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The *N*-body problem in spaces with uniformly varying curvature

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We generalize the curved N-body problem to spheres and hyperbolic spheres whose curvature κ varies in time. Unlike in the particular case when the curvature is constant, the equations of motion are non-autonomous. We first briefly consider the analog of the Kepler problem and then investigate homographic orbits for any number of bodies, proving the existence of several such classes of solutions on spheres. Allowing the curvature to vary in time offers some insight into the effect of an expanding universe in the context of the curved N-body problem, when κ satisfies Hubble's law. The study of these equations also opens the possibility of finding new connections between classical mechanics and general relativity. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4983681]

I. INTRODUCTION

In the 1830s, János Bolyai and Nikolai Lobachevsky independently thought that the laws of physics depend on the geometry of the universe, so they sought a natural extension of gravity to hyperbolic space, Refs. 2 and 17. This idea led to the study of the Kepler problem and the 2-body problem in the framework of hyperbolic and elliptic geometry. Unlike in Euclidean space, the equations describing them are not equivalent, since the latter system is not integrable, Ref. 20. More recently, the problem was generalized to any number N of bodies, leading to works such as Refs. 3–15 and 18–24. In the light of Hubble's law,[16] a non-flat universe (i.e., a 3-sphere or a hyperbolic 3-sphere) would have uniformly varying curvature $\kappa = \kappa(t)$ as the universe expands, meaning that at a given time the curvature is the same at every point. Therefore, by modifying the equations of the curved N-body problem to allow for uniformly varying curvature, we can construct a gravitational model that accounts for an expanding universe without requiring general relativity. Of course, we do not claim that the model we will introduce here could replace general relativity in cosmological studies. We are mostly interested in the mathematical aspects of a curved N-body problem on expanding or contracting spheres and hyperbolic spheres, a problem that, to our knowledge, has not been considered before in the framework of classical mechanics.

We are not only deriving here the equations of motion of this N-body problem but will also examine how the uniformly varying curvature affects the system's behaviour and the existence of certain solutions. In Section II we find the equations of the N-body problem with uniformly varying curvature on the variable 3-sphere,

$$\mathbb{S}^3_\kappa := \mathbb{S}^3_\kappa(t) = \{(x, y, z, w) \in \mathbb{R}^4 : x^2 + y^2 + z^2 + w^2 = \kappa^{-1}(t),\ \kappa(t) > 0\},$$

and the variable hyperbolic 3-sphere,

$$\mathbb{H}^3_\kappa := \mathbb{H}^3_\kappa(t) = \{(x, y, z, w) \in \mathbb{R}^{3,1} : x^2 + y^2 + z^2 - w^2 = \kappa^{-1}(t),\ \kappa(t) < 0\},$$

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where $\mathbb{R}^{3,1}$ is the Minkowski space, by generalizing the derivation of the curved N-body equations with cotangent potential, as done in Ref. 3, and perform a change of coordinates to reduce the problem to the study of the motion projected onto the unit manifolds $\mathbb{S}^3$ and $\mathbb{H}^3$, respectively. We then seek the first integrals of the equations and find a Lagrangian for the projected coordinates. In Section III, we derive the equations of the Kepler problem, a two-body system where one body is fixed, and rule out some of the solutions typically expected in such problems. In Section IV we first define the concept of a homographic solution, for which the configuration of the particles remains similar to itself during the motion while the curvature of the space changes in time. We then show that such orbits exist in $\mathbb{S}^3_\kappa$, but not in $\mathbb{H}^3_\kappa$, and that the homographic solutions of $\mathbb{S}^3_\kappa$ correspond to the special central configurations in $\mathbb{S}^3$, which was studied in Ref. 13. This observation allows us to reformulate the question of existence of homographic solutions on a variable 3-sphere as a problem of existence of special central configurations in the unit sphere $\mathbb{S}^3$. Each special central configuration found in Ref. 13 gives rise to a homographic solution of the N-body problem in spaces with uniformly varying positive curvature. In Section V, we provide several new special central configurations in $\mathbb{S}^3$. We first show the existence of a double Lagrangian special central configuration for 6 bodies and the existence of a double tetrahedron special central configuration for 8 bodies. Then we give the criteria for the general 4- and 5-body special central configurations.

We would like to mention that the idea of introducing and studying the N-body problem in spaces with uniformly varying curvature came to us from Sergio Benenti's book in progress, Ref. 1. In his manuscript, Benenti develops a remarkable axiomatic setting for isotropic cosmological models, considering the spaces $\mathbb{S}_\kappa(t)$ and $\mathbb{H}_\kappa(t)$ as defined above. However, he shows no interest in deriving the equations of motion of an N-body problem, focusing instead on some cosmological questions which he treats with relativistic techniques.

II. EQUATIONS OF MOTION

In order to study the N-body problem in spaces with uniformly varying curvature, it is first necessary to generalize the equations of motion from the constant curvature case by applying the Euler-Lagrange equations to the Lagrangian used in Ref. 3, where κ is a non-zero differentiable function of time. The goal of this section is to obtain the new system of equations and its basic integrals of motion.

A. Deriving the equations of motion

Let the curvature $\kappa : [0,\infty) \to \mathbb{R}$ be a non-zero differentiable function of time. Take $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N)$, with $\mathbf{q}_i \in \mathbb{R}^4$, if $\kappa(t) > 0$, but the Minkowski 4-space is $\mathbb{R}^{3,1}$, if $\kappa(t) < 0$. We define the potential energy to be $-U_\kappa$, where U_k is the force function

$$U_\kappa(\mathbf{q}) = \sum_{1\le i<j\le N} \frac{m_i m_j |\kappa|^{1/2} \kappa \mathbf{q}_i \cdot \mathbf{q}_j}{[\sigma - \sigma(\kappa \mathbf{q}_i \cdot \mathbf{q}_j)^2]^{1/2}}, \tag{1}$$

where σ denotes the sign of κ, and $\cdot$ is the standard inner product for $\kappa > 0$ and the Lorentz product $\mathbf{q}_i \cdot \mathbf{q}_j = x_i x_j + y_i y_j + z_i z_j - w_i w_j$ for $\kappa < 0$. When κ is constant, U_κ offers the natural extension of Newton's law to curved spaces, see Ref. 3.

We define the kinetic energy as

$$T_\kappa(\dot{\mathbf{q}}) = \frac{1}{2} \sum_{i=1}^{N} m_i \dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i, \tag{2}$$

so the Lagrangian function is $L_\kappa = T_\kappa + U_\kappa$. Consequently we can obtain in the standard manner the Euler-Lagrange equations with holonomic constraints,

$$\frac{d}{dt}\frac{\partial L_\kappa}{\partial \dot{\mathbf{q}}_i} - \frac{\partial L_\kappa}{\partial \mathbf{q}_i} - \lambda_\kappa^i \frac{\partial f_\kappa^i}{\partial \mathbf{q}_i} = 0,\ i = 1, \ldots, N, \tag{3}$$

where $\kappa \neq 0$ and $f_\kappa^i = \mathbf{q}_i \cdot \mathbf{q}_i - \frac{1}{\kappa} = 0,\ i = 1, \dots, N$ are the constraints that keep the particle system on $\mathbb{S}_\kappa^3$ or $\mathbb{H}_\kappa^3$, respectively. The above system then becomes

$$m_i \ddot{\mathbf{q}}_i = \nabla_{\mathbf{q}_i} U_\kappa + 2\lambda_\kappa^i \mathbf{q}_i,\ i = 1, \dots, N. \tag{4}$$

Dot-multiplying these equations by $\mathbf{q}_i$ leads to

$$m_i \ddot{\mathbf{q}}_i \cdot \mathbf{q}_i = \nabla_{\mathbf{q}_i} U_\kappa \cdot \mathbf{q}_i + 2\lambda_\kappa^i \mathbf{q}_i \cdot \mathbf{q}_i,\ i = 1, \dots, N. \tag{5}$$

Since U_κ is a homogeneous function of degree 0, it follows by Euler's formula for homogeneous functions that $\nabla_{\mathbf{q}_i} U_\kappa \cdot \mathbf{q}_i = 0$. As $f_\kappa^i = 0$, we also have

$$\dot{f}_\kappa^i = 2\dot{\mathbf{q}}_i \cdot \mathbf{q}_i + \frac{\dot{\kappa}}{\kappa^2} = 0,\ i = 1, \dots, N$$

and

$$\ddot{f}_\kappa^i = 2\ddot{\mathbf{q}}_i \cdot \mathbf{q}_i + 2\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i + \frac{\ddot{\kappa}}{\kappa^2} - 2\frac{\dot{\kappa}^2}{\kappa^3} = 0,\ i = 1, \dots, N.$$

Substituting these into (5) gives

$$-m_i \dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i - \frac{m_i \ddot{\kappa}}{2\kappa^2} + \frac{m_i \dot{\kappa}^2}{\kappa^3} = 2\frac{\lambda_\kappa^i}{\kappa},\ i = 1, \dots, N,$$

so

$$\lambda_\kappa^i = -\frac{m_i \kappa}{2} \dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i - \frac{m_i \ddot{\kappa}}{4\kappa} + \frac{m_i \dot{\kappa}^2}{2\kappa^2},\ i = 1, \dots, N. \tag{6}$$

If we insert (6) into (4), we obtain that

$$m_i \ddot{\mathbf{q}}_i = \nabla_{\mathbf{q}_i} U_\kappa - m_i \kappa (\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i) \mathbf{q}_i - m_i \frac{\ddot{\kappa}}{2\kappa} \mathbf{q}_i + m_i \frac{\dot{\kappa}^2}{\kappa^2} \mathbf{q}_i,\ i = 1, \dots, N, \tag{7}$$

with the constraints

$$\kappa \mathbf{q}_i \cdot \mathbf{q}_i = 1,\ i = 1, \dots, N,\ \kappa \neq 0. \tag{8}$$

The change of variables $\mathbf{q}_i = |\kappa|^{-1/2} \overline{\mathbf{q}}_i$ projects the system on $\mathbb{S}^3$ and $\mathbb{H}^3$. We obtain

$$\dot{\mathbf{q}}_i = -\frac{\sigma \dot{\kappa} \overline{\mathbf{q}}_i}{2|\kappa|^{3/2}} + \frac{\dot{\overline{\mathbf{q}}}_i}{|\kappa|^{1/2}},$$

$$\ddot{\mathbf{q}}_i = -\frac{\sigma \ddot{\kappa} \overline{\mathbf{q}}_i}{2|\kappa|^{3/2}} + \frac{3\dot{\kappa}^2 \overline{\mathbf{q}}_i}{4|\kappa|^{5/2}} - \frac{\sigma \dot{\kappa} \dot{\overline{\mathbf{q}}}_i}{|\kappa|^{3/2}} + \frac{\ddot{\overline{\mathbf{q}}}_i}{|\kappa|^{1/2}}.$$

