CRITERIA FOR $\bar{d}$-CONTINUITY

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11th October 2006

Abstract. Bernoullicity is the strongest mixing property that a measure-theoretic dynamical system can have. This is known to be intimately connected to the so-called $\bar{d}$ metric on processes introduced by Ornstein. In this paper, we consider families of measures arising in a number of contexts and give conditions under which the measures depend $\bar{d}$-continuously on the parameters. At points where there is $\bar{d}$-continuity, it is often straightforward to establish that the measures have the Bernoulli property.

Introduction

One of the most standard examples of a measure-preserving transformation in ergodic theory is that of a Bernoulli shift. To construct these, one starts with a finite set S equipped with a measure m and defines the shift transformation T on the space $X = S^{\mathbb{Z}}$ equipped with the measure $m^{\mathbb{Z}}$, the direct product of copies of m indexed by the integers. These transformations model the simple situation which occurs in probability theory of having a sequence of independent identically distributed random variables (Bernoulli trials). For this reason, Bernoulli shifts are often referred to as the most random possible dynamical systems.

A fundamental question which often occurs in ergodic theory is when a pair of measure-preserving transformations are measure-theoretically isomorphic. These questions are known to be hard, and until the mid 1970s, the results on this were few. It then came as a great surprise when Ornstein showed in a series of papers (collected together as [11] - see also [16]) that there is a very simple classification of Bernoulli shifts (namely that two Bernoulli shifts are isomorphic if and only if they have the same entropy) and further that gave verifiable conditions to check whether other systems are also isomorphic to Bernoulli shifts. This resulted in an explosion of work, showing that a large number of 'naturally occurring' measure-preserving

1991 *Mathematics Subject Classification.* 28D05, 60G10.
Key words and phrases. Bernoulli, coupling, g-measure.

systems were isomorphic to Bernoulli shifts. When this is the case, it is often said that the system has the Bernoulli property. Some of the results in this paper fall into this category.

A convenient way of establishing Bernoullicity in our context is provided by means of the $\bar{d}$ distance introduced by Ornstein. A useful description of this offered by joinings (see Rudolph's book [14] for details). A joining of (X, ν_1) and (X, ν_2) is a shift-invariant measure on $X \times X$ with the property that $\mu(A \times X) = \nu_1(A)$ and $\mu(X \times B) = \nu_2(B)$ for any measurable subsets A and B of X. Writing $J(\nu_1, \nu_2)$ for the set of joinings of (X, ν_1) and (X, ν_2), the $\bar{d}$ distance is given by

$$\bar{d}(\nu_1, \nu_2) = \inf_{\mu \in J(\nu_1, \nu_2)} \int \delta(x_1, x_2)\ d\mu(x_1, x_2),$$

where $\delta(x, y)$ is 1 if the zeroth coordinates of x and y agree, and 0 otherwise. We will make use of this definition in the first section of this paper. The attraction of the $\bar{d}$ metric is that many ergodic properties are particularly well-behaved with respect to this metric. In particular, the set of processes which are isomorphic to Bernoulli shifts is $\bar{d}$-closed (see [11]). This will let us show that certain systems have the Bernoulli property by showing that they can be arbitrarily well approximated in the $\bar{d}$ metric by processes which have the Bernoulli property. It is important to note that Ornstein's results upon all of which this is based apply only to invertible dynamical systems. Thus, when applied to systems which are inherently one-sided, the above machinery will yield conclusions about their natural extensions. This is all that one can realistically expect in almost all cases, as the property of being one-sided isomorphic to a Bernoulli shift is known to be very unusual (with such nice examples as Markov chains failing to have this property - see [12]).

In the first section of the paper, we will consider the $\bar{d}$ distance between certain g-measures and will be able to deduce Bernoullicity results as a corollary. The notion of a g-measure was introduced to ergodic theory by Keane in 1974 (see [8]), but has a long history in probability theory, where it is variously called a chain with complete connections and a uniform martingale. We will take a non-standard definition of g-measures which is equivalent to the more usual definition in the case where the function g is continuous. Let Σ denote the full shift space $\{0, \dots, k-1\}^{\mathbb{Z}^+}$. For $x \in \Sigma$, and $i \in \{0, \dots, k-1\}$, let ix denote the compound sequence defined by $(ix)_n = i$ if $n = 0$ and x_{n-1} otherwise. The function g will be called a g-function if g is a function from Σ to $(0, 1)$ and $\sum_{i=0}^{k-1} g(ix) = 1$. We will only consider those g which are continuous here. The cylinder of those points which agree with x for the first $n+1$ places (that is the set $\{y\colon y_i = x_i,\ \forall i \leqslant n\}$) will be denoted by $[x]^n$. The measure μ is a g-measure if

$$\lim_{n\to\infty} \frac{\mu([ix]^{n+1})}{\mu([x]^n)} = g(ix)$$

for all $x \in \Sigma$.

