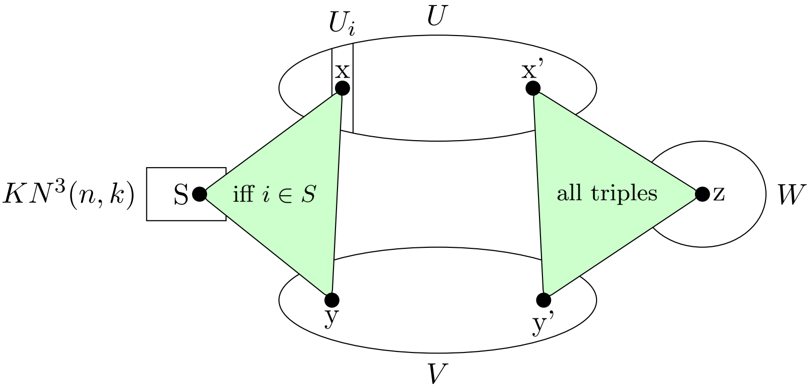

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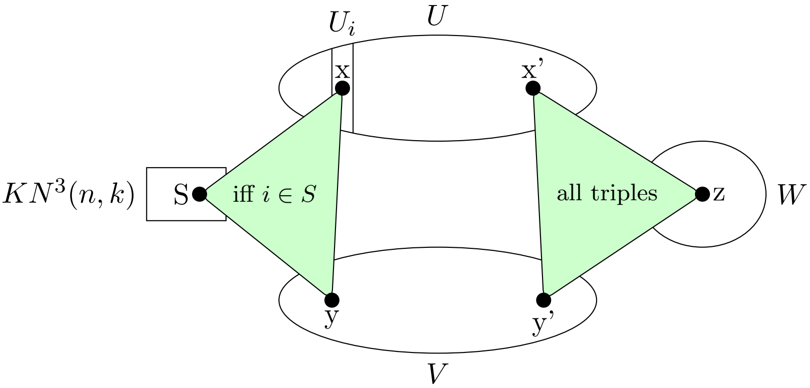

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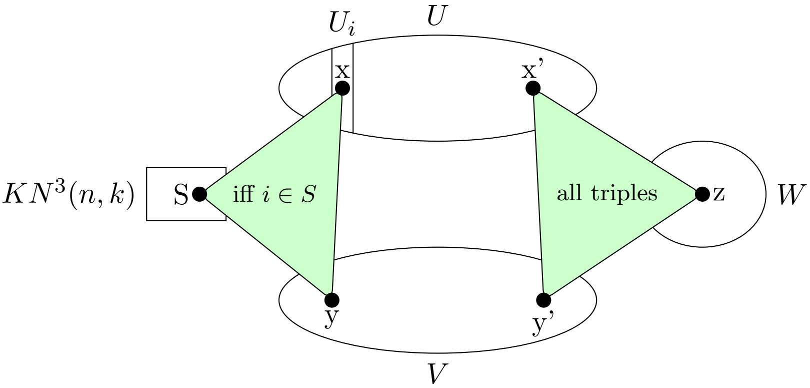

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More constructions

Other constructions use a generalized Kneser graph:

Definition

$\mathrm{KN}_s^r(n,k)$ is the r-uniform hypergraph with vertex set $\binom{[n]}{k}$ in which r vertices $F_1, \ldots, F_r$ form an edge if and only if no element of $[n]$ is contained in F_i for more than s distinct i.
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Let $c > 0$; then for any integers r, t, there exists $K_0 = K_0(c,r,t)$ such that if $k \geq K_0$, $s = r - 1$, and $n = (r/s + c)k$, then $\chi\left(\mathrm{KN}_s^r(n,k)\right) > t$.

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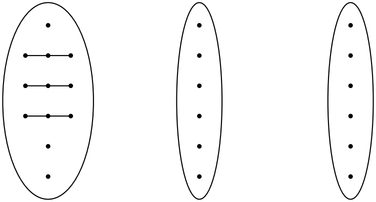
A graph is near bipartite if it is triangle-free and there is a partition of its vertices into two classes, V_1 and V_2, such that V_1 is an independent set and $H[V_2]$ is a partial matching.

Hypergraph cycles

Definition (Near r-partite)

Let H be an r-uniform hypergraph. H is near r-partite if there exists a partition $V_1 \cup \cdots \cup V_r$ of $V(H)$ such that all edges of H either cross the partition or are contained completely in V_1, and in addition $H[V_1]$ is a partial matching.