Define $\overline{U}(\overline{\mathbf{q}}) = \sum_{1 \le i < j \le N} \frac{m_i m_j \sigma \overline{\mathbf{q}}_i \cdot \overline{\mathbf{q}}_j}{[\sigma - \sigma(\overline{\mathbf{q}}_i \cdot \overline{\mathbf{q}}_j)^2]^{1/2}}$. It is easy to see that $\nabla_{\overline{\mathbf{q}}_i} \overline{U} = |\kappa|^{-1} \nabla_{\mathbf{q}_i} U_\kappa$. Thus the equations of motion take the form

$$m_i \ddot{\overline{\mathbf{q}}}_i = |\kappa|^{3/2} \nabla_{\overline{\mathbf{q}}_i} \overline{U} - \sigma m_i (\dot{\overline{\mathbf{q}}}_i \cdot \dot{\overline{\mathbf{q}}}_i) \overline{\mathbf{q}}_i + \frac{m_i \dot{\kappa}}{\kappa} \dot{\overline{\mathbf{q}}}_i,\ i = 1, \dots, N, \tag{9}$$

with constraints

$$\overline{\mathbf{q}}_i \cdot \overline{\mathbf{q}}_i = \sigma,\ \overline{\mathbf{q}}_i \cdot \dot{\overline{\mathbf{q}}}_i = 0,\ i = 1, \dots, N,\ \kappa \neq 0. \tag{10}$$

In Refs. 5 and 13, the explicit form of $\nabla_{\overline{\mathbf{q}}_i} \overline{U}$ is written as

$$\nabla_{\overline{\mathbf{q}}_i} \overline{U} = \sum_{j=1, j \neq i}^{N} \frac{m_i m_j (\overline{\mathbf{q}}_j - \sigma(\overline{\mathbf{q}}_i \cdot \overline{\mathbf{q}}_j) \overline{\mathbf{q}}_i)}{(\sigma - \sigma(\overline{\mathbf{q}}_i \cdot \overline{\mathbf{q}}_j)^2)^{3/2}} = \sum_{j=1, j \neq i}^{N} \frac{m_i m_j (\overline{\mathbf{q}}_j - \mathrm{csn} d_{ij} \overline{\mathbf{q}}_i)}{\mathrm{sn}^3 d_{ij}}, \tag{11}$$

where $\mathrm{sn}(x) = \sin(x)$ or $\sinh(x)$, $\mathrm{csn}(x) = \cos(x)$ or $\cosh(x)$, and d_{ij} is the distance between $\overline{\mathbf{q}}_i$ and $\overline{\mathbf{q}}_j$ in $\mathbb{S}^3$ or $\mathbb{H}^3$. It is $d_{ij} := \mathrm{arccsn}(\sigma \overline{\mathbf{q}}_i \cdot \overline{\mathbf{q}}_j)$. Following Ref. 5, we define

$$\mathrm{ctn}(x) = \mathrm{csn}(x)/\mathrm{sn}(x) = \cot(x) \text{ or } \coth(x),$$
$$\mathrm{csct}(x) = 1/\mathrm{sn}(x) = 1/\sin(x) \text{ or } 1/\sinh(x).$$

We denote by $\mathbf{F}_i$ the term $\nabla_{\overline{\mathbf{q}}_i} \overline{U}$. Physically, it is the attraction force on $\overline{\mathbf{q}}_i$.

B. Integrals of the total angular momentum

We can now obtain the integrals of the total angular momentum. Consider the wedge product, $\wedge$, of $\overline{\mathbf{q}}_i$ and the ith equations of (9). For details on the wedge product in the current context, see Ref. 5, p. 31. Dividing by $|\kappa|$, and summing the equations over i, we obtain

$$\sum_{i=1}^{N} \frac{m_i}{|\kappa|}\ddot{\overline{\mathbf{q}}}_i \wedge \overline{\mathbf{q}}_i = \sum_{i=1}^{N} |\kappa|^{1/2}(\nabla_{\overline{\mathbf{q}}_i}\overline{U}) \wedge \overline{\mathbf{q}}_i - \sum_{i=1}^{N}\left[\frac{m_i}{\kappa}(\dot{\overline{\mathbf{q}}}_i \cdot \dot{\overline{\mathbf{q}}}_i)\overline{\mathbf{q}}_i \wedge \overline{\mathbf{q}}_i + \frac{m_i\dot{\kappa}}{\sigma\kappa^2}\dot{\overline{\mathbf{q}}}_i \wedge \overline{\mathbf{q}}_i\right].$$

Since wedge-product expressions are skew-symmetric, $\sum_{i=1}^{N} |\kappa|^{1/2}(\nabla_{\overline{\mathbf{q}}_i}\overline{U}) \wedge \overline{\mathbf{q}}_i = 0$. Combining this property with the fact that $\overline{\mathbf{q}}_i \wedge \overline{\mathbf{q}}_i = 0$, we obtain that

$$\sum_{i=1}^{N}\left[\frac{m_i}{|\kappa|}\ddot{\overline{\mathbf{q}}}_i \wedge \overline{\mathbf{q}}_i - \frac{\sigma m_i\dot{\kappa}}{\kappa^2}\dot{\overline{\mathbf{q}}}_i \wedge \overline{\mathbf{q}}_i\right] = 0. \tag{12}$$

This is the negative of the time derivative of the system's angular momentum about the origin, $\mathbf{L} = \sum_{i=1}^{N} \frac{m_i}{|\kappa|}\overline{\mathbf{q}}_i \wedge \dot{\overline{\mathbf{q}}}_i$, and provides the six integrals

$$L_{wx} = \sum_{i=1}^{N}\frac{m_i}{|\kappa|}(y_i\dot{z}_i - z_i\dot{y}_i), \qquad L_{wy} = \sum_{i=1}^{N}\frac{m_i}{|\kappa|}(x_i\dot{z}_i - z_i\dot{x}_i),$$

$$L_{wz} = \sum_{i=1}^{N}\frac{m_i}{|\kappa|}(x_i\dot{y}_i - y_i\dot{x}_i), \qquad L_{xy} = \sum_{i=1}^{N}\frac{m_i}{|\kappa|}(w_i\dot{z}_i - z_i\dot{w}_i),$$

$$L_{xz} = \sum_{i=1}^{N}\frac{m_i}{|\kappa|}(w_i\dot{y}_i - y_i\dot{w}_i), \qquad L_{yz} = \sum_{i=1}^{N}\frac{m_i}{|\kappa|}(w_i\dot{x}_i - x_i\dot{w}_i).$$

C. The $\overline{\mathbf{q}}$-Lagrangian

If we have a Lagrangian in both the normal and projected coordinates, we may obtain the projected equations without having to first derive the full equations. The Lagrangian of the system in $\overline{\mathbf{q}}$ coordinates is

$$\overline{L} = \sum_{i=1}^{N}\frac{m_i(\dot{\overline{\mathbf{q}}}_i \cdot \dot{\overline{\mathbf{q}}}_i)}{2|\kappa|} + \sum_{1\le i<j\le N} |\kappa|^{1/2}\frac{\sigma m_j m_i(\overline{\mathbf{q}}_j \cdot \overline{\mathbf{q}}_i)}{(\sigma - \sigma(\overline{\mathbf{q}}_j \cdot \overline{\mathbf{q}}_i)^2)^{1/2}}, \tag{13}$$

with constraints $f_i = \overline{\mathbf{q}}_i \cdot \overline{\mathbf{q}}_i - \sigma = 0$. Applying the Euler-Lagrange equations in terms of $\overline{\mathbf{q}}_i$ gives system (9).

III. THE KEPLER PROBLEM

The simplest system that can be derived from the N-body problem is the Kepler problem, which describes the gravitational motion of a point mass m about a fixed-point mass M. Without loss of generality, we will assume that M is fixed at position $N = (0, 0, 0, 1)$ in terms of $\overline{\mathbf{q}}$ coordinates.

We will use the 3-spherical/hyperbolic coordinates $(\alpha, \theta, \varphi)$ by taking

$$\overline{\mathbf{q}} = (x, y, z, w) = (\mathrm{sn}\alpha \sin\theta\cos\varphi, \mathrm{sn}\alpha\sin\theta\sin\varphi, \mathrm{sn}\alpha\cos\theta, \mathrm{csn}\alpha),$$

so Equation (13) becomes

$$\overline{L} = \frac{m(\dot{\alpha}^2 + \dot{\theta}^2\mathrm{sn}^2\alpha + \dot{\varphi}^2\mathrm{sn}^2\alpha\sin^2\theta)}{2|\kappa|} + |\kappa|^{1/2}mM\mathrm{ctn}\alpha \tag{14}$$

with no constraint. Then the conjugate momenta for the system are

$$p_\alpha = \frac{m\dot{\alpha}}{|\kappa|},\ p_\theta = \frac{m\dot{\theta}\mathrm{sn}^2\alpha}{|\kappa|},\ p_\varphi = \frac{m\dot{\varphi}\mathrm{sn}^2\alpha\sin^2\theta}{|\kappa|},$$

and the Hamiltonian has the form

$$H = \frac{|\kappa|}{2m}(p_\alpha^2 + p_\theta^2 \mathrm{csct}^2\alpha + p_\varphi^2 \mathrm{csct}^2\alpha \csc^2\theta) - |\kappa|^{1/2} mM \mathrm{ctn}\alpha. \tag{15}$$