For continuous g-functions, it is easy to show that there is at least one g-measure. In the case where g is Hölder continuous, the g-measure is known to be unique and to have a Bernoulli natural extension. This statement also holds if the variations of g are summable (where $\mathrm{var}_n g = \sup_{\{x,y\colon x_i = y_i,\ \forall i<n\}} |g(x) - g(y)|$) (see [17] for details of the above statements). One of the weakest conditions known which guarantees

uniqueness of g-measures was provided by Berbee (see [1]). Berbee also proves Bernoullicity. Our conditions are very similar, but slightly weaker in some cases and slightly stronger in others. A very minor modification shows that our results remain valid under Berbee's condition. It was for a long time an open question whether or not for every continuous g-function there is a unique g-measure, but this was settled recently by Bramson and Kalikow (see [3]) who constructed an example of a continuous g-function having two distinct g-measures.

The interpretation which we will use for g-measures is that $g(ix)$ is a Markov transition probability, giving the probability of moving from x to ix. A g-measure is then an invariant measure for this Markov process. For details of this interpretation, the reader is referred to [13]. A closely related description is that a g-measure is a stationary distribution for a sequence $(X_n)_{n\in\mathbb{Z}}$ of random variables taking values in S satisfying

$$\mathbb{P}(X_n = i | X_{n-1} = x_1, X_{n-2} = x_2, \dots) = g(ix).$$

The second section of the paper gives a condition for uniqueness of equilibrium states. For a continuous function ϕ defined on a two-sided shift space X, an equilibrium state for ϕ is an invariant measure maximizing the quantity $\int \phi\, d\mu + h_\mu(T)$ where T is the shift map. By Walters ([17]), it is known that if ϕ is a one-sided function (that is one dependent only on those x_i with $i \geqslant 0$) with the property that $\sum_n \mathrm{var}_n\phi < \infty$ then there is a unique equilibrium state for ϕ whose natural extension is Bernoulli. Extending the definition of variation to functions which are two-sided by defining $\mathrm{var}_n\phi = \sup\{|\phi(x) - \phi(y)|\colon x_i = y_i\, \forall |i| \leqslant n\}$ and using the technique of Bowen ([2]) by which to each two-sided function with summable variation, one associates a one-sided function, one may check that if the two-sided function ϕ satisfies $\sum n\mathrm{var}_n\phi < \infty$, then ϕ has a unique equilibrium state which is Bernoulli. Moreover, if ϕ satisfies $\sum \mathrm{var}_n\phi < \infty$, then ϕ has a unique equilibrium state which is known to have the K property. This suggests the conjecture that in fact the condition $\sum \mathrm{var}_n\phi < \infty$ is sufficient to ensure Bernoullicity of the unique equilibrium state. This turns out to be the case as we will show in Theorem 2.

In the final section, we will consider Gibbs measures. These were introduced by Dobrushin, Lanford and Ruelle (and are thus sometimes called DLR measures). For a detailed general reference, the reader is referred to [6].

We will consider exclusively Gibbs measures where the index set is the one-dimensional lattice $\mathbb{Z}$. The state space at each site is a fixed set S and so the configuration space or state space for the whole system is $S^{\mathbb{Z}}$. The Gibbs states are defined approximately by assigning energy $H(x)$ to configurations and defining a measure which is in some sense proportional to $\exp(-H(x))$ to the state x.

To be more precise, the energy of a configuration is defined by specifying the contribution to the total energy of each finite part of the configuration. Given Λ, a finite subset of $\mathbb{Z}$, Φ_Λ is a function of $S^{\mathbb{Z}}$ which is dependent only on $\{x_i\colon i \in \Lambda\}$. In a given state x, the contribution to the energy due to the interaction of x_i with itself and the other sites is given by $\sum_{i\in\Lambda} \Phi_\Lambda(x)$. This series is required to be summable, so the requirement that $\sum_{i\in\Lambda} \|\Phi_\Lambda\|_\infty < \infty$ is imposed. From this it follows that given any finite set Λ, the sum $\sum_{\{\Lambda'\colon \Lambda'\cap\Lambda\neq\emptyset\}} \|\Phi_{\Lambda'}\|_\infty < \infty$. The collection (Φ_Λ) is called a Gibbs interaction potential. The potential is called translation-invariant if it satisfies $\Phi_\Lambda(x) = \Phi_{\Lambda-1}(Tx)$ where $\Lambda - 1$ denotes the set $\{i-1\colon i \in \Lambda\}$ and T denotes the left-shift map. We will consider only translation-invariant interaction potentials in what follows.

