The Game of Revolutionaries and Spies

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- All graphs in this talk will be simple and undirected.
- For a vertex $v \in V(G)$, let $N(v)$ denote its neighborhood.
- K_n is the complete graph on n vertices, C_n the cycle on n vertices, and $K_{n,m}$ is the complete bipartite graph having partite sets size n and m.

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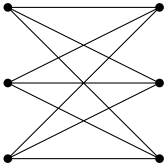
$G = K_{3,3},\ r = 4,\ s = 1,\ m = 2.$



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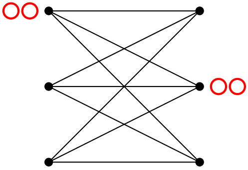

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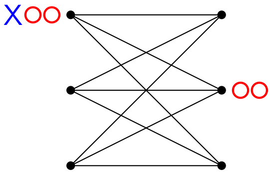

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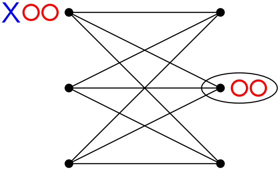

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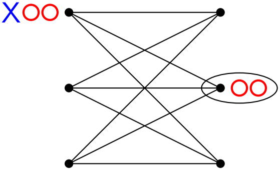

Fact

If $s < \lfloor \frac{r}{m} \rfloor \leq |V(G)|$ then spies lose in Round 0.

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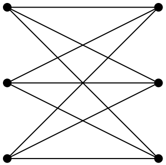

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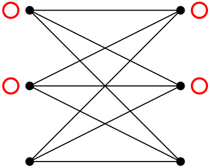

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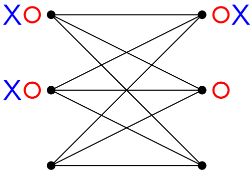

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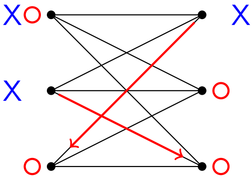

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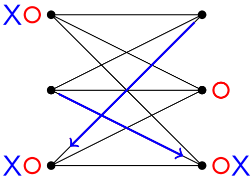

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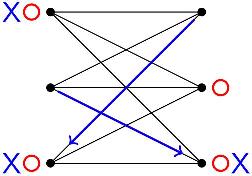

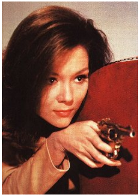

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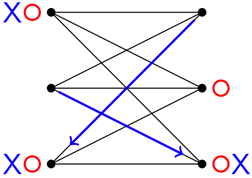

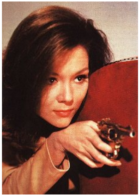

Fact

If $s \geq r - m + 1$ then spies will never lose.

Bias

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I don't really support the revolutionaries.

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I usually root for the spies

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For a given G, r, s, m we say that

- spies win if there exists a strategy for Player 2 by which she can prevent Player 1 from ever winning.
- spies lose if there exists a strategy for Player 1 by which he will win after a finite number of rounds.

Introduced by Jozef Beck

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- *When G is a tree with more than s vertices, spies lose if and only if $s \leq \frac{r}{m} - 1$.*
- *When G is the infinite grid and $m = 2$, spies lose if $s \leq \frac{3}{4}(r-1)$, but spies win if $s \geq r - 2$.*

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How many spies are needed to win? We already know $s \geq \lfloor \frac{r}{m} \rfloor$ spies are necessary, and $r - m + 1$ spies suffice.

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What graphs are good for spies or good for revolutionaries?

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Call the partite sets V_1 and V_2.

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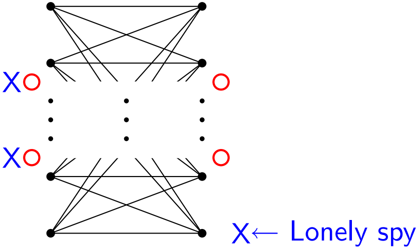

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- Suppose at the end of Round j there are ℓ_i lonely spies in V_i.
- Unless $s_i \geq \lfloor \frac{r - s_{3-i} + \ell_{3-i}}{2} \rfloor$ for $i \in [2]$, the revolutionaries can win in Round $j + 1$.