Then the equations of motion become

$$\dot\alpha = \frac{\partial H}{\partial p_\alpha} = \frac{|\kappa| p_\alpha}{m}, \tag{16}$$

$$\dot\theta = \frac{\partial H}{\partial p_\theta} = \frac{|\kappa| p_\theta \mathrm{csct}^2\alpha}{m}, \tag{17}$$

$$\dot\varphi = \frac{\partial H}{\partial p_\varphi} = \frac{|\kappa| p_\varphi \mathrm{csct}^2\alpha \csc^2\theta}{m}, \tag{18}$$

$$\dot p_\alpha = -\frac{\partial H}{\partial \alpha} = \frac{|\kappa| \mathrm{ctn}\alpha \mathrm{csct}^2\alpha}{m}(p_\theta^2 + p_\varphi^2 \csc^2\theta) - |\kappa|^{1/2} mM \mathrm{csct}^2\alpha, \tag{19}$$

$$\dot p_\theta = -\frac{\partial H}{\partial \theta} = \frac{|\kappa| \mathrm{csct}^2\alpha \csc^2\theta \cot\theta p_\varphi^2}{m}, \tag{20}$$

$$\dot p_\varphi = -\frac{\partial H}{\partial \varphi} = 0. \tag{21}$$

From (21) we have that $A = p_\varphi$ is a constant, and direct computation leads to

$$\begin{aligned} L_{wz} &= p_\varphi, \\ L_{wx} &= p_\theta \sin\varphi + p_\varphi \cot\theta \cos\varphi, \\ L_{wy} &= p_\theta \cos\varphi + p_\varphi \cot\theta \sin\varphi. \end{aligned}$$

Notice that $L = p_\theta^2 + p_\varphi^2 \csc^2\theta$ is a constant. Using this property, we can eliminate (20) and (21) and obtain the equations of motion in the form

$$\dot\alpha = \frac{|\kappa| p_\alpha}{m}, \tag{22}$$

$$\dot\theta = \pm \frac{|\kappa| \sqrt{(L - A^2 \csc^2\theta)} \mathrm{csct}^2\alpha}{m}, \tag{23}$$

$$\dot\varphi = \frac{|\kappa| A \mathrm{csct}^2\alpha \csc^2\theta}{m}, \tag{24}$$

$$\dot p_\alpha = \frac{|\kappa| L \mathrm{ctn}\alpha \mathrm{csct}^2\alpha}{m} - |\kappa|^{1/2} mM \mathrm{csct}^2\alpha. \tag{25}$$

If we insert (22) into (25), we get the second order equation

$$\ddot\alpha = \frac{\kappa^2 L \mathrm{ctn}\alpha \mathrm{csct}^2\alpha}{m^2} - |\kappa|^{3/2} M \mathrm{csct}^2\alpha + \frac{\dot\kappa \dot\alpha}{\kappa}. \tag{26}$$

A. Necessary condition on κ for circular solutions

The simplest solution of the Kepler problem in the Euclidean and constant curvature cases is the circular solution, i.e., an orbit for which the moving mass is at a constant distance from the fixed mass. We find that such solutions do not exist for non-constant curvature.

Proposition 1. Circular orbits occur only in systems with constant curvature.

Proof. Obviously, a circular orbit occurs when α has a constant value throughout the motion. By (22), in order for this to be the case, we must have $p_\alpha = 0$. But then, by (25), $0 = \frac{1}{m}|\kappa| L \mathrm{ctn}\alpha \mathrm{csct}^2\alpha - |\kappa|^{1/2} mM \mathrm{csct}^2\alpha$. If we isolate κ, we find that

$$|\kappa| = \left(\frac{m^2 M}{L \mathrm{ctn}\alpha}\right)^2.$$

Since the right-hand side consists only of constants, it follows that the system has circular orbits only if κ does not depend on time. □

We can also prove the following related result.

Proposition 2. A system has non-fixed T-periodic solutions in phase space for some $T > 0$ only if κ is T-periodic.

Proof. Let $(\alpha(t), \varphi(t), \theta(t), p_\alpha)$ be a solution to the curved Kepler problem with curvature $\kappa(t)$, such that for every $t \in [0, \infty)$, we have

$$\begin{aligned} \alpha(t+T) &= \alpha(t), & p_\alpha(t+T) &= p_\alpha(t), \\ \theta(t+T) &= \theta(t), & \varphi(t+T) &= \varphi(t) + 2n\pi, \end{aligned}$$

for some $T \in \mathbb{R}, n \in \mathbb{Z}$. If $A \neq 0$, then by (24)

$$0 = \dot{\varphi}(t+T) - \dot{\varphi}(t) = \frac{(|\kappa(t+T)| - |\kappa(t)|)A \cdot \mathrm{csct}^2\alpha(t)\csc^2\theta(t)}{m},$$

so $|\kappa(t+T)| = |\kappa(t)|$. Since κ is continuous and non-zero, there are no t_1, t_2 such that $\kappa(t_1) = -\kappa(t_2)$, so $\kappa(t+T) = \kappa(t)$.

If $A = 0$, then by (23)

$$0 = \dot{\theta}(t+T) - \dot{\theta}(t) = \pm\frac{(|\kappa(t+T)| - |\kappa(t)|)\sqrt{L} \cdot \mathrm{csct}^2\alpha(t)}{m},$$

so by the same argument as above, $\kappa(t+T) = \kappa(t)$. Therefore T-periodic solutions occur only when κ is T-periodic. □

IV. HOMOGRAPHIC ORBITS

In this section, we study a class of rigid motions (rigid motions in terms of $\overline{\mathbf{q}}$ coordinates). We found that they exist in $\mathbb{S}^3$, but not in $\mathbb{H}^3$, and that they are related to special central configurations, a concept introduced in Refs. 5 and 13.

A. Homographic orbits in $\mathbb{S}^3$

In $\mathbb{S}^3$, a solution of the form $\mathbf{q}(t) = A^{-1}e^{\xi(t)}A\mathbf{q}(0)$ is called a homographic orbit, where A is a constant matrix in $SO(4)$, and

$$\xi(t) = \begin{bmatrix} 0 & -\alpha(t) & 0 & 0 \\ \alpha(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta(t) \\ 0 & 0 & \beta(t) & 0 \end{bmatrix},$$

$\alpha(t), \beta(t) \in C^1(\mathbb{R}),\ \alpha(0) = \beta(0) = 0$. Since Equations (9) with $\kappa > 0$ are invariant under the $SO(4)$-action, it is sufficient to consider the case $A = id_{SO(4)}$.

Definition 1 (Ref. 13). *Consider the masses $m_1, \ldots, m_N > 0$ in $\mathbb{S}^3$. Then a configuration*

$$\overline{\mathbf{q}} = (\overline{\mathbf{q}}_1, \overline{\mathbf{q}}_2 \ldots \overline{\mathbf{q}}_N),\ \overline{\mathbf{q}}_i = (x_i, y_i, z_i, w_i),\ i = 1, \ldots, N$$

is called a special central configuration if it is a critical point of the force function $\overline{U}$, i.e.,

$$\nabla_{\overline{\mathbf{q}}_i}\overline{U}(\overline{\mathbf{q}}) = 0,\ i = 1, \ldots, N.$$

In $\mathbb{S}^3$, special central configurations lead to fixed-point solutions. The next result shows that homographic orbits can be derived from the special central configurations in $\mathbb{S}^3$, i.e., finding homographic solutions on spheres with variable curvature is equivalent to finding fixed-point solutions in the unit sphere $\mathbb{S}^3$.

Proposition 3. Let $\overline{\mathbf{q}}(t) = (\overline{\mathbf{q}}_1(t), \overline{\mathbf{q}}_2(t) \ldots \overline{\mathbf{q}}_N(t))$ be a homographic orbit in $\mathbb{S}^3_\kappa$. Then $\overline{\mathbf{q}}(t)$ is a solution to the N-body problem with time varying curvature if and only if $\overline{\mathbf{q}}$ is a special central configuration and $\overline{\mathbf{q}}_i(t) = e^{\xi(t)}\overline{\mathbf{q}}_i$ for $i = 1, 2, \ldots, N$, where

$$\xi(t) = \begin{bmatrix} 0 & -cK(t) & 0 & 0 \\ cK(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm cK(t) \\ 0 & 0 & \mp cK(t) & 0 \end{bmatrix}, c \in \mathbb{R}, \tag{27}$$

with $K(t) = \int_0^t \kappa(\tau)d\tau$.

Proof. Let $\overline{\mathbf{q}}_i(t) = e^{\xi(t)}\overline{\mathbf{q}}_i$, where $\overline{\mathbf{q}}$ is a special central configuration, and $\xi(t)$ is defined as in (27). Then Equations (9) become

$$m_i(\ddot{\xi}(t) + \dot{\xi}^2(t))\overline{\mathbf{q}}_i = -m_i(\dot{\xi}(t)\overline{\mathbf{q}}_i \cdot \dot{\xi}(t)\overline{\mathbf{q}}_i)\overline{\mathbf{q}}_i + \frac{m_i\dot{\kappa}(t)}{\kappa(t)}\dot{\xi}(t)\overline{\mathbf{q}}_i,$$

by using the fact that $\dot{\xi}(t)$ commutes with $e^{\xi(t)}$ and that $\nabla_{\overline{\mathbf{q}}_i}U = 0$. Notice that

$$\dot{\xi}^2(t)\overline{\mathbf{q}}_i = -c^2\kappa^2(t)\overline{\mathbf{q}}_i, \quad \frac{\dot{\kappa}(t)}{\kappa(t)}\dot{\xi}(t) = \ddot{\xi}(t),$$

so we have

$$\begin{aligned} m_i\ddot{\xi}(t)\overline{\mathbf{q}}_i - m_ic^2\kappa^2(t)\overline{\mathbf{q}}_i &= -m_ic^2\kappa^2(x_i^2 + y_i^2 + z_i^2 + w_i^2)\overline{\mathbf{q}}_i + m_i\ddot{\xi}(t)\overline{\mathbf{q}}_i \\ &= m_i\ddot{\xi}(t)\overline{\mathbf{q}}_i - m_ic^2\kappa^2(t)\overline{\mathbf{q}}_i, \end{aligned}$$

therefore $\overline{\mathbf{q}}(t)$ is a solution of the N-body problem with time varying curvature when $\kappa > 0$.