A Gibbs state is then defined by specifying the conditional probabilities of the various configurations on S^Λ conditional on the external configuration:

$$\mathbb{P}(x_\Lambda | x_{\Lambda^c}) = \frac{\exp(-H_\Lambda(x_\Lambda x_\Lambda^c))}{\sum_{y_\Lambda} \exp(-H_\Lambda(y_\Lambda x_\Lambda^c))},$$

where

$$H_\Lambda(x) = \sum_{\{\Lambda' \colon \Lambda' \cap \Lambda \neq \emptyset\}} \Phi_\Lambda(x).$$

The question of whether for a given interaction potential, there exists a Gibbs state and whether it is unique is a classical problem of statistical mechanics. It can be shown that in all the cases which we consider, there is at least one Gibbs state. Dobrushin gave a condition which implies uniqueness (see [4]). A second condition which is known to imply uniqueness is that the 'interaction energy of two half-lines' is finite, that is

$$\sum_{\{\Lambda \colon \Lambda \cap (-\infty, 0) \neq \emptyset,\, \Lambda \cap [0,\infty) \neq \emptyset\}} \mathrm{var}(\Phi_\Lambda) < \infty \tag{1}$$

where $\mathrm{var}(\Phi_\Lambda) = \max(\Phi_\Lambda) - \min(\Phi_\Lambda)$. For a reference, see [6]. In Theorem 4, we show that this condition is sufficient to imply Bernoullicity of the unique Gibbs state.

Previous results giving conditions for Bernoullicity were given by Gallavotti (see [5]) and Ledrappier (see [9]). They showed that if the interaction potential satisfies $\sum_{\Lambda \ni 0} \|\Phi_\Lambda\|_\infty \,\mathrm{diam}(\Lambda) < \infty$ then there is a unique Gibbs state which is Bernoulli. The result which we prove below is an extension of their work.

Note that by Ruelle's work (see [15]), there is a close connection between Gibbs states and equilibrium states. This and a result of Walters on equilibrium states are the main ingredients of the proof in the final section.

Note that throughout the paper, the results are proven in the case of full shifts on a symbol set S. This is primarily for simplicity of exposition, and the results remain true in the case of subshifts of finite type. In the second section, it has been pointed out to us that the proof remains valid even in the case of a more general subshift.

1. $\bar{d}$ distances and g-measures

Let Σ be $\{0, \dots, k-1\}^{\mathbb{Z}^+}$, the one-sided full shift space on k symbols.

Theorem 1. *Suppose g is a continuous g-function on Σ with the property that*

$$\sum_{n=r}^{\infty} \prod_{i=r}^{n} (1 - a_i) = \infty \tag{2}$$

for some $r \geqslant 1$, where $a_i = \frac{k}{2}\mathrm{var}_i(g)$ and let ν_g be a g-measure. Then as $\|h-g\|_\infty \to 0$, $\bar{d}(\nu_g, \nu_h) \to 0$ where ν_h is any h-measure.

Proof. Let ν_g be a g-measure, let $\|h - g\|_\infty = \delta$ and let ν_h be an h-measure. We will then produce a joining of ν_g and ν_h to estimate the $\bar{d}$ distance between them.

Let $F\colon \{0,\dots,k-1\}\times\Sigma\times\Sigma \to (0,1)$ be defined by $F(i,x,y) = \min(g(ix),h(iy))$. Define $\Delta\colon \Sigma\times\Sigma \to [0,1)$ by $\Delta(x,y) = 1-\sum_i F(i,x,y)$ and note that Δ is bounded above by a constant α which is strictly smaller than 1. (We can show that Δ is bounded above by $1-\inf_x g(x)$). Writing X_{n-} for the sequence $X_{n-1}, X_{n-2}, \ldots \in \Sigma$ and defining Y_{n-} similarly, we define a transition probability by