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- Now $\ell_1 \geq r/2 - s_2$; say $\ell_1 = r/2 - s_2$ (to simplify proof).
- To avoid losing in Round 2, spies need

$$s_1 \geq \lfloor \frac{r - s_2}{2} \rfloor \text{ and } s_2 \geq \lfloor \frac{r - s_1 + \ell_1}{2} \rfloor.$$

Which together imply that $5s \geq \frac{7}{2}r - 3$, and so $s \geq \frac{7}{10}r - 3/5$.

Complete bipartite graphs, cont.

Theorem (B-C-P-W-Z (2011+))

For $s, r \in \mathbb{N}$ *and* $G = K_{n,n}$*, where* $n \gg s + r$*:*

Complete bipartite graphs, cont.

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For $s, r \in \mathbb{N}$ and $G = K_{n,n}$, where $n \gg s + r$:

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- *We conjecture that $s \geq \frac{3}{2}\frac{r}{m}$ spies suffice.*
- *If $r \geq s\left(\frac{m}{2} + \frac{\lceil m/3 \rceil}{2}\right) + 2m$, spies lose.*

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If G has a dominating vertex then when $s > \frac{r}{m} - 1$, spies win.

Player 2 will keep any “off-duty” spy (one who isn’t currently covering a meeting) on the dominating vertex. Can show that she then has enough off-duty spies at any time to cover any future meetings.

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Corollary (B-C-P-W-Z (2011+))

Fix $n > r$. Then for every $0 \leq m \leq \binom{n}{2}$, there exists a graph $G(i)$ having n vertices and i edges such that if $s \geq \frac{r}{m} - 1$ then spies win the game played on $G(i)$.

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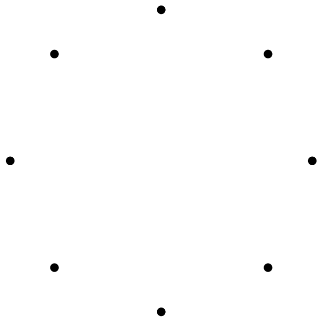

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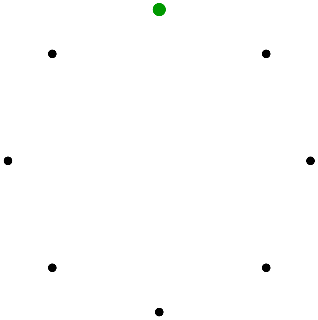

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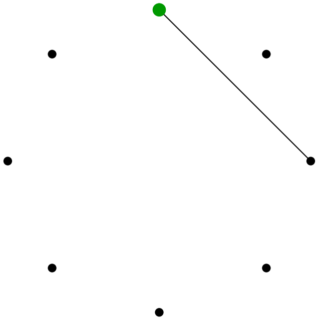

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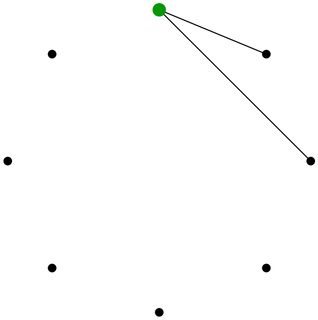

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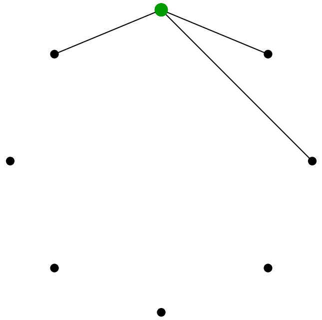

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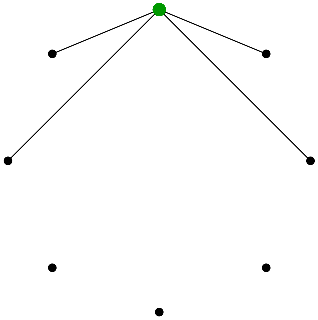