Conversely, suppose $\overline{\mathbf{q}}(t) = (\overline{\mathbf{q}}_1(t), \overline{\mathbf{q}}_2(t) \ldots \overline{\mathbf{q}}_N(t))$ is a solution of the N-body problem with uniformly varying positive curvature that is a homographic orbit in $\mathbb{S}^3_\kappa$. Then

$$\overline{\mathbf{q}}_i(t) = \begin{bmatrix} x_i(t) \\ y_i(t) \\ z_i(t) \\ w_i(t) \end{bmatrix} = \begin{bmatrix} x_i\cos(\alpha(t)) - y_i\sin(\alpha(t)) \\ x_i\sin(\alpha(t)) + y_i\cos(\alpha(t)) \\ z_i\cos(\beta(t)) - w_i\sin(\beta(t)) \\ z_i\sin(\beta(t)) + w_i\cos(\beta(t)) \end{bmatrix},$$

where $\overline{\mathbf{q}}_i(0) = (x_i, y_i, z_i, w_i)^T$, and α, β are real differentiable functions such that $\alpha(0) = \beta(0) = 0$. Notice that

$$\begin{aligned} \dot{x}_i(t) &= -\dot{\alpha}(t)y_i(t), \\ \dot{y}_i(t) &= \dot{\alpha}(t)x_i(t), \\ \dot{z}_i(t) &= -\dot{\beta}(t)w_i(t), \\ \dot{w}_i(t) &= \dot{\beta}(t)z_i(t). \end{aligned}$$

If we look at the angular momentum integrals in the xy and zw directions, we find that

$$\begin{aligned} L_{xy} &= \frac{1}{\kappa(t)}\sum_{i=1}^N m_i(x_i(t)\dot{y}_i(t) - \dot{x}_i(t)y_i(t)) \\ &= \frac{\dot{\alpha}(t)}{\kappa(t)}\sum_{i=1}^N m_i((x_i\cos(\alpha(t)) - y_i\sin(\alpha(t)))^2 + (x_i\sin(\alpha(t)) + y_i\cos(\alpha(t)))^2) \\ &= \frac{\dot{\alpha}(t)}{\kappa(t)}\sum_{i=1}^N m_i(x_i^2 + y_i^2) \end{aligned}$$

and

$$\begin{aligned} L_{zw} &= \frac{1}{\kappa(t)} \sum_{i=1}^{N} m_i(x_i(t)\dot{y}_i(t) - \dot{x}_i(t)y_i(t)) \\ &= \frac{\dot{\beta}(t)}{\kappa(t)} \sum_{i=1}^{N} m_i\big((z_i\cos(\beta(t)) - w_i\sin(\beta(t)))^2 + (z_i\sin(\beta(t)) + w_i\cos(\beta(t))^2\big) \\ &= \frac{\dot{\beta}(t)}{\kappa(t)} \sum_{i=1}^{N} m_i(z_i^2 + w_i^2). \end{aligned}$$

For L_{xy} and L_{zw} to be constant, we must have one of the following three conditions satisfied:

(1) $x_i = y_i = 0$ for all $i = 1, \ldots, N$ and $\beta(t) = cK(t)$ for some $c \in \mathbb{R}$;
(2) $z_i = w_i = 0$ for all $i = 1, \ldots, N$ and $\alpha(t) = cK(t)$ for some $c \in \mathbb{R}$;
(3) $\alpha(t) = aK(t)$ for some $a \in \mathbb{R}$ and $\beta(t) = bK(t)$ for some $b \in \mathbb{R}$.

The first and second cases are proved the same way, so we will look at the first and third cases only. In the first case, since $\overline{\mathbf{q}}_i(t)$ is a solution of the N-body problem with uniformly varying positive curvature, the following equation is satisfied for $i = 1, \ldots, N$:

$$m_i c\dot{\kappa} \begin{bmatrix} 0 \\ 0 \\ -w_i(t) \\ z_i(t) \end{bmatrix} - m_i c^2\kappa^2 \overline{\mathbf{q}}_i(t) = \kappa^{3/2}(t)\nabla_{\overline{\mathbf{q}}_i} U - m_i c^2\kappa^2\overline{\mathbf{q}}_i(t) + m_i c\dot{\kappa} \begin{bmatrix} 0 \\ 0 \\ -w_i(t) \\ z_i(t) \end{bmatrix}.$$

Then $\nabla_{\overline{\mathbf{q}}_i} U = 0$, so $\overline{\mathbf{q}}_i$ is a special central configuration, and

$$\overline{\mathbf{q}}_i(t) = \begin{bmatrix} 0 \\ 0 \\ z_i\cos(cK) - w_i\sin(cK) \\ z_i\sin(cK) + w_i\cos(cK) \end{bmatrix} = e^{\xi(t)}\overline{\mathbf{q}}_i,$$

where $\xi(t)$ is defined in (27).

In the third case, if we notice that $\frac{\dot{\kappa}(t)}{\kappa(t)}\dot{\xi}(t) = \ddot{\xi}(t)$, we know that the following equation is satisfied by $\overline{\mathbf{q}}_i(t)$:

$$\begin{aligned} \kappa^{3/2}\nabla_{\overline{\mathbf{q}}_i} U &= m_i\kappa^2(t) \begin{bmatrix} (a^2(x_i^2+y_i^2) + b^2(z_i^2+w_i^2) - a^2)x_i \\ (a^2(x_i^2+y_i^2) + b^2(z_i^2+w_i^2) - a^2)y_i \\ (a^2(x_i^2+y_i^2) + b^2(z_i^2+w_i^2) - b^2)z_i \\ (a^2(x_i^2+y_i^2) + b^2(z_i^2+w_i^2) - b^2)w_i \end{bmatrix} \\ &= m_i\kappa^2(t)(b^2 - a^2) \begin{bmatrix} (z_i^2+w_i^2)x_i \\ (z_i^2+w_i^2)y_i \\ -(x_i^2+y_i^2)z_i \\ -(x_i^2+y_i^2)w_i \end{bmatrix}. \end{aligned}$$

Assuming that κ is not constant, this equation can only hold if $\overline{\mathbf{q}}_i(t)$ satisfies condition (1) or (2), or if $a = \pm b$ and $\overline{\mathbf{q}}_i$ is a special configuration. In either case, the hypothesis holds, and

$$\overline{\mathbf{q}}_i(t) = \begin{bmatrix} x_i\cos(aK) - y_i\sin(aK) \\ x_i\sin(aK) + y_i\cos(aK) \\ z_i\cos(aK) - w_i\sin(\pm aK) \\ z_i\sin(\pm aK) + w_i\cos(aK) \end{bmatrix} = e^{\xi(t)}\overline{\mathbf{q}}_i.$$

This remark completes the proof. □

B. Homographic orbits in $\mathbb{H}^3$

In $\mathbb{H}^3$, a solution of the form $\mathbf{q}(t) = B^{-1} e^{\xi_j(t)} B\mathbf{q}(0)$, $j = 1, 2$, is called a homographic orbit, where B is a constant matrix in $SO(3, 1)$, and

$$\xi_1(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\eta(t) & \eta(t) \\ 0 & \eta(t) & 0 & 0 \\ 0 & \eta(t) & 0 & 0 \end{bmatrix}, \xi_2(t) = \begin{bmatrix} 0 & -\alpha(t) & 0 & 0 \\ \alpha(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta(t) \\ 0 & 0 & \beta(t) & 0 \end{bmatrix},$$

$\alpha(t), \beta(t), \eta(t) \in C^1(\mathbb{R})$, $\alpha(0) = \beta(0) = \eta(0) = 0$. Since Equations (9) with $\kappa < 0$ are invariant under the $SO(3, 1)$-action, it is sufficient to consider the case $B = id_{SO(3,1)}$.

As all homographic solutions for $\kappa > 0$ correspond to fixed-point solutions, or special central configurations in $\mathbb{S}^3$, and there are no fixed-point solutions in $\mathbb{H}^3$,[5,13] we expect that there are no homographic solutions for $\kappa < 0$. We will now show that this is indeed the case.

Proposition 4. There are no homographic solutions of the N-body problem with negative uniformly varying curvature.

Proof. If a solution is homographic, then it has the form

$$\overline{\mathbf{q}}_i(t) = e^{\xi_j(t)}\overline{\mathbf{q}}_i,\ j = 1, 2,\ i = 1, \ldots, N.$$

We will rule out the two possible cases separately.

Case 1. $\xi = \xi_1$. In this case, solutions will take the form

$$\overline{\mathbf{q}}_i(t) = \begin{bmatrix} x_i \\ y_i - z_i\eta(t) + w_i\eta(t) \\ z_i + y_i\eta(t) - z_i\eta^2(t)/2 + w_i\eta^2(t)/2 \\ w_i + y_i\eta(t) - z_i\eta^2(t)/2 + w_i\eta^2(t)/2 \end{bmatrix},$$

where η is a differentiable function, and $\overline{\mathbf{q}}_i = \begin{bmatrix} x_i \\ y_i \\ z_i \\ w_i \end{bmatrix}$ is the initial position of the ith particle. If we look at the angular momentum integrals in the xy and yz directions, we find after some simple calculations that

$$L_{xy} = \frac{\dot\eta(t)}{\kappa(t)} \sum_{i=1}^N m_i x_i (w_i - z_i), \tag{28}$$

$$L_{yz} = \frac{\dot\eta(t)}{\kappa(t)} \sum_{i=1}^N m_i\Big(y_i^2 + z_i^2 - z_i w_i + \eta(t) y_i (w_i - z_i) + \frac{\eta^2(t)}{2}(w_i - z_i)^2\Big). \tag{29}$$

Note that $\eta(t)$ is not constant. Otherwise, we get $\eta(t) = \eta(0) = 0$, and we obtain a fixed-point solution in $\mathbb{H}^3$, which is not possible.[5,13] Thus either $\dot\eta(t) = c\kappa(t)$ for some $c \neq 0$ or

$$\eta^2(t) \sum_{i=1}^N \frac{m_i(w_i - z_i)^2}{2} + \eta(t) \sum_{i=1}^N m_i y_i (w_i - z_i) + \sum_{i=1}^N m_i (y_i^2 + z_i^2 - z_i w_i) = 0$$

and

$$\sum_{i=1}^N m_i x_i (w_i - z_i) = 0$$

for all $t \in [0, \infty)$. In the first case, in order for L_{yz} to be constant, it would be necessary that $\sum_{i=1}^N \frac{m_i(w_i - z_i)^2}{2} = 0$, so $w_i = z_i$ for all $i = 1, \ldots, N$. But if this is the case, then $x_i^2 + y_i^2 + z_i^2 - w_i^2 = x_i^2 + y_i^2 = -1$, which is impossible. In the second case, each of $\sum_{i=1}^N m_i x_i (w_i - z_i)$, $\sum_{i=1}^N \frac{m_i(w_i - z_i)^2}{2}$, $\sum_{i=1}^N m_i y_i (w_i - z_i)$, and $\sum_{i=1}^N m_i (y_i^2 + z_i^2 - z_i w_i)$ must be equal to zero. This is possible only if $y_i = 0$,

$z_i = w_i$ for $i = 1, \ldots, N$. But then $x_i^2 + y_i^2 + z_i^2 - w_i^2 = x_i^2 = -1$, which is impossible. Therefore there are no homographic orbits for ξ_1.