$$\mathbb{P}(X_n = i, Y_n = i | X_{n-} = x, Y_{n-} = y) = F(i,x,y)$$

and

$$\mathbb{P}(X_n = i, Y_n = j | X_{n-} = x, Y_{n-} = y) = \frac{(g(ix) - F(i,x,y))^+(h(jy) - F(j,x,y))^+}{\Delta(x,y)}$$

when $i \neq j$ and a^+ denotes $\max(a,0)$. It may then be checked that the marginal transition probabilities of the two processes are correct. This transition probability defines a mapping of the distributions of $(X_i,Y_i)_{i<n}$ to the distributions of $(X_i,Y_i)_{i<n+1}$. Identifying each of these spaces with $\mathcal{P}(\Sigma\times\Sigma)$ (where $\mathcal{P}(X)$ denotes the space of probability measures on X), the transition probability gives rise to a continuous affine map of $\mathcal{P}(\Sigma\times\Sigma)$. One can verify that this evolution leaves invariant the set of measures whose marginals on the first and second coordinates are respectively ν_g and ν_h. Using the Schauder-Tychonoff theorem, it follows that there exists a joining μ of the measures ν_g and ν_h which is shift-invariant and has the property that the transition probabilities are given by the above formulae.

We proceed to get an upper estimate for $\bar{d}(\nu_g,\nu_h)$ by estimating $\int \delta(x,y)\, d\mu$. To do this, define a map π from $\Sigma\times\Sigma$ to $\mathbb{Z}^+$ sending the pair (x,y) to $\sup\{n \geqslant 0\colon x_i = y_i,\ \forall i < n\}$ and note that $\delta(x,y) = 1$ if and only if $\pi(x,y) = 0$. We then get a process $(Z_n)_{n\in\mathbb{Z}}$ taking values in $\mathbb{Z}^+$ by defining $Z_n = \pi(X_{n-},Y_{n-})$. To estimate $\int \delta(x,y)\, d\mu$, we see that it is sufficient to estimate $\int \chi_{[0]}\, d\mu'$ where μ' is the induced measure on the $\mathbb{Z}^+$-valued process. Note also that in the $\mathbb{Z}^+$-valued process, the only valid transitions from n are to 0 and $n+1$. We then estimate the transition probability of going from n to 0. We have that the transition probability is bounded above by $\sup_{\{x,y\colon [x]^{n-1}=[y]^{n-1}\}} \Delta(x,y)$. But we have $\Delta(x,y) \leqslant \alpha$ and also

$$\begin{aligned}
\Delta(x,y) &= 1 - \sum_i \min(g(ix),h(iy)) \\
&= \sum_i (g(ix) - h(iy))^+ \\
&\leqslant \sum_i (g(ix) - g(iy))^+ + (g(iy) - h(iy))^+ \\
&\leqslant \tfrac{k}{2}(\mathrm{var}_{n+1} g + \|g-h\|_\infty).
\end{aligned}$$

So we let $q_n = \min(\alpha, \frac{k}{2}(\mathrm{var}_{n+1} g + \|g-h\|_\infty))$. This is an upper bound for the transition probability of going from n to 0. This allows us to get an upper bound on the proportion of time the $\mathbb{Z}^+$-valued system spends in the 0 state, thus giving an estimate on the $\bar{d}$ distance from ν_g to ν_h. We have seen that the proportion of time spent in the 0 state for the $\mathbb{Z}^+$-valued process is bounded above by the proportion of time spent in the 0 state for the Markov chain with probabilities calculated above. It is a straightforward calculation that for this Markov chain, the limiting proportion of time spent in the 0 state is given by

$$\begin{aligned} p_0 &= \left(1+\sum_{n=1}^{\infty}\prod_{i=1}^{n}(1-q_i)\right)^{-1} \\ &= \left(1+\sum_{n=1}^{\infty}\prod_{i=1}^{n}\min(1-\alpha, 1-a_i-\tfrac{k}{2}\|g-h\|_\infty)\right)^{-1}. \end{aligned}$$

One then checks that since we have monotone convergence that as $\|g-h\|_\infty$ converges to 0,

$$p_0 \to \left(1+\sum_{n=1}^{\infty}\prod_{i=1}^{n}\min(1-\alpha, 1-a_i)\right)^{-1}.$$

We therefore have continuity of the $\bar{d}$ distance provided that

$$\sum_{n=1}^{\infty}\prod_{i=1}^{n}\min(1-\alpha, 1-a_i) = \infty.$$

This is easily seen to be equivalent to (2). □

Note that one can also derive the theorem under Berbee's conditions ([1]). It is then necessary to use a different bound for $\Delta(x,y)$. Namely one needs to use the result that $\Delta(x,y) \leqslant 1 - s_n + \|(h-g)/g\|$ where $s_n = \inf_{[x]^n=[y]^n} g(x)/g(y)$.