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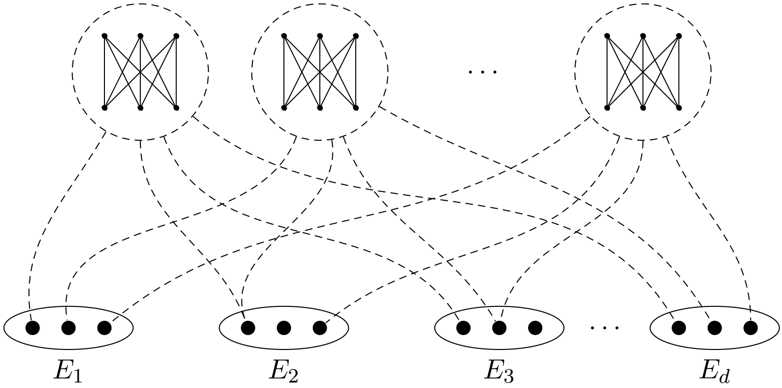

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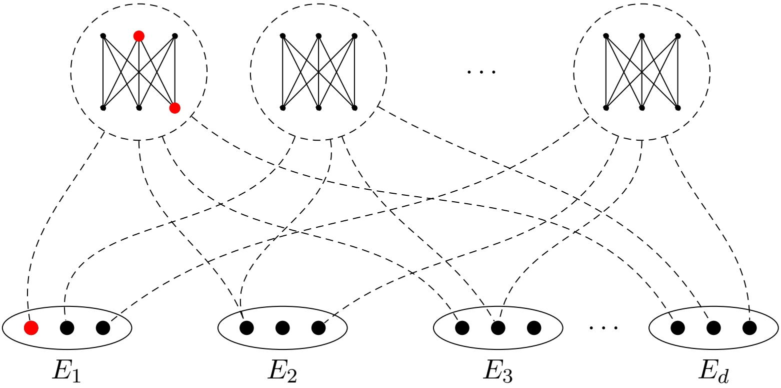

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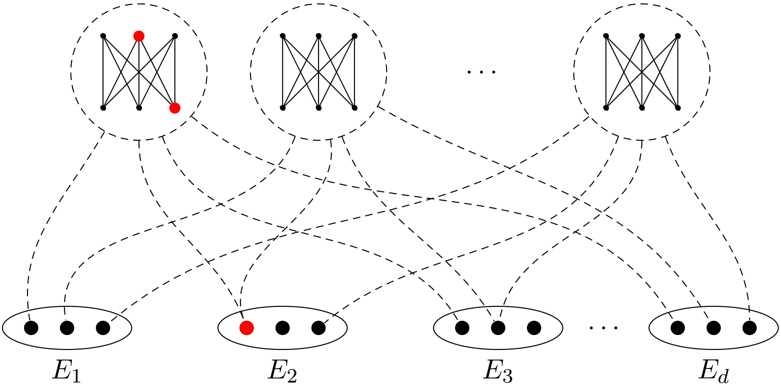
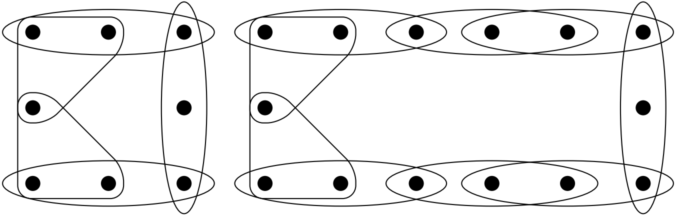


Hypergraph Cycles

Let C^r_{2k+1} be the r-uniform cycle with $2k+1$ edges formed by arranging $rk+(r-1)$ vertices in a circle and arranging edges to contain r consecutive vertices where the overlap between edges alternates between 1 and $r-1$.



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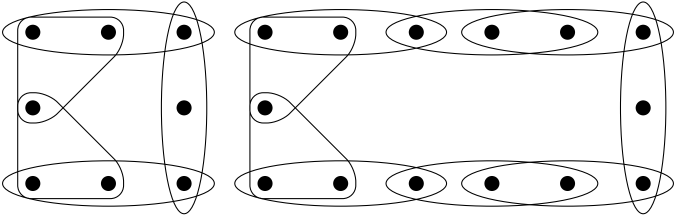

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The theorem is extended to a family of critical graphs.

Thank you