Mantel's Theorem for Random Hypergraphs

Jane Butterfield

University of Minnesota - Twin Cities
and
József Balogh, University of Illinois at Urbana-Champaign
Ping Hu, University of Illinois at Urbana-Champaign
John Lenz, University of Illinois at Chicago

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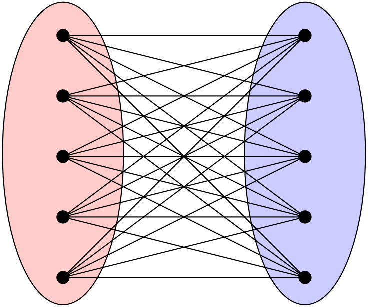
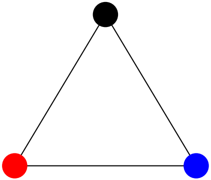
Avoiding triangles





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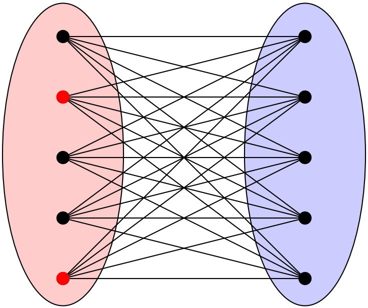
Every K_3-free graph on n vertices has at most $\lfloor \frac{n^2}{4} \rfloor$ edges.



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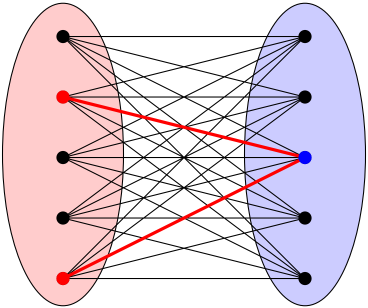

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Theorem (DeMarco, Kahn (2013+))

If $p > K\sqrt{\log(n)/n}$ then with high probability every maximum K_3-free subgraph of $G(n,p)$ is bipartite.

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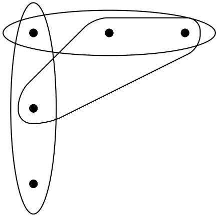

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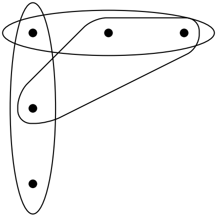

Example: F_5 is a 3-uniform hypergraph.

An r-uniform hypergraph is r-partite if we can partition the vertices into r sets such that every edge contains exactly one vertex from each set.

Example: F_5 is not tripartite.

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We seek a sparse Mantel-like result for F_5.

Main result

Theorem (Balogh-B-Hu-Lenz (2013+))

There exists a positive constant K such that if $p > K\log(n)/n$ then w.h.p. every maximum F_5-free subhypergraph of $G^3(n,p)$ is tripartite.

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Note: we know the *structure*, not only *size*.

Proof idea

Recent results of Conlon–Gowers (2013+), Schacht (2013+), Balogh–Morris–Samotij (2013+), Saxton–Thomason (2013+) and Samotij (2012) lead to...

Theorem

For every $\delta > 0$ there exist positive constants K and ϵ such that if $p \geq K/n$, then w.h.p. the following holds. Every F_5-free subgraph of $G^3(n,p)$ with at least $(2/9 - \epsilon)\binom{n}{3}p$ edges admits a partition (V_1, V_2, V_3) of $[n]$ such that all but at most $\delta n^3 p$ edges have one vertex in each V_i.

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- In an "optimal" partition into three parts, not many cross-edges.
- Want to show that there are *no* cross-edges.

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Note: $|\mathcal{F}| \geq t(\mathcal{G})$.

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Lemma

There exist positive constants K and δ such that if $p > K\log(n)/n$ then the following is true. Let $\mathcal{F}$ be an F_5-free subhypergraph of $\mathcal{G}$ and $\Pi = (A_1, A_2, A_3)$ be a balanced 3-partition maximizing $|\mathcal{F}[\Pi]|$. Let $\mathcal{B}_i = \{e \in \mathcal{F} : |e \cap A_i| \geq 2\}$. If the following conditions hold then w.h.p.