The theorem has the following corollary.

Corollary. *Suppose g is a continuous g-function which satisfies (2). Then μ_g is the unique g-measure and has a Bernoulli natural extension.*

Proof. Note that it is sufficient to show that any g-measure has a Bernoulli natural extension. Then since the g-measures form a compact convex set, the non-extreme points of which are not even ergodic, the uniqueness will follow.

Now, let ν_g be a g-measure and let h_n be a sequence of Hölder continuous g-functions converging to g uniformly. Then we have that the h_n-measures ν_{h_n} have Bernoulli natural extensions and by the theorem converge to ν_g in the $\bar{d}$ metric. Since the Bernoulli processes are closed in the $\bar{d}$ metric, it follows that ν_g has a Bernoulli natural extension as required. □

2. Bernoullicity of Equilibrium States

Theorem 2. *Suppose ϕ is a two-sided function satisfying $\sum \mathrm{var}_n\phi < \infty$, then there is a continuous one-sided function $\tilde{\phi}$ with the properties*

(1) $\sum \mathrm{var}_n\tilde{\phi} < \infty$;
(2) *$\tilde{\phi}$ is cohomologous to ϕ.*

It follows that the equilibrium state of ϕ is unique and is Bernoulli.

Proof. If ϕ is constant, the conclusion holds trivially, so we assume that ϕ is not constant. Since ϕ is necessarily continuous, it follows that for all sufficiently large i, there exist numbers n_i such that $\mathrm{var}_{n_i}(\phi) \leqslant 2^{-i}$, but $\mathrm{var}_{n_i-1}(\phi) > 2^{-i}$. Set n_0 and all undefined n_i to be 0. Let n_{i_0} be the first non-zero value. Note that since

$\mathrm{var}_n(\phi)$ is a decreasing sequence, it follows that $\sum \mathrm{var}_n(\phi) \geqslant \sum_{i=i_0}^{\infty}(n_i - n_{i-1})2^{-i} = \sum_{i=0}^{\infty} 2^{-(i+1)} n_i$. In particular, we see that

$$\sum_{i=0}^{\infty} n_i 2^{-i} < \infty. \tag{3}$$

Now let ϕ_i be defined by $\phi_i(x) = \inf_{\{y\colon y_{-n_i}^{n_i} = x_{-n_i}^{n_i}\}} \phi(y)$ for $i \geqslant i_0$ and $\phi_i(x) = A$ where $A = \inf_y \phi(y)$ otherwise. Then we see that ϕ_i is a sequence of functions which converges uniformly and monotonically to ϕ. Next set $\psi_i(x) = \phi_i(x) - \phi_{i-1}(x)$. Then we have $\phi(x) = A + \sum_{i \geqslant i_0} \psi_i(x)$. Further, since $\phi_{i-1}(x) \leqslant \phi_i(x) \leqslant \phi(x)$, we see $\|\psi_i\|_\infty \leqslant 2^{-(i-1)}$ when $i > i_0$.

We then define

$$\tilde{\phi}(x) = A + \sum_{i=i_0}^{\infty} \psi_i \circ T^{n_i}.$$

This is easily seen only to depend on $(x_i)_{i \geqslant 0}$, so is one-sided as required. It is also the uniform limit of continuous functions, so is itself continuous. We then get a bound on the variations by noting

(1) $\mathrm{var}_n(\psi_i \circ T^{n_i}) = 0$ for $n > 2n_i$;
(2) $\mathrm{var}_n(\psi_i \circ T^{n_i}) \leqslant 2^{-(i-1)}$ for $n \leqslant 2n_i$ when $i > i_0$.