Case 2. $\xi = \xi_2$. In this case, solutions will take the form

$$\overline{\mathbf{q}}_i(t) = \begin{bmatrix} x_i \cos(\alpha(t)) - y_i \sin(\alpha(t)) \\ x_i \sin(\alpha(t)) + y_i \cos(\alpha(t)) \\ z_i \cosh(\beta(t)) + w_i \sinh(\beta(t)) \\ z_i \sinh(\beta(t)) + w_i \cosh(\beta(t)) \end{bmatrix},$$

where α, β are real-valued differentiable functions such that $\alpha(0) = \beta(0) = 0$, and $\overline{\mathbf{q}}_i = \begin{bmatrix} x_i \\ y_i \\ z_i \\ w_i \end{bmatrix}$ is the initial position of the ith particle. Notice that

$$\begin{aligned} \dot{x}_i(t) &= -\dot{\alpha}(t) y_i(t), \\ \dot{y}_i(t) &= \dot{\alpha}(t) x_i(t), \\ \dot{z}_i(t) &= \dot{\beta}(t) w_i(t), \\ \dot{w}_i(t) &= \dot{\beta}(t) z_i(t). \end{aligned}$$

If we look at the angular momentum integrals in the xy and zw directions, we find that

$$L_{xy} = \frac{1}{\kappa(t)} \sum_{i=1}^{N} m_i (x_i(t)\dot{y}_i(t) - \dot{x}_i(t) y_i(t))$$

$$= \frac{\dot{\alpha}(t)}{\kappa(t)} \sum_{i=1}^{N} m_i \left((x_i \cos(\alpha(t)) - y_i \sin(\alpha(t)))^2 + (x_i \sin(\alpha(t)) + y_i \cos(\alpha(t)))^2 \right)$$

$$= \frac{\dot{\alpha}(t)}{\kappa(t)} \sum_{i=1}^{N} m_i (x_i^2 + y_i^2),$$

$$L_{zw} = \frac{1}{\kappa(t)} \sum_{i=1}^{N} m_i (z_i(t)\dot{w}_i(t) - \dot{z}_i(t) w_i(t))$$

$$= \frac{\dot{\beta}(t)}{\kappa(t)} \sum_{i=1}^{N} m_i ((z_i \cosh(\beta(t)) + w_i \sinh(\beta(t)))^2 - (z_i \sinh(\beta(t)) + w_i \cosh(\beta(t)))^2)$$

$$= \frac{\dot{\beta}(t)}{\kappa(t)} \sum_{i=1}^{N} m_i (z_i^2 - w_i^2).$$

Since $z_i^2 - w_i^2$ is always negative, L_{zw} is constant only if $\beta(t) = bK(t)$ for some $b \in \mathbb{R}$. L_{xy} is constant if either $\alpha(t) = aK(t)$ for some $a \in \mathbb{R}$ or $x_i = y_i = 0$ for all $i = 1, \ldots, N$. If $x_i = y_i = 0$ for all $i = 1, \ldots, N$, then the system satisfies the equation

$$m_i b \dot{\kappa} \begin{bmatrix} 0 \\ 0 \\ w_i \\ z_i \end{bmatrix} + m_i b^2 \kappa^2 \overline{\mathbf{q}}_i = \kappa^{3/2} \nabla_{\overline{\mathbf{q}}_i} U + m_i b^2 \kappa^2 \overline{\mathbf{q}}_i + m_i b \dot{\kappa} \begin{bmatrix} 0 \\ 0 \\ w_i \\ z_i \end{bmatrix}.$$

Consequently $\nabla_{\overline{\mathbf{q}}_i} U = 0$, which is impossible for $\kappa < 0$,.[5,13] If $\alpha(t) = aK(t)$, notice that $\frac{\dot{\kappa}(t)}{\kappa(t)} \dot{\xi}(t) = \ddot{\xi}(t)$, so $\overline{\mathbf{q}}_i(t)$ satisfies the following equation:

$$\kappa^{3/2}(t)\nabla_{\bar{\mathbf{q}}_i} U = m_i \kappa^2(t) \begin{bmatrix} (b^2(z_i^2 - w_i^2) - a^2(x_i^2 + y_i^2) - a^2)x_i \\ (b^2(z_i^2 - w_i^2) - a^2(x_i^2 + y_i^2) - a^2)y_i \\ (b^2 + b^2(z_i^2 - w_i^2) - a^2(x_i^2 + y_i^2))z_i \\ (b^2 + b^2(z_i^2 - w_i^2) - a^2(x_i^2 + y_i^2))w_i \end{bmatrix}$$

$$= m_i(a^2 + b^2)\kappa^2(t) \begin{bmatrix} (z_i^2 - w_i^2)x_i \\ (z_i^2 - w_i^2)y_i \\ -(x_i^2 + y_i^2)z_i \\ -(x_i^2 + y_i^2)w_i \end{bmatrix}.$$

Assuming that κ is not constant, this equation can only hold if $\nabla_{\bar{\mathbf{q}}_i} U = 0$, which is impossible for $\kappa < 0$. Therefore there are no homographic orbits for ξ_2. □

V. SPECIAL CENTRAL CONFIGURATIONS

We have shown in Sec. IV that there is a strong link between homographic orbits and special central configurations in $\mathbb{S}^3$. We will now look at several examples of special central configurations and provide a rough classification of all 4-body special central configurations. We will assume that $\mathbf{q} \in \mathbb{S}^3$ if no further confusion arises.

A configuration of N bodies is singular if there exists some $1 \le i < j \le N$ such that $\mathbf{q}_i = \pm\mathbf{q}_j$. In that case, the attractive force on $\mathbf{q}_i$ exerted by $\mathbf{q}_j$ is

$$\frac{m_i m_j(\mathbf{q}_j - \cos \mathbf{q}_i)}{\sin^3 d_{ij}} = \infty.$$

Recall that a non-singular configuration $\mathbf{q}$ of N bodies is a *special central configuration* if it is a critical point of U, i.e.,

$$\mathbf{F}_i = \nabla_{\mathbf{q}_i} U(\mathbf{q}) = 0, \ \ i = 1, \ldots, N.$$

In this section, we will make use of several results about special central configurations that have been proved in Ref. 13 as follows:

(1) No special central configuration in $\mathbb{S}^3$ has all masses lying in any closed hemisphere, unless all masses lie on a great 2-sphere.
(2) No special central configuration on $\mathbb{S}^2$ has all masses lying on any closed hemisphere, unless all masses lie on a great circle.
(3) If $\mathbf{q}$ is a special central configuration in $\mathbb{S}^3$, and $g \in SO(4)$, then the configuration $g\mathbf{q}$, resulting from the action of g on $\mathbf{q}$, is also a special central configuration.

A. Double Lagrangian special central configurations on $\mathbb{S}^2_{xyz}$

Let

$$\mathbb{S}^2_{xyz} := \{(x, y, z, w) \in \mathbb{R}^4 \ | \ x^2 + y^2 + z^2 = 1, w = 0\}.$$

One of the simplest central configurations is the Lagrange solution, consisting of 3 bodies of equal masses evenly spaced around a circle.[5,13] We now look at the special central configurations consisting of two parallel Lagrangian central configurations, which we will call double Lagrangian central configurations.

Proposition 5. In the 6-body problem on the sphere, there are infinitely many double Lagrangian special central configurations, i.e., configurations of the form

$$\mathbf{q}_1 = \begin{bmatrix} r_1 \\ 0 \\ c_1 \\ 0 \end{bmatrix}, \mathbf{q}_2 = \begin{bmatrix} -\frac{r_1}{2} \\ \frac{\sqrt{3}r_1}{2} \\ c_1 \\ 0 \end{bmatrix}, \mathbf{q}_3 = \begin{bmatrix} -\frac{r_1}{2} \\ -\frac{\sqrt{3}r_1}{2} \\ c_1 \\ 0 \end{bmatrix},$$

$$\mathbf{q}_4 = \begin{bmatrix} r_2 \\ 0 \\ c_2 \\ 0 \end{bmatrix}, \mathbf{q}_5 = \begin{bmatrix} -\frac{r_2}{2} \\ \frac{\sqrt{3}r_2}{2} \\ c_2 \\ 0 \end{bmatrix}, \mathbf{q}_6 = \begin{bmatrix} -\frac{r_2}{2} \\ -\frac{\sqrt{3}r_2}{2} \\ c_2 \\ 0 \end{bmatrix},$$

$$m_1 = m_2 = m_3 = 1,$$

$$m_4 = m_5 = m_6 = m,$$

where $c_1 \in (0,1)$, $c_2 \in (-1,0)$, $r_1 = \sqrt{1-c_1^2}$, $r_2 = \sqrt{1-c_2^2}$, *and* $m \in (0,\infty)$.

Proof. To obtain a special central configuration in the 6-body problem on the sphere, we must have

$$\nabla_{\mathbf{q}_i} U = \sum_{j=1, j\neq i}^{6} \frac{m_i m_j(\mathbf{q}_j - (\mathbf{q}_i \cdot \mathbf{q}_j)\mathbf{q}_i)}{(1-(\mathbf{q}_i \cdot \mathbf{q}_j)^2)^{3/2}} = 0$$

for $i = 1, 2, 3, 4, 5, 6$. By symmetry arguments, it is sufficient for the equations to hold for $\nabla_{\mathbf{q}_1} U$ and $\nabla_{\mathbf{q}_4} U$.

