Online Ramsey Games for Triangles

Jane Butterfield

University of Minnesota - Twin Cities
and József Balogh, University of Illinois at Urbana-Champaign

September 21st, 2012

Graph theory terms

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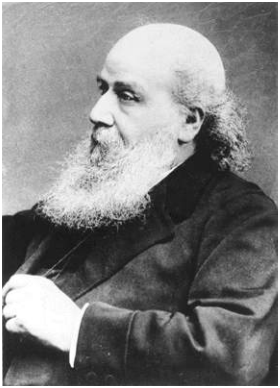

James Joseph Sylvester

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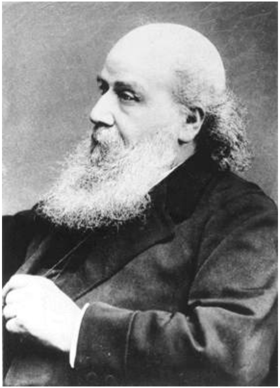

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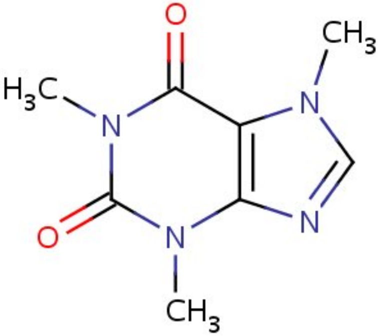

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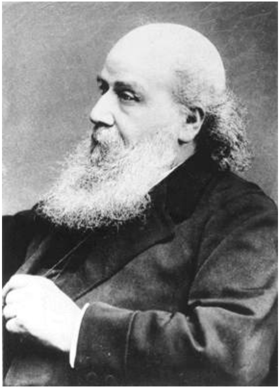

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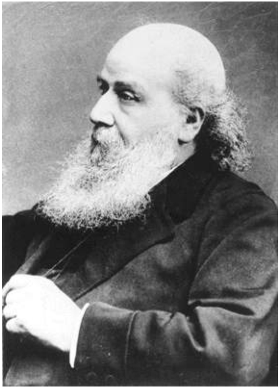

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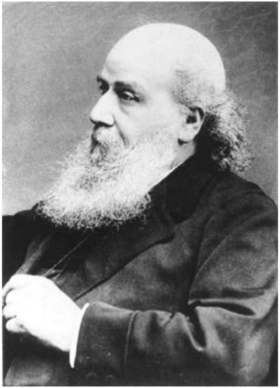

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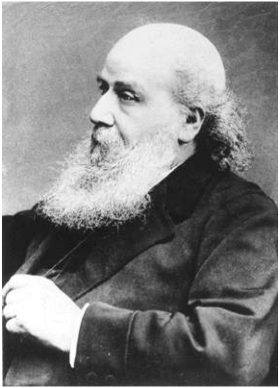

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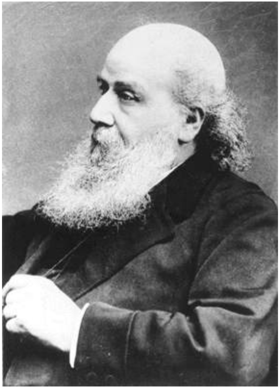

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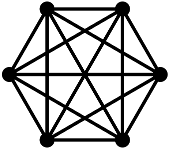
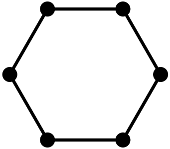
Some important graphs



K_6



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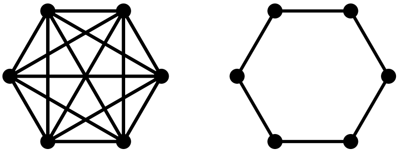

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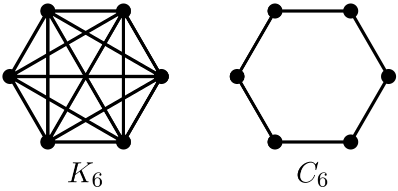

- The complete graph, K_ℓ, has ℓ vertices, and every two vertices form an edge.
- The cycle, C_ℓ, has vertices $v_1, \ldots, v_\ell$, and $v_i v_j$ is an edge iff $|i - j| = 1 (\text{mod } \ell)$.