Now summing, we have $\mathrm{var}_{2n_i+1}(\tilde{\phi}) \leqslant 2^{-(i-2)}$ for $i \geqslant i_0$. Now since the variation is decreasing, we see

$$\sum_{n=0}^{\infty} \mathrm{var}_n(\tilde{\phi}) \leqslant (1 + 2n_{i_0})(\|\psi_{i_0}\|_\infty + 2) + \sum_{i=i_0}^{\infty} 2(n_{i+1} - n_i)2^{-(i-2)}.$$

In particular, it follows from (3) that $\tilde{\phi}$ has summable variation as required. It therefore follows from Walters' theorem ([17]) that the equilibrium states for $\tilde{\phi}$ are unique and Bernoulli as required. It remains only to show that $\tilde{\phi}$ is cohomologous to ϕ. From this, it follows that the equilibrium states for ϕ and $\tilde{\phi}$ are the same (see [18]) To this end, define

$$F(x) = \sum_{i=i_0}^{\infty} \sum_{j=0}^{n_i - 1} \psi_i \circ T^j.$$

First, we note that F is the uniform limit of continuous functions: to see this, note that $\|\psi_i \circ T^j\|_\infty \leqslant 2^{-(i-1)}$ when $i > i_0$ so that $\sum_{i=i_0}^{\infty} \sum_{j=0}^{n_i-1} \|\psi_i \circ T^j\|_\infty \leqslant n_{i_0}\|\psi_{i_0}\|_\infty + \sum_{i=i_0+1}^{\infty} n_i 2^{-(i-1)}$. We then see that $\tilde{\phi} = \phi + F \circ T - F$. It follows that $\tilde{\phi}$ is cohomologous to ϕ as claimed and the result follows. □

The reader should note that the above construction can also be used to show that any two-sided potential has a continuous one-sided potential which is the limit of potentials cohomologous to the original two-sided potential.

We now use the earlier section to find sufficient conditions for the equilibrium state to vary $\bar{d}$-continuously on the potential. Set $A = \{\psi\colon \sum_n n\mathrm{var}_n\psi < \infty\}$ and $A_0 = \{\psi\colon P(\psi) = 0, \sum_n n\mathrm{var}_n\psi < \infty\}$. Equip the spaces A and A_0 with the metric d arising from the norm:

$$\|\phi\| = \|\phi\|_\infty + \sum_n n\mathrm{var}_n\phi.$$

Theorem 3. *The map $E: A \to M(X)$ sending a potential to its equilibrium state is continuous with respect to the metrics d on A and $\bar{d}$ on $M(X)$.*

Proof. First note that the map $\kappa: A \to A_0$ defined by $\kappa(\psi) = \psi - P(\psi)$ is continuous with respect to d since P is Lipschitz with respect to the uniform norm. Further, ψ and $\kappa(\psi)$ have the same equilibrium state, so it is sufficient to work with potentials having pressure 0.

We fix a potential ϕ in A_0 and note that an argument similar to that in Theorem 2 shows that since $\sum_n n\mathrm{var}_n\phi < \infty$ then ϕ is cohomologous to a one-sided potential $\tilde{\phi}$ which satisfies the same condition. We then observe that given a sequence of potentials ϕ_n which converge to ϕ in the metric d_A, one may choose one-sided potentials $\tilde{\phi}_n$ cohomologous to ϕ_n which converge to $\tilde{\phi}$ in this metric. Once again, the equilibrium states for ϕ_n coincide with those for $\tilde{\phi}_n$.

It is now clearly sufficient to show that the map sending a one-sided potential with pressure 0 to its equilibrium state is continuous. We now show this. Pick $M > 0$ and let B denote the collection of one-sided potentials ψ such that the pressure $P(\psi)$ is 0 and $\sum_n n\mathrm{var}_n\psi < M$. Then pick $\tau \in B$ and consider the Ruelle-Perron-Frobenius operator $\mathcal{L}_\tau: C(X) \to C(X)$ defined by $\mathcal{L}_\tau(f)(x) = \sum_i \exp(\tau(ix))f(ix)$. It is well known (see [17]) that such an operator has a simple eigenvalue of 1 (since the pressure was taken to be 0) with an eigenfunction h_τ. Further there is a measure ν_τ with the property that for any continuous function f, $\mathcal{L}_\tau^n(f) \to h_\tau \int f\, d\nu_\tau$ where the convergence is in the supremum norm. Another property of this operator which we will need is that it is power bounded: there exists a K such that $\|\mathcal{L}^n\|_\infty \leqslant K$ for all n.