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(i) $\sum_i |\mathcal{B}_i| \leq \delta p n^3$,

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(iii) *the shadow graph of $\mathcal{B}_1$ is disjoint from $\mathcal{Q}(\Pi)$.*

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A hiccough

Unfortunately, we cannot prove that $Q(\Pi) = \emptyset$ for all balanced Π.
We *can* prove that if $Q(\Pi) \neq \emptyset$ then Π is far from maximal:

Lemma

There exist positive constants K and δ such that if $p > K \log(n)/n$, Π is a balanced 3-partition, and $\mathcal{Q}(\Pi) \neq \emptyset$, then w.h.p.

$$t(\mathcal{G}) > |\mathcal{G}[\Pi]| + |\mathcal{Q}(\Pi)|\delta n^2 p^2.$$

Proof: similar to the proof of Lemma 5.2 in DeMarco-Kahn.
We can conclude that $t(\mathcal{G}) > |\mathcal{G}[\Pi]| + |\mathcal{Q}(\Pi)|\delta n^2 p^2$. for every balanced 3-partition Π.

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|\tilde{\mathcal{F}}| &\leq |\tilde{\mathcal{F}}[\Pi]| + 3|\tilde{\mathcal{B}}_1| \\
&= |\mathcal{F}[\Pi]| + 3|\mathcal{B}_1| + 3|\tilde{\mathcal{F}} \cap B(\Pi)| \\
&\leq |\mathcal{G}[\Pi]| + 3|\mathcal{B}(\Pi)| \text{– first key lemma} \\
&\leq |\mathcal{G}[\Pi]| + 3 \cdot 2|Q(\Pi)|np \text{ – Chernoff} \\
&\leq |\mathcal{G}[\Pi]| + |Q(\Pi)|\delta n^2 p^2 \\
&\leq t(\mathcal{G}).
\end{aligned}$$

- Let $\tilde{\mathcal{F}}$ be a maximum F_5-free subhypergraph of $\mathcal{G}$.
- Let Π be a balanced 3-partition maximizing $\tilde{\mathcal{F}}[\Pi]$.
- Let $\mathcal{B}(\Pi) = \{e \in \mathcal{G} : \exists (u, v) \in \mathcal{Q}(\Pi) \text{ s.t. } u, v \in e\}$.
- Let $\mathcal{F} = \tilde{\mathcal{F}} \setminus \mathcal{B}(\Pi)$.
- ...we can show that $\mathcal{F}$ and Π satisfy the conditions of the key lemmas, so:

$$\begin{aligned}
|\tilde{\mathcal{F}}| &\leq |\tilde{\mathcal{F}}[\Pi]| + 3|\tilde{\mathcal{B}}_1| \\
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&\leq |\mathcal{G}[\Pi]| + |Q(\Pi)|\delta n^2 p^2 \\
&\leq t(\mathcal{G}). \text{ – second key lemma}
\end{aligned}$$

- But $|\tilde{\mathcal{F}}| \geq t(\mathcal{G})$, so equality holds throughout.

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- But $\mathcal{B}_1 \neq \emptyset$ contradicts one of the equalities.
- Also, $\mathcal{Q}(\Pi) \neq \emptyset$ contradicts one of the equalities.
- It follows that $\tilde{\mathcal{B}_1} = \emptyset$.
- Since $\tilde{\mathcal{B}_1} \geq \tilde{\mathcal{B}_2}, \tilde{\mathcal{B}_3}$, it follows that $\tilde{\mathcal{F}}$ is tripartite.

Future work

Conjecture

The right threshold is probably $p = K\sqrt{\log(n)}/n$.

Idea: if $p > 0.01\sqrt{\log(n)}/n$ then w.h.p. we can find a copy of K_4^- whose edges are not contained in any F_5.

Thank you.

butter@umn.edu