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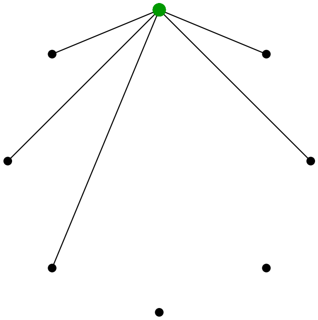

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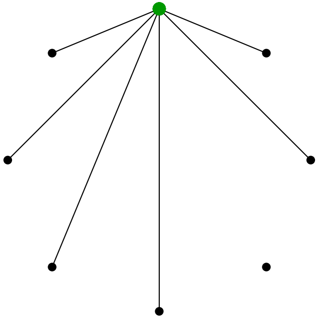

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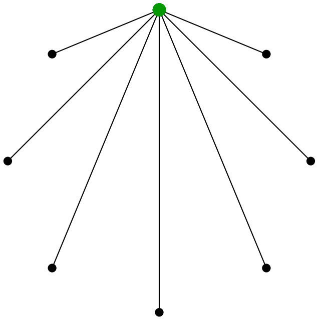

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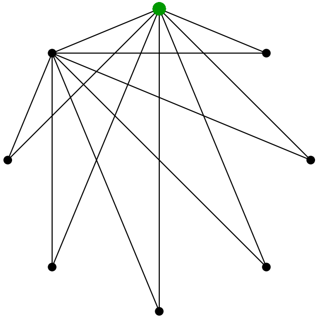

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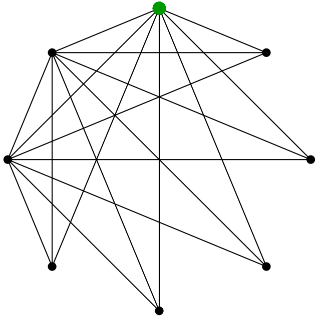

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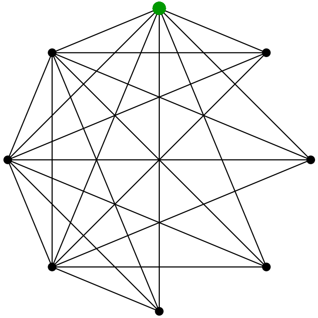

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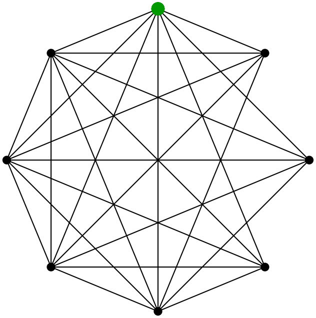

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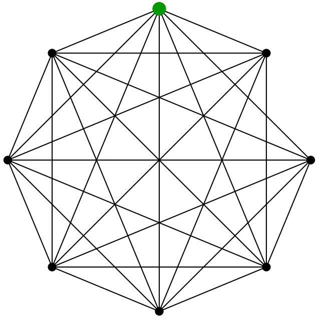
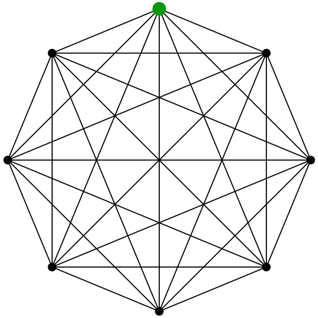


$G(i)$



Graph parameters, cont.

A graph is unicyclic if it contains at most one cycle.

Theorem (B-C-P-W-Z (2011+))

If G is a cycle and $s \geq r/m$, then spies win.
If G is a cycle of length ℓ and $r/m > s > r/m - 1 \geq 0$, then spies lose if and only if $\ell > s + 2$.

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Suppose G contains exactly one cycle, C_ℓ, and $|V(G)| - \ell = t$. If $s + 1 > r/m > s \geq 1$ then spies lose if and only if $\ell \geq \max\{s - t + 3, 4\}$.

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- We have started to consider $K_{n,n,n}$. We know that as $k \to \infty$ the complete k-partite graph with parts of size n (for $n > s, r$) becomes good for spies (i.e. $s = \frac{r}{m}$ spies suffice to win.)