For normalization, we have assumed that $\int h_\tau\, d\nu_\tau = 1$. The equilibrium state is given by $d\mu_\tau = h_\tau\, d\nu_\tau$. This is a g-measure with g-function $g_\tau = \exp(\tau)h_\tau/h_\tau \circ T$. By the proof of Bowen (see [2]), we see that the function h_τ has summable variation and in particular, we see that g_τ satisfies (2). Applying a similar argument to the above, we have that the equilibrium state for $\psi \in B$ is a g-measure with g-function g_ψ, defined by $g_\psi = \exp(\psi)h_\psi/h_\psi \circ T$ (where we normalize h_ψ by assuming $\int h_\psi\, d\mu_\tau = 1$). To check $\bar{d}$-closeness of μ_τ and μ_ϕ, it is therefore sufficient, by Theorem 1, to show that h_ψ depends continuously in the supremum norm on ψ.

To this end, note that $\psi \in B$ implies (by the proof of Bowen) that h_ψ satisfies $h_\psi(x)/h_\psi(y) \leqslant \exp\bigl(\sum_{n+1}^\infty \mathrm{var}_i(\psi)\bigr)$ when x and y coincide in the first n coordinates. From this, we deduce that $|h_\psi(x) - h_\psi(y)| \leqslant \|h_\psi\|_\infty |\exp(\sum_{n+1}^\infty \mathrm{var}_i\psi) - 1|$ which may in turn be bounded above (using the mean value theorem and the fact that $\|h_\psi\|_\infty \leqslant \exp M$) by $\exp(2M)\sum_{n+1}^\infty \mathrm{var}_i\psi$. Summing, we see that $\sum_{n=0}^\infty n\mathrm{var}_n\psi \leqslant M\exp(2M)$ for each $\psi \in B$. In particular, the h_ψ for ψ in B form a relatively compact set by the Arzelà-Ascoli theorem. Let C denote the closure (with respect to the supremum norm) of the collection of h_ψ for $\psi \in B$. Then C is compact and for each $\epsilon > 0$ and $f \in C$, there exists an $N > 0$ such that $\|\mathcal{L}_\tau^N f - h_\tau\|_\infty < \epsilon/K$. There then exists an open neighbourhood U of f in C such that for each $g \in U$, $\|\mathcal{L}_\tau^N g - h_\tau\|_\infty < \epsilon/K$. It then follows from the power-boundedness of $\mathcal{L}_\tau$ that $\|\mathcal{L}_\tau^n g - h_\tau\|_\infty < \epsilon$ for all $n > N$. Now by compactness, we see that there is an N such that for each $f \in C$, $\|\mathcal{L}_\tau^N f - h_\tau\|_\infty < \epsilon$.

We then have

$$\begin{aligned}\|h_\tau - h_\psi\|_\infty &\leqslant \|h_\tau - \mathcal{L}_\tau^N h_\psi\|_\infty + \|\mathcal{L}_\tau^N h_\psi - \mathcal{L}_\psi^N h_\psi\|_\infty \\ &\leqslant \epsilon + \|\mathcal{L}_\tau^N h_\psi - \mathcal{L}_\psi^N h_\psi\|_\infty.\end{aligned}$$

Since the right term converges to 0 as ψ tends to τ, the desired conclusion follows and the theorem is proved. □

3. Bernoullicity of Gibbs states

In this section, we appeal once more to the result of Walters about summability of variations implying Bernoullicity, this time in the context of Gibbs states.

This is satisfactory in that it is an extension of the results of Gallavotti ([5]) and Ledrappier ([9]). It provides another example of the situation where all general conditions known to imply uniqueness of a measure in a certain class also imply its Bernoullicity. The result may be stated as follows:

Theorem 4. *Suppose $(\Phi_\Lambda)_{\Lambda\in\mathcal{F}}$ is a translation-invariant interaction potential with the property that*

$$\sum_{\{\Lambda:\ \min\Lambda=0\}} \operatorname{diam}(\Lambda)\operatorname{var}(\Phi_\Lambda) < \infty. \tag{4}$$

Then there is a unique equilibrium state for (Φ_Λ) and this equilibrium state has the Bernoulli property.

Note that the uniqueness referred to above is demonstrated in [6]. The earlier results of Gallavotti and Ledrappier proved Bernoullicity under stronger conditions which are equivalent to

$$\sum_{\{\Lambda:\ \min\Lambda=0\}} |\Lambda|\operatorname{diam}(\Lambda)\operatorname{var}(\Phi_\Lambda) < \infty.$$

Proof. We will use the work of Ruelle to relate the question to one of equilibrium states which will then have a very simple solution.