Since $d_{12} = d_{13}$ and $d_{15} = d_{16}$,

$$\mathbf{F}_1 = \frac{\mathbf{q}_2 + \mathbf{q}_3 - 2\cos d_{12}\mathbf{q}_1}{\sin^3 d_{12}} + m\frac{\mathbf{q}_4 - \cos d_{14}\mathbf{q}_1}{\sin^3 d_{14}} + m\frac{\mathbf{q}_5 + \mathbf{q}_6 - 2\cos d_{15}\mathbf{q}_1}{\sin^3 d_{15}}.$$

Thus the y and w components of $\nabla_{\mathbf{q}_1} U$ are zero. Similarly, the y and w components of $\nabla_{\mathbf{q}_4} U$ are zero. Also, we have

$$\mathbf{q}_i \cdot \nabla_{\mathbf{q}_i} U = \sum_{j=1, j\neq i}^{N} m_i m_j \frac{\mathbf{q}_j \cdot \mathbf{q}_i - \mathbf{q}_j \cdot \mathbf{q}_i}{(1-(\mathbf{q}_i \cdot \mathbf{q}_j)^2)^{3/2}} = 0,$$

so $\nabla_{\mathbf{q}_i} U$ is orthogonal to $\mathbf{q}_i$. Therefore the z components of $\nabla_{\mathbf{q}_1} U$ and $\nabla_{\mathbf{q}_4} U$ are zero if and only if the x components of them are zero. So it is sufficient to have the following two equations satisfied:

$$0 = \frac{3r_1^2 c_1}{(1-(c_1^2 - \frac{r_1^2}{2})^2)^{3/2}} + \frac{m(c_2 - (c_1c_2 + r_1r_2)c_1)}{(1-(c_1c_2 + r_2r_1)^2)^{3/2}} + \frac{m(2c_2 - (2c_1c_2 - r_1r_2)c_1)}{(1-(c_1c_2 - \frac{r_1r_2}{2})^2)^{3/2}}, \tag{30}$$

$$0 = \frac{c_1 - (c_1c_2 + r_1r_2)c_2}{(1-(c_1c_2 + r_2r_1)^2)^{3/2}} + \frac{2c_1 - (2c_1c_2 - r_1r_2)c_2}{(1-(c_1c_2 - \frac{r_1r_2}{2})^2)^{3/2}} + \frac{3mr_2^2 c_2}{(1-(c_2^2 - \frac{r_2^2}{2})^2)^{3/2}}. \tag{31}$$

By isolating m in (31), we get

$$m = -\frac{(1-(c_2^2 - \frac{r_2^2}{2})^2)^{3/2}}{3r_2^2 c_2}\left(\frac{c_1 - (c_1c_2 + r_1r_2)c_2}{(1-(c_1c_2 + r_2r_1)^2)^{3/2}} + \frac{2c_1 - (2c_1c_2 - r_1r_2)c_2}{(1-(c_1c_2 - \frac{r_1r_2}{2})^2)^{3/2}}\right). \tag{32}$$

If (c_1, c_2, m) satisfy the requirements for a special configuration, then by symmetry so do $(-c_2, -c_1, \frac{1}{m})$, so we can find all special central configurations with $c_1 \geq -c_2$ and then obtain the equivalent special central configurations with $c_1 < -c_2$. Let

$$B = \{(a,b) \in (0,1) \times (-1,0) : a \geq -b\}.$$

Consider the function

$$f : B \to \mathbb{R},$$

$$f(c_1,c_2) = \frac{3r_1^2 c_1}{(1-(c_1^2-\frac{r_1^2}{2})^2)^{3/2}} + \frac{m(c_2-(c_1c_2+r_1r_2)c_1)}{(1-(c_1c_2+r_2r_1)^2)^{3/2}} + \frac{m(2c_2-(2c_1c_2-r_1r_2)c_1)}{(1-(c_1c_2-\frac{r_1r_2}{2})^2)^{3/2}},$$

where m is as in (32). Since B is path-connected, there exists a path

$$p:[0,1]\to B$$

such that $p(0)=(\frac{1}{10},-\frac{1}{10})$ and $p(1)=(\frac{9}{10},-\frac{1}{2})$. The function f is continuous on its domain, so $f\circ p$ is continuous on $[0, 1]$. Since $f(p(0)) < 0$ and $f(p(1)) > 0$, we have by the intermediate value theorem that there exists $a\in[0,1]$ such that $f(p(a)) = 0$. This is true for any such path p. Define

$$A = \{(c_1,c_2)\in B : f(c_1,c_2)=0\}.$$

Then the set $B \setminus A$ must have $(\frac{9}{10},-\frac{1}{2})$ in a different path component than $(\frac{1}{10},-\frac{1}{10})$. No finite set can path-disconnect B, so $f(c_1, c_2) = 0$ has infinitely many solutions. But (c_1, c_2) is a special central configuration if $(c_1,c_2)\in A$ and $m(c_1,c_2)>0$. If $(c_1,c_2)\in B$, then $m(c_1,c_2)>0$ if and only if

$$\frac{c_1-(c_1c_2+r_1r_2)c_2}{(1-(c_1c_2+r_2r_1)^2)^{3/2}} + \frac{2c_1-(2c_1c_2-r_1r_2)c_2}{(1-(c_1c_2-\frac{r_1r_2}{2})^2)^{3/2}} > 0.$$

Note that we have $|c_1c_2+r_1r_2|<1$ since $c_1c_2+r_1r_2=\cos(d_{14})$, and $|2c_1c_2-r_1r_2|<2$ since $c_1c_2-\frac{r_1r_2}{2} = \cos(d_{15})$. Therefore, we have

$$\frac{c_1-(c_1c_2+r_1r_2)c_2}{(1-(c_1c_2+r_2r_1)^2)^{3/2}} + \frac{2c_1-(2c_1c_2-r_1r_2)c_2}{(1-(c_1c_2-\frac{r_1r_2}{2})^2)^{3/2}} > \frac{c_1+c_2}{(1-(c_1c_2+r_1r_2)^2)^{3/2}} + \frac{2c_1+2c_2}{(1-(c_1c_2-\frac{r_1r_2}{2})^2)^{3/2}} \geq 0$$

since $c_1\geq -c_2$ for $(c_1,c_2)\in B$. Then m is always positive in B, so every element $(c_1,c_2)\in A$ corresponds to the special central configuration

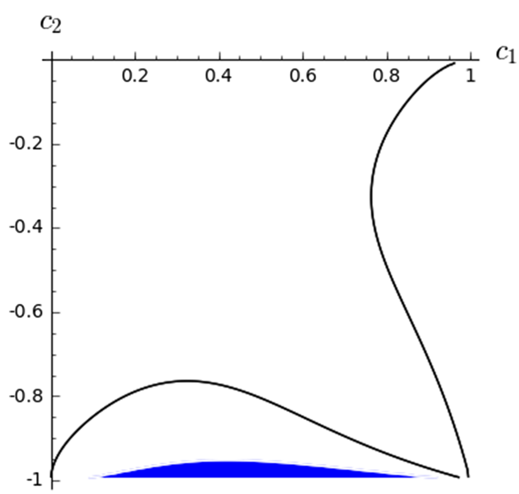

FIG. 1. The set of solutions to $f(c_1, c_2) = 0$.

$$\mathbf{q}_1 = \begin{bmatrix} \sqrt{1-c_1^2} \\ 0 \\ c_1 \\ 0 \end{bmatrix}, \mathbf{q}_2 = \begin{bmatrix} -\frac{\sqrt{1-c_1^2}}{2} \\ \frac{\sqrt{3}\sqrt{1-c_1^2}}{2} \\ c_1 \\ 0 \end{bmatrix}, \mathbf{q}_3 = \begin{bmatrix} -\frac{\sqrt{1-c_1^2}}{2} \\ -\frac{\sqrt{3}\sqrt{1-c_1^2}}{2} \\ c_1 \\ 0 \end{bmatrix},$$

$$\mathbf{q}_4 = \begin{bmatrix} \sqrt{1-c_2^2} \\ 0 \\ c_2 \\ 0 \end{bmatrix}, \mathbf{q}_5 = \begin{bmatrix} -\frac{\sqrt{1-c_2^2}}{2} \\ \frac{\sqrt{3}\sqrt{1-c_2^2}}{2} \\ c_2 \\ 0 \end{bmatrix}, \mathbf{q}_6 = \begin{bmatrix} -\frac{\sqrt{1-c_2^2}}{2} \\ -\frac{\sqrt{3}\sqrt{1-c_2^2}}{2} \\ c_2 \\ 0 \end{bmatrix},$$

$$m_1 = m_2 = m_3 = 1,$$

$$m_4 = m_5 = m_6 = m(c_1, c_2),$$

where $m(c_1, c_2)$ is as defined in (32). This remark completes the proof. □

To get a visual understanding of the roots of f, we insert (32) into (30) and implicitly plot the solutions of the resulting equation (see Figure 1). We can then see the set of solutions to $f(c_1, c_2) = 0$, where the curves are solutions, and the shaded region is formed by the (c_1, c_2) values for which $m(c_1, c_2) \leq 0$. Since no solution occurs in the shaded region, all these solutions represent special central configurations. As we showed above, the right branch of the solution set is a path-disconnecting subset of B, the solutions are symmetric about $c_1 = -c_2$, and m is positive on B.

B. Double tetrahedron special central configurations in $\mathbb{S}^3$

We now extend the previous case from two triangles on $\mathbb{S}^2_{xyz}$ to two tetrahedra in $\mathbb{S}^3$. We will call such a solution of the 8-body problem of the sphere a double tetrahedron special central configuration.

Proposition 6. In the 8-body problem in $\mathbb{S}^3$, there exist infinitely many double tetrahedron special central configurations, i.e., configurations of the form

$$\mathbf{q}_1 = \begin{bmatrix} r_1 \\ 0 \\ 0 \\ c_1 \end{bmatrix}, \mathbf{q}_2 = \begin{bmatrix} -\frac{r_1}{3} \\ \frac{2\sqrt{2}r_1}{3} \\ 0 \\ c_1 \end{bmatrix}, \mathbf{q}_3 = \begin{bmatrix} -\frac{r_1}{3} \\ -\frac{\sqrt{2}r_1}{3} \\ \frac{\sqrt{6}r_1}{3} \\ c_1 \end{bmatrix}, \mathbf{q}_4 = \begin{bmatrix} -\frac{r_1}{3} \\ -\frac{\sqrt{2}r_1}{3} \\ -\frac{\sqrt{6}r_1}{3} \\ c_1 \end{bmatrix},$$

$$\mathbf{q}_5 = \begin{bmatrix} r_2 \\ 0 \\ 0 \\ c_2 \end{bmatrix}, \mathbf{q}_6 = \begin{bmatrix} -\frac{r_2}{3} \\ \frac{2\sqrt{2}r_2}{3} \\ 0 \\ c_2 \end{bmatrix}, \mathbf{q}_7 = \begin{bmatrix} -\frac{r_2}{3} \\ -\frac{\sqrt{2}r_2}{3} \\ \frac{\sqrt{6}r_2}{3} \\ c_2 \end{bmatrix}, \mathbf{q}_8 = \begin{bmatrix} -\frac{r_2}{3} \\ -\frac{\sqrt{2}r_2}{3} \\ -\frac{\sqrt{6}r_2}{3} \\ c_2 \end{bmatrix},$$

$$m_1 = m_2 = m_3 = m_4 = 1,$$

$$m_5 = m_6 = m_7 = m_8 = m,$$

where $c_1 \in (0, 1), c_2 \in (-1, 0), m \in (0, \infty), r_1 = \sqrt{1-c_1^2}$, and $r_2 = \sqrt{1-c_2^2}$.