Ramsey theory

Frank Plumpton Ramsey

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"How big does something have to be for some particular structure to show up inside it?"

- How large can n be before any 2-coloring of $[n]$ contains a monochromatic arithmetic progression of length ℓ?
- How many vertices can K_n have before any 2-coloring of its edges induces a monochromatic copy of K_ℓ?

Three Ramsey games

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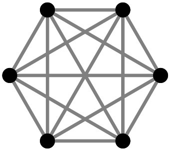

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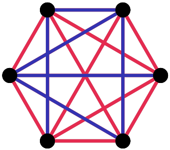

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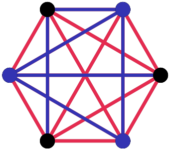

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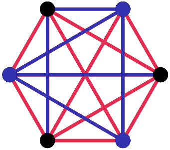

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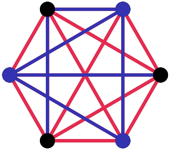

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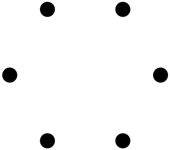

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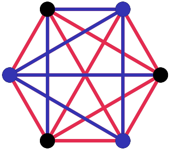

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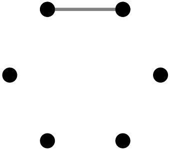

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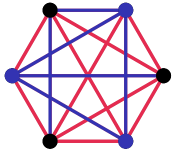

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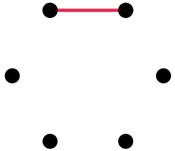

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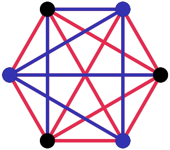

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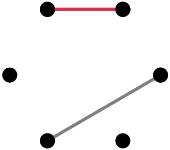

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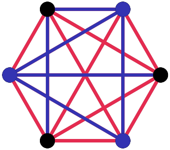

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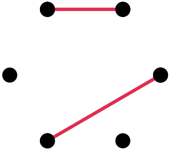

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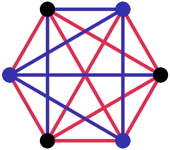

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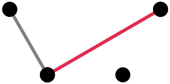

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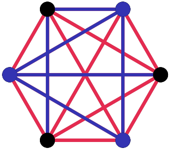

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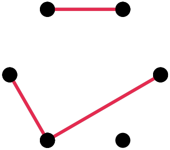

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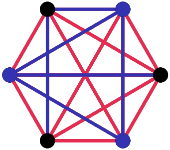

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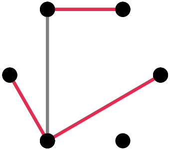

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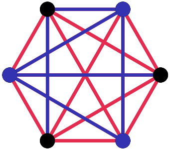

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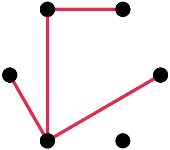

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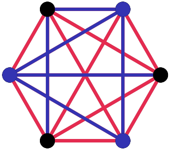

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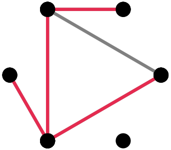

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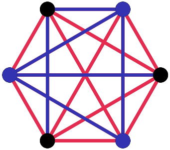

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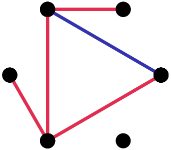

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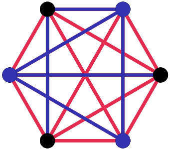

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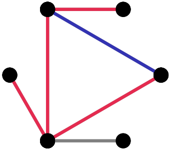

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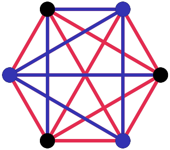

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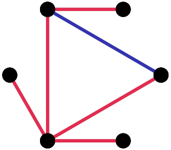

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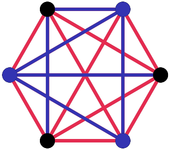

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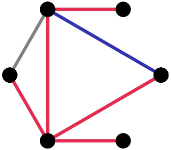

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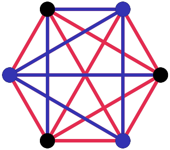

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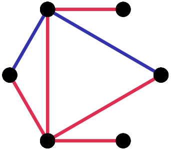

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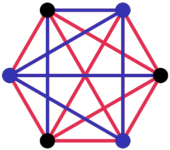

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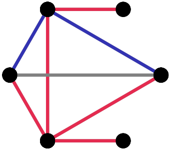

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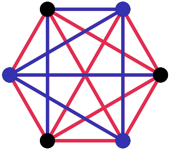

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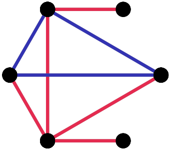

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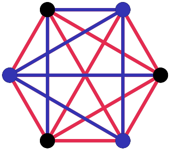

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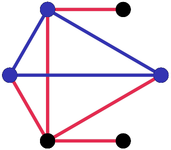

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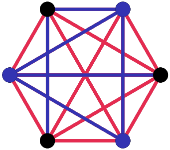

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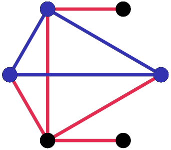

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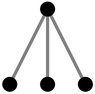

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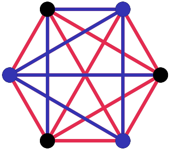

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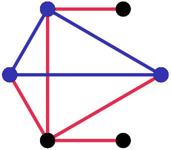

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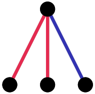

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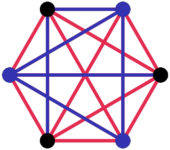

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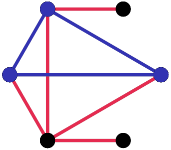

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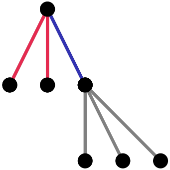

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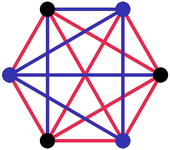

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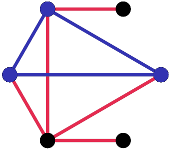

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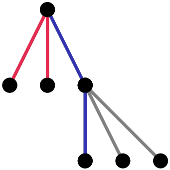

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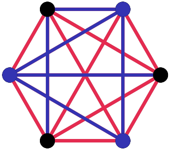

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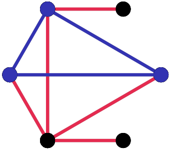

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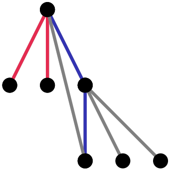

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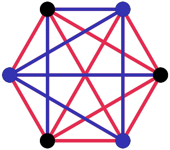

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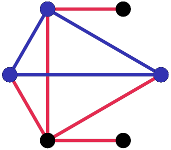

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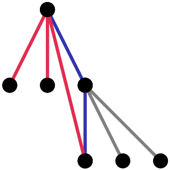

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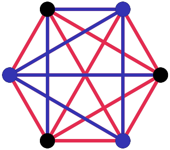

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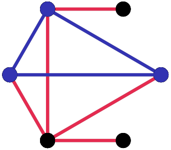

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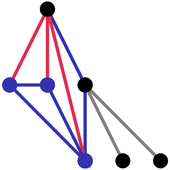

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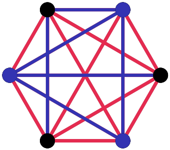

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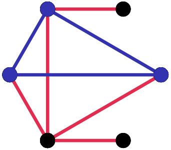

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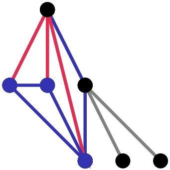
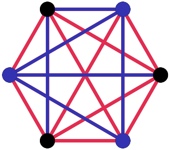


On-line two-player Ramsey game Builder presents edges one-at-a-time, and Painter must color each as it appears. The underlying graph must be in some family $\mathcal{F}$.

Three Ramsey games



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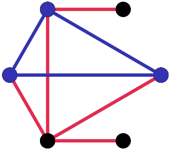

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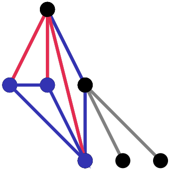

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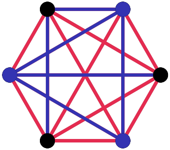

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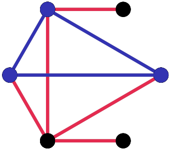

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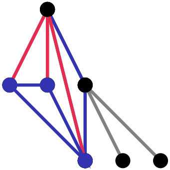

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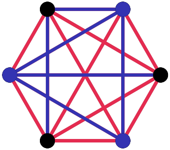
Three Ramsey games



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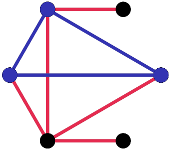

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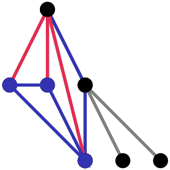

On-line two-player Ramsey game For which families can Painter avoid creating a monochromatic triangle?

Three Ramsey games

Classical Ramsey Theory:

$2^{\ell/2} < R_2(K_\ell) < 2^{2\ell}$.

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Example family: $\mathcal{F}_\Delta = \{G : \Delta(G) \leq \Delta\}$.	**On-line two-player Ramsey game** For which families can Painter avoid creating a monochromatic triangle?

The one-player game

Definition (Threshold)

We will call $N_0(F, r, n)$ a *threshold* for the r-color on-line one-player F-avoidance game if there are positive numbers c and C such that when $N < cN_0(F, r, n)$ there exists a strategy such that Painter almost surely wins the game played with N edges by following the strategy, and when $N > CN_0(F, r, n)$ Painter almost surely loses the game played with N edges.

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Theorem (Marciniszyn-Spöhel-Steger (2009))

For every graph F and integer $r > 0$ the threshold $N_0(F, r, n)$ exists.

Simple strategies in the one-player game

- The greedy strategy: Painter orders the colors, and always uses the least-indexed color that will not close a monochromatic copy of F.

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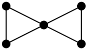
  - For $F =$  and $r = 2$, Marciniszyn-Spöhel-Steger proved this strategy is not optimal.

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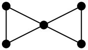
  - For $F =$  and $r = 2$, Marciniszyn-Spöhel-Steger proved this strategy is not optimal.
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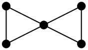
  - For $F =$  and $r = 2$, Marciniszyn-Spöhel-Steger proved this strategy is not optimal.
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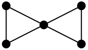
  - For $F =$  and $r = 2$, let F_1 be a triangle and let $F_2 = F$. Gives a better bound than greedy.

Known bounds for $N_0(F, r, n)$

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The parameter in the lower bound is some sort of density function, defined recursively.

$F = K_k$ or C_k

Conjecture (Marciniszyn-Spöhel-Steger 2009)

For cliques and cycles, the known lower bound is sharp. In particular, for all $r \geq 1$ and $k \geq 2$,

$$N_0(K_k, r, n) = n^{\left(2-\frac{2}{k+1}\right)\left(1-\binom{k}{2}^{-r}\right)}$$

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Marciniszyn-Spöhel-Steger proved true for $r = 2$, **remains open** for $r \geq 3$.

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Theorem (Belfrage, Mütze, Spöhel (2010+))

Fix any integer $r \geq 2$ and any real number $d > 0$. If Builder has a winning strategy in the on-line r-color Ramsey game $(F, \mathcal{H}_d)$, then the threshold for the r-color on-line one-player Ramsey game satisfies

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- In fact, given a Builder strategy in $(F, \mathcal{H}_d)$, the random graph almost surely reproduces his strategy in the first N steps!
- If there is a winning Builder strategy, therefore, Painter will almost surely lose after N steps in the one-player random game.

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$$N_0(K_3, 2, n) = n^{\frac{4}{3}}, \tag{1}$$

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$$n^{\frac{3}{2}-\frac{1}{18}} \leq N_0(K_3, 3, n) \leq n^{\frac{3}{2}}.$$

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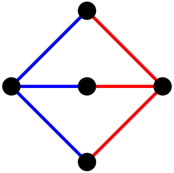

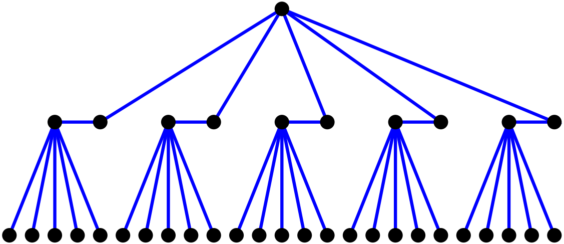

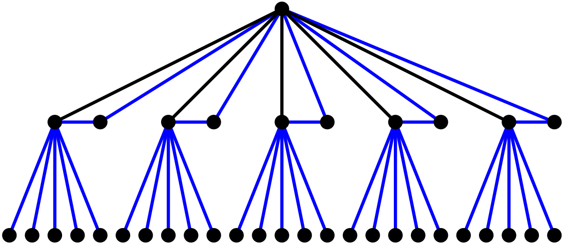

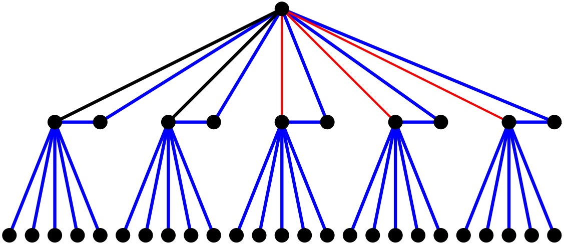

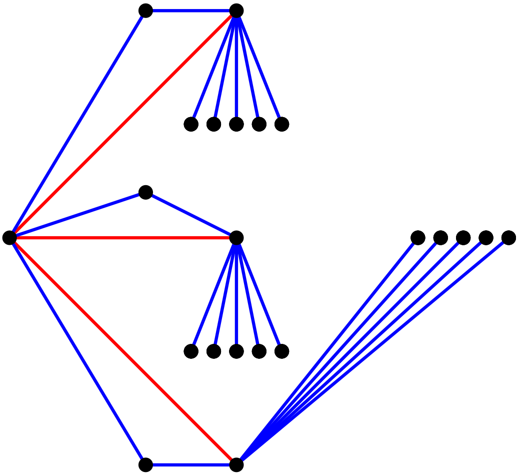

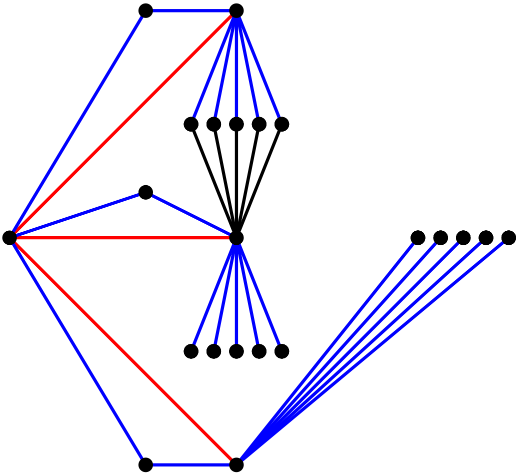

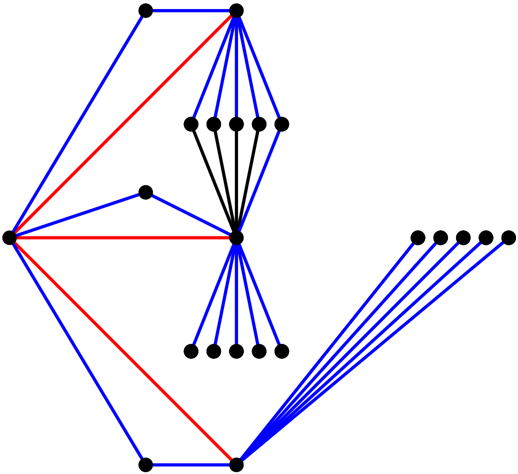

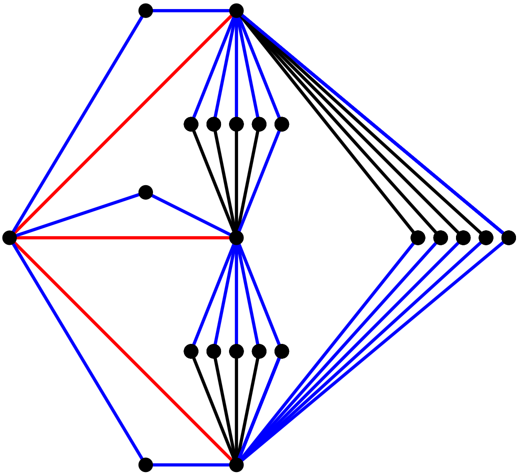

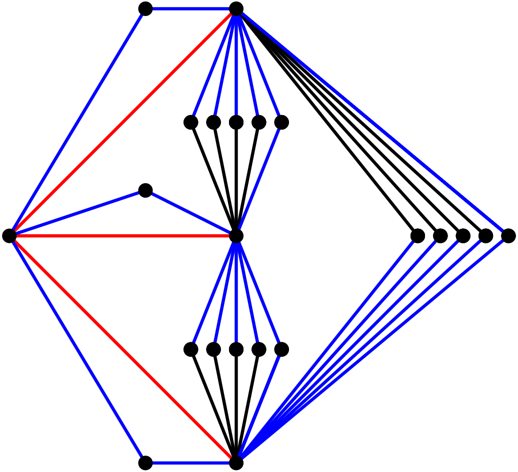

Vertices: 22
Edges: 42 (once final triangle is placed)

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UMN undergraduate research question.

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...or cast doubt on the conjecture or the method.

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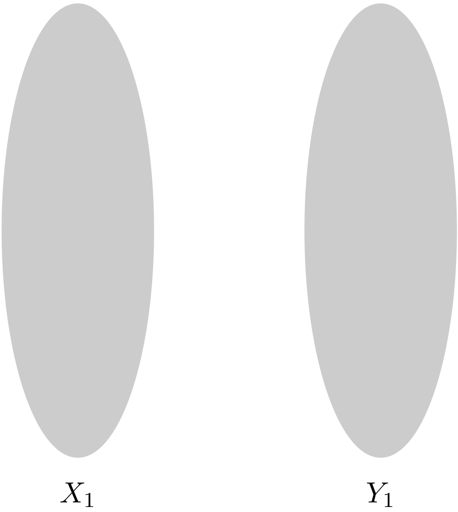
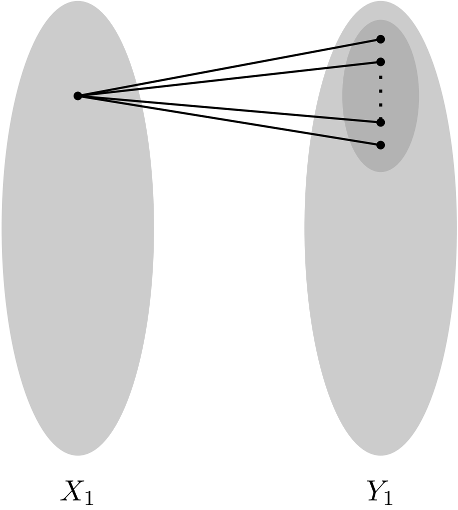
- Graph Ramsey Theory: for any n_r, there exists m_r large enough that any r-coloring of the edges of K_{m_r} results in a monochromatic copy of K_{n_r}.



X_1
Y_1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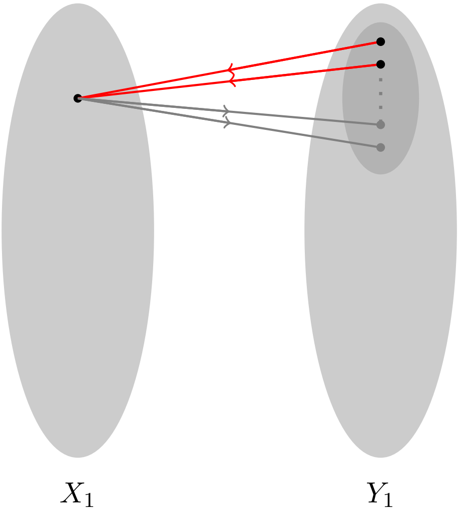
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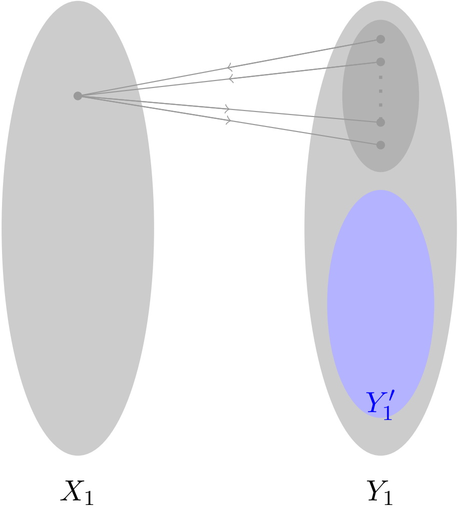
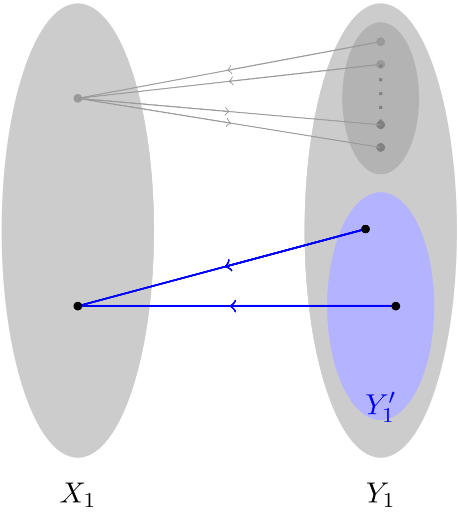

Y_1'
X_1
Y_1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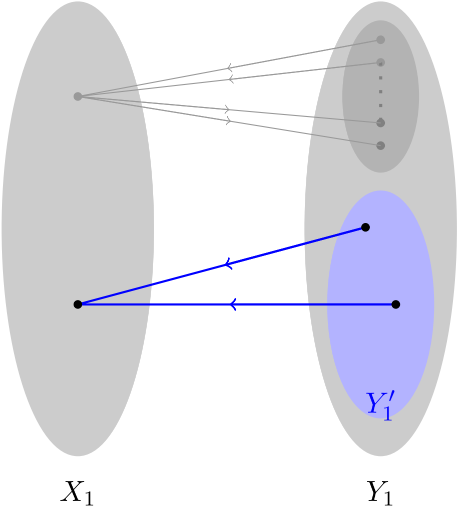

If there are n_{r-1} vertices in Y_1', Builder can use Strategy S_{r-1}; fixing $X_2, Y_2 \subseteq Y_1'$.
Density: every subgraph has average out-degree strictly less than 2.

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Thank you!

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