Suppose the interaction potential (Φ_Λ) has the property (4). We construct an equilibrium potential ϕ, which we also write as $A((\Phi_\Lambda))$, from (Φ_Λ).

$$\phi(x) = \sum_{\min\Lambda=0} \Phi_\Lambda(x)$$

Note that A is a linear map from the space of interaction potentials to the space of equilibrium potentials. Then we first demonstrate that ϕ has summable variations. First, we note that

$$\operatorname{var}_n\phi \leqslant \sum_{\min\Lambda=0,\,\max\Lambda\geqslant n} \operatorname{var}(\Phi_\Lambda).$$

Summing, we see that

$$\begin{aligned}
\sum_{n=0}^{\infty}\operatorname{var}_n\phi &\leqslant \sum_{n=0}^{\infty}\sum_{\min\Lambda=0,\,\max\Lambda\geqslant n} \operatorname{var}(\Phi_\Lambda)\\
&= \sum_{n=0}^{\infty}\sum_{\min\Lambda=0,\,\max\Lambda=n} (n+1)\operatorname{var}(\Phi_\Lambda)\\
&= \sum_{\min\Lambda=0} (1+\operatorname{diam}(\Lambda))\operatorname{var}(\Phi_\Lambda) < \infty.
\end{aligned}$$

It follows that ϕ has a unique equilibrium state which has the Bernoulli property ([17]), but using [6] and [15], we see that the unique Gibbs state is the same as the equilibrium state for ϕ completing the proof. $\square$

Finally, we remark that it is possible to get a result about $\bar{d}$ continuity of Gibbs states with respect to variation of the Gibbs potential in a suitable norm from the analagous result about equilibrium states:

Define a norm on the Gibbs potentials by

$$\|(\Phi_\Lambda)\| = \sum_{\min\Lambda=0} (\|\Phi_\Lambda\|_\infty + (\text{diam }(\Lambda)+1)^2 \text{var}\Phi_\Lambda).$$

We then claim that the map A sending Gibbs potentials to equilibrium potentials has the property that it sends potentials with $\|(\Phi_\Lambda)\| < \infty$ to equilibrium potentials ϕ with $\|\phi\| < \infty$.

In fact, we can say more: Pick (Φ_Λ) with the property that $\|(\Phi_\Lambda)\| < \infty$ and let $\phi = A((\Phi_\Lambda))$. Then we see as before that $\text{var}_n\phi \leqslant \sum_{\{\Lambda:\ \min\Lambda=0,\ \max\Lambda\geqslant n\}} \text{var}(\Phi_\Lambda)$ so we have

$$\begin{aligned}\sum n\text{var}_n\phi &\leqslant \sum_{\min\Lambda=0} \sum_{i=0}^{\text{diam}(\Lambda)} i\text{var}(\Phi_\Lambda)\\ &\leqslant \sum_{\min\Lambda=0} (\text{diam}\Lambda)^2 \text{var}(\Phi_\Lambda)\end{aligned}$$

Similarly, we have $\|\phi\|_\infty \leqslant \sum_{\min\Lambda=0} \|\Phi_\Lambda\|_\infty$, which together tell us that $\|\phi\| \leqslant \|(\Phi_\Lambda)\|$ so that the linear operator A has norm at most one when restricted to the space of Gibbs potentials of finite norm.

This tells us that the one-sided equilibrium potential depends continuously on the Gibbs potential and we can therefore apply Theorem 3 to deduce

Theorem 5. *Let $\mathcal{G}$ denote the collection of Gibbs potentials for which $\|(\Phi_\Lambda)\| < \infty$. Then restricted to $\mathcal{G}$ with norm $\|\cdot\|$, the map sending a Gibbs potential to its Gibbs state is continuous with respect to $\bar{d}$.*

It would be interesting to see whether Bernoullicity could be established for Gibbs states satisfying the weaker conditions introduced by Minlos and Natapov in [10] which they show are sufficient to guarantee uniqueness of the Gibbs state.

In addition, it is somewhat unsatisfactory that conditions for the $\bar{d}$-continuity of the equilibrium states and Gibbs states are stronger than those required for the Bernoullicity. It seems likely that $\bar{d}$-continuity would hold under weaker conditions than those which we have been able to find so far.

The first named author would like to thank the Royal Society for financial support and the Mathematics Institute of the University of Warwick for their hospitality during part of the time while this work was being completed.

We thank Klaus Schmidt for suggesting the problem of summability of variations implying Bernoullicity of equilibrium state, and understand that he has independently obtained a different proof.

We would like to express our thanks to Peter Walters for his careful reading of this paper, and his many helpful comments which have helped improve the results and streamline the exposition.

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