Proof. In order to have a special central configuration in the 8-body problem on the sphere, we must have

$$\nabla_{\mathbf{q}_i} U = \sum_{j=1, j\neq i}^{8} \frac{m_i m_j (\mathbf{q}_j - (\mathbf{q}_i \cdot \mathbf{q}_j)\mathbf{q}_i)}{(1-(\mathbf{q}_i \cdot \mathbf{q}_j)^2)^{3/2}} = 0 \tag{33}$$

for $i = 1, 2, 3, 4, 5, 6, 7, 8$. Let

$$g = \begin{bmatrix} -\frac{1}{3} & -\frac{\sqrt{2}}{3} & \frac{\sqrt{6}}{3} & 0 \\ \frac{2\sqrt{2}}{3} & -\frac{1}{6} & \frac{\sqrt{3}}{6} & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathrm{SO}(4), \quad h = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathrm{SO}(4).$$

The action of $\langle g, h\rangle$ on $\mathbf{q}$ is the permutation group

$$\langle(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)(\mathbf{q}_5, \mathbf{q}_6, \mathbf{q}_7), (\mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4)(\mathbf{q}_6, \mathbf{q}_7, \mathbf{q}_8)\rangle,$$

so by the symmetries of $\langle g, h\rangle$ it is sufficient for (33) to hold for $i = 1, 5$. For these two vertices, (33) becomes

$$\begin{aligned}\nabla_{\mathbf{q}_1} U &= \frac{\mathbf{q}_2 + \mathbf{q}_3 + \mathbf{q}_4 - 3(c_1^2 - \frac{r_1^2}{3})\mathbf{q}_1}{(1 - (c_1^2 - \frac{r_1^2}{3})^2)^{3/2}} + m\frac{\mathbf{q}_5 - (c_1c_2 + r_1r_2)\mathbf{q}_1}{(1 - (c_1c_2 + r_1r_2)^2)^{3/2}} \\ &\quad + m\frac{\mathbf{q}_6 + \mathbf{q}_7 + \mathbf{q}_8 - 3(c_1c_2 - \frac{r_1r_2}{3})\mathbf{q}_1}{(1 - (c_1c_2 - \frac{r_1r_2}{3})^2)^{3/2}}, \\ \nabla_{\mathbf{q}_5} U &= m\frac{\mathbf{q}_2 + \mathbf{q}_3 + \mathbf{q}_4 - 3(c_1c_2 - \frac{r_1r_2}{3})\mathbf{q}_5}{(1 - (c_1c_2 - \frac{r_1r_2}{3})^2)^{3/2}} + m\frac{\mathbf{q}_1 - (c_1c_2 + r_1r_2)\mathbf{q}_5}{(1 - (c_1c_2 + r_1r_2)^2)^{3/2}} \\ &\quad + m^2\frac{\mathbf{q}_6 + \mathbf{q}_7 + \mathbf{q}_8 - 3(c_2^2 - \frac{r_2^2}{3})\mathbf{q}_5}{(1 - (c_2^2 - \frac{r_2^2}{3})^2)^{3/2}}.\end{aligned}$$

Since

$$\mathbf{q}_2 + \mathbf{q}_3 + \mathbf{q}_4 = \begin{bmatrix} -r_1 \\ 0 \\ 0 \\ 3c_1 \end{bmatrix} \text{ and } \mathbf{q}_6 + \mathbf{q}_7 + \mathbf{q}_8 = \begin{bmatrix} -r_2 \\ 0 \\ 0 \\ 3c_2 \end{bmatrix},$$

we can see that the y and z coordinates of $\nabla_{\mathbf{q}_1} U, \nabla_{\mathbf{q}_5} U$ are identically 0. Also, by the identity $\mathbf{q}_i \cdot \nabla_{\mathbf{q}_i} U = 0$, we see that the w components of $\nabla_{\mathbf{q}_1} U$ and $\nabla_{\mathbf{q}_5} U$ are zero if and only if their x components are zero. Therefore, it is sufficient to have the following two equations satisfied:

$$0 = \frac{4r_1^2c_1}{(1 - (c_1^2 - \frac{r_1^2}{3})^2)^{3/2}} + \frac{m(c_2 - (c_1c_2 + r_1r_2)c_1)}{(1 - (c_1c_2 + r_2r_1)^2)^{3/2}} + \frac{m(3c_2 - (3c_1c_2 - r_1r_2)c_1)}{(1 - (c_1c_2 - \frac{r_1r_2}{3})^2)^{3/2}}, \tag{34}$$

$$0 = \frac{c_1 - (c_1c_2 + r_1r_2)c_2}{(1 - (c_1c_2 + r_2r_1)^2)^{3/2}} + \frac{3c_1 - (3c_1c_2 - r_1r_2)c_2}{(1 - (c_1c_2 - \frac{r_1r_2}{3})^2)^{3/2}} + \frac{4mr_2^2c_2}{(1 - (c_2^2 - \frac{r_2^2}{4})^2)^{3/2}}. \tag{35}$$

By isolating m in (35), we obtain

$$m = -\frac{(1 - (c_2^2 - \frac{r_2^2}{3})^2)^{3/2}}{4r_2^2c_2}\left(\frac{c_1 - (c_1c_2 + r_1r_2)c_2}{(1 - (c_1c_2 + r_2r_1)^2)^{3/2}} + \frac{3c_1 - (3c_1c_2 - r_1r_2)c_2}{(1 - (c_1c_2 - \frac{r_1r_2}{3})^2)^{3/2}}\right). \tag{36}$$

If the elements (c_1, c_2, m) satisfy the requirements for a special configuration, then by the $SO(4)$ rotation $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$, so do the elements $(-c_2, -c_1, \frac{1}{m})$. Therefore, we can find all special central configurations with $c_1 \geq -c_2$ and then obtain the equivalent special central configurations with $c_1 < -c_2$. Consider the set

$$B = \{(a, b) \in (0, 1) \times (-1, 0) : a \geq -b\}.$$

Define

$$g : B \to \mathbb{R},$$

$$g(c_1,c_2)=\frac{4r_1^2c_1}{(1-(c_1^2-\frac{r_1^2}{3})^2)^{3/2}}+\frac{m(c_2-(c_1c_2+r_1r_2)c_1)}{(1-(c_1c_2+r_2r_1)^2)^{3/2}}$$
$$+\frac{m(3c_2-(3c_1c_2-r_1r_2)c_1)}{(1-(c_1c_2-\frac{r_1r_2}{3})^2)^{3/2}},$$

where m is as in (36). Since B is path-connected, there exists a path

$$p:[0,1]\to B$$

such that $p(0)=(\frac{1}{10},-\frac{1}{10})$ and $p(1)=(\frac{9}{10},-\frac{1}{2})$. The function g is continuous on its domain, so $g\circ p$ is a continuous function on $[0, 1]$, and since $g(p(0))<0$ and $g(p(1))>0$, we have by the intermediate value theorem that there exists an $a\in[0,1]$ such that $g(p(a)) = 0$. This is true for any such path p. Define

$$C=\{(c_1,c_2)\in B: g(c_1,c_2)=0\}.$$

Then the set $B\setminus C$ must have $(\frac{9}{10},-\frac{1}{2})$ in a different path component than $(\frac{1}{10},-\frac{1}{10})$. No finite set can path-disconnect B, so $g(c_1, c_2) = 0$ has infinitely many solutions. But (c_1, c_2) is a special central configuration if $(c_1,c_2)\in C$ and $m(c_1,c_2)>0$. If $(c_1,c_2)\in B$, then $m(c_1,c_2)>0$ if and only if

$$\frac{c_1-(c_1c_2+r_1r_2)c_2}{(1-(c_1c_2+r_2r_1)^2)^{3/2}}+\frac{3c_1-(3c_1c_2-r_1r_2)c_2}{(1-(c_1c_2-\frac{r_1r_2}{3})^2)^{3/2}}>0.$$

Note that we have $|c_1c_2+r_1r_2|<1$, since $c_1c_2+r_1r_2=\cos(d_{15})$, and $|3c_1c_2-r_1r_2|<3$, since

$$c_1c_2-\frac{r_1r_2}{3}=\cos(d_{16}).$$

Therefore, we have

$$\frac{c_1-(c_1c_2+r_1r_2)c_2}{(1-(c_1c_2+r_2r_1)^2)^{3/2}}+\frac{3c_1-(3c_1c_2-r_1r_2)c_2}{(1-(c_1c_2-\frac{r_1r_2}{3})^2)^{3/2}}$$
$$>\frac{c_1+c_2}{(1-(c_1c_2+r_1r_2)^2)^{3/2}}+\frac{3c_1+3c_2}{(1-(c_1c_2-\frac{r_1r_2}{3})^2)^{3/2}}\ge 0$$

because $c_1\ge -c_2$ for $(c_1,c_2)\in B$. Then m is always positive in B, so every element $(c_1,c_2)\in C$ corresponds to the special central configuration of the form

$$\mathbf{q}_1=\begin{bmatrix}\sqrt{1-c_1^2}\\0\\0\\c_1\end{bmatrix},\ \mathbf{q}_2=\begin{bmatrix}-\frac{\sqrt{1-c_1^2}}{3}\\\frac{2\sqrt{2}\sqrt{1-c_1^2}}{3}\\0\\c_1\end{bmatrix},\ \mathbf{q}_3=\begin{bmatrix}-\frac{\sqrt{1-c_1^2}}{3}\\-\frac{\sqrt{2}\sqrt{1-c_1^2}}{3}\\\frac{\sqrt{6}\sqrt{1-c_1^2}}{3}\\c_1\end{bmatrix},\ \mathbf{q}_4=\begin{bmatrix}-\frac{\sqrt{1-c_1^2}}{3}\\-\frac{\sqrt{2}\sqrt{1-c_1^2}}{3}\\-\frac{\sqrt{6}\sqrt{1-c_1^2}}{3}\\c_1\end{bmatrix},$$

$$\mathbf{q}_5=\begin{bmatrix}\sqrt{1-c_2^2}\\0\\0\\c_2\end{bmatrix},\ \mathbf{q}_6=\begin{bmatrix}-\frac{\sqrt{1-c_2^2}}{3}\\\frac{2\sqrt{2}\sqrt{1-c_2^2}}{3}\\0\\c_2\end{bmatrix},\ \mathbf{q}_7=\begin{bmatrix}-\frac{\sqrt{1-c_2^2}}{3}\\-\frac{\sqrt{2}\sqrt{1-c_2^2}}{3}\\\frac{\sqrt{6}\sqrt{1-c_2^2}}{3}\\c_2\end{bmatrix},\ \mathbf{q}_8=\begin{bmatrix}-\frac{\sqrt{1-c_2^2}}{3}\\-\frac{\sqrt{2}\sqrt{1-c_2^2}}{3}\\-\frac{\sqrt{6}\sqrt{1-c_2^2}}{3}\\c_2\end{bmatrix},$$

$$m_1=m_2=m_3=m_4=1,$$

$$m_5=m_6=m_7=m_8=m(c_1,c_2),$$

where $m(c_1, c_2)$ is as defined in (36). This remark completes the proof. □

To get a visual understanding of the solutions to g, we insert (36) into (34) and implicitly plot the solutions to the resulting equation (see Figure 2). We can then see the set of solutions to $g(c_1, c_2) = 0$. As expected, the solutions are symmetric about $c_1 = -c_2$ and the right branch of the solution is a path-disconnecting set of B.

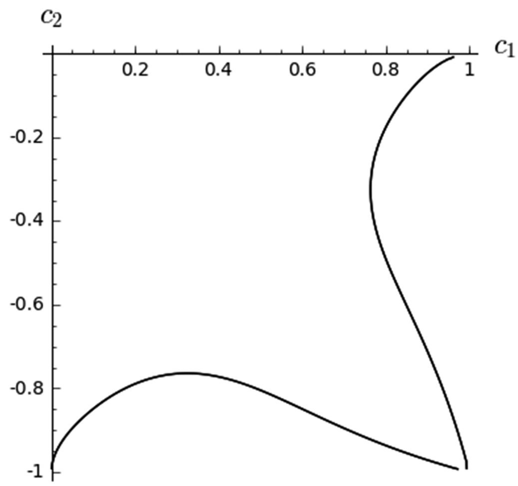

FIG. 2. The set of solutions to $g(c_1, c_2) = 0$.

C. Special central configurations for four bodies in $\mathbb{S}^3$

We first show that every special central configuration of the 4-body problem in $\mathbb{S}^3$ occurs on a great 2-sphere and then prove that there are no quadrilateral special central configurations on $\mathbb{S}^1$. Finally, we derive a necessary and sufficient condition for the existence of tetrahedral special central configurations.

Proposition 7. Every 4-body special central configuration in $\mathbb{S}^3$ occurs on a great 2-sphere.

Proof. Let $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_4)$ be a special central configuration in $\mathbb{S}^3$. Then $\mathbf{F}_i = 0$, $i = 1, \ldots, 4$. Recall Equation (11). We obtain

$$0 = \mathbf{F}_1 = \sum_{j=2}^{4} \frac{m_1 m_j (\mathbf{q}_j - \cos d_{1j} \mathbf{q}_1)}{\sin^3 d_{1j}} = \sum_{j=2}^{4} \frac{m_1 m_j \mathbf{q}_j}{\sin^3 d_{1j}} - \sum_{j=2}^{4} \frac{m_1 m_j \cos d_{1j}}{\sin^3 d_{1j}} \mathbf{q}_1.$$

This implies that the four vectors $\mathbf{q}_1, \ldots, \mathbf{q}_4$ are linearly dependent. Thus the dimension of the space spanned by $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$, and $\mathbf{q}_4$ is at most 3, and they must lie on a great 2-sphere. □

Proposition 8. There are no 4-body special central configurations on a great circle.

Proof. We first derive a necessary condition on the mutual distances and then show that no non-singular configurations satisfy the condition.

We may assume that the positions of masses are given by the polar coordinates $0 = \varphi_1 < \varphi_2 < \varphi_3 < \varphi_4 < 2\pi$ (see Figure 3). Recall that $\mathbf{q}$ is a non-singular configuration on $\mathbb{S}^3$ if $\mathbf{q}_i \neq \pm\mathbf{q}_j$ for all $i \neq j$. Thus, $\varphi_i \neq \pi$ for $i = 2, 3, 4$. Since they could not be on one half-circle,[13] there are two possibilities: two bodies are on the upper half-circle $\varphi \in (0, \pi)$ and one on the lower half-circle $\varphi \in (\pi, 2\pi)$; one body is on the upper half-circle $\varphi \in (0, \pi)$ and two on the lower half-circle $\varphi \in (\pi, 2\pi)$. By a reflection about the x-axis, the two cases become one. So we assume that two bodies are on the upper half-circle $\varphi \in (0, \pi)$ and one on the lower half-circle $\varphi \in (\pi, 2\pi)$, i.e., $\varphi_2 < \varphi_3 < \pi$. Then there are two possibilities for φ_4: $\varphi_4 \in (\pi, \pi + \varphi_2)$ and $\varphi_4 \in (\pi + \varphi_2, \pi + \varphi_3)$. The cases $\varphi \in (\pi, \pi + \varphi_2)$ and $\varphi \in (\pi + \varphi_2, \pi + \varphi_3)$ differ only by a rotation $-\varphi_4$ and the relabeling

$$\mathbf{q}_1' = \mathbf{q}_4,\ \mathbf{q}_2' = \mathbf{q}_1,\ \mathbf{q}_3' = \mathbf{q}_2,\ \mathbf{q}_4' = \mathbf{q}_3,$$

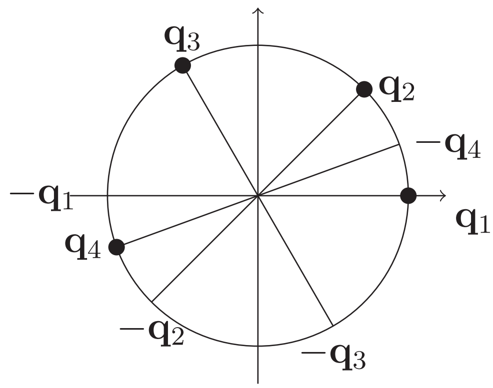

FIG. 3. A configuration for four masses on a great circle.

so it is sufficient to consider the case $\varphi_4 \in (\pi, \pi + \varphi_2)$. Note that the force function (1) on $\mathbb{S}^3$ is $U = \sum_{1\le i<j\le N} m_i m_j \cot d_{ij}$. In this case, we can write it as

$$U(\varphi_1, \dots, \varphi_4) = m_1 m_2 \cot(\varphi_2 - \varphi_1) + m_1 m_3 \cot(\varphi_3 - \varphi_1) - m_1 m_4 \cot(\varphi_4 - \varphi_1) + m_2 m_3 \cot(\varphi_3 - \varphi_2) + m_2 m_4 \cot(\varphi_4 - \varphi_2) + m_3 m_4 \cot(\varphi_4 - \varphi_3).$$

As $d_{14} = 2\pi - (\varphi_4 - \varphi_1)$, the sign of the term involving $m_1 m_4$ is negative. Since a special central configuration is a critical point of U, by taking the derivative with respect to φ_1, we have

$$\frac{m_1 m_2}{\sin^2(\varphi_2 - \varphi_1)} + \frac{m_1 m_3}{\sin^2(\varphi_3 - \varphi_1)} - \frac{m_1 m_4}{\sin^2(\varphi_4 - \varphi_1)} = 0.$$

Similarly, we obtain

$$\frac{m_2}{r_{12}} + \frac{m_3}{r_{13}} = \frac{m_4}{r_{14}}, \tag{37}$$

$$\frac{m_3}{r_{23}} + \frac{m_4}{r_{24}} = \frac{m_1}{r_{12}}, \tag{38}$$

$$\frac{m_1}{r_{13}} + \frac{m_2}{r_{23}} = \frac{m_4}{r_{34}}, \tag{39}$$

$$\frac{m_2}{r_{24}} + \frac{m_3}{r_{34}} = \frac{m_1}{r_{14}}, \tag{40}$$

where $r_{ij} = \sin^2 d_{ij} = \sin^2(\varphi_i - \varphi_j)$. Multiplying (37) by $\frac{1}{r_{34}}$ and subtracting (39) multiplied by $\frac{1}{r_{14}}$ lead to

$$m_2 \left(\frac{1}{r_{12} r_{34}} - \frac{1}{r_{23} r_{14}} \right) + \frac{m_3}{r_{13} r_{34}} = \frac{m_1}{r_{13} r_{14}}.$$

From (40) we can conclude that

$$\frac{m_2}{r_{24} r_{13}} + \frac{m_3}{r_{13} r_{34}} = \frac{m_1}{r_{13} r_{14}},$$

and we thus obtain the necessary condition

$$\frac{1}{r_{12} r_{34}} = \frac{1}{r_{23} r_{14}} + \frac{1}{r_{13} r_{24}}. \tag{41}$$

We now show that Equation (41) can never be satisfied. Note that r_{ij} is also equal to $\sin^2 d(\pm \mathbf{q}_i, \pm \mathbf{q}_j)$. Let us look now at the upper semicircle determined by $\mathbf{q}_2$ and $-\mathbf{q}_2$. Between the two boundary points, there lie $\mathbf{q}_3$, $-\mathbf{q}_1$, and $\mathbf{q}_4$ consecutively. Thus

$$0 < d(\mathbf{q}_2, \mathbf{q}_3) < d(\mathbf{q}_2, -\mathbf{q}_1) < d(\mathbf{q}_2, \mathbf{q}_4) < \pi,$$

and we get

$$r_{12} = \sin^2 d(\mathbf{q}_2, -\mathbf{q}_1) > \min\{\sin^2 d(\mathbf{q}_2, \mathbf{q}_3), \sin^2 d(\mathbf{q}_2, \mathbf{q}_4)\} = \min\{r_{23}, r_{24}\}.$$

Similarly, by focusing on other semicircles determined by $\mathbf{q}_i$ and $-\mathbf{q}_i$, we obtain other similar inequalities,

$$r_{12} > \min\{r_{13}, r_{14}\}, \qquad \text{i.e.,} \quad \frac{1}{r_{12}} < \max\left\{\frac{1}{r_{13}}, \frac{1}{r_{14}}\right\}, \tag{42}$$

$$r_{12} > \min\{r_{23}, r_{24}\}, \qquad \text{i.e.,} \quad \frac{1}{r_{12}} < \max\left\{\frac{1}{r_{23}}, \frac{1}{r_{24}}\right\}, \tag{43}$$

$$r_{34} > \min\{r_{13}, r_{23}\}, \qquad \text{i.e.,} \quad \frac{1}{r_{34}} < \max\left\{\frac{1}{r_{13}}, \frac{1}{r_{23}}\right\}, \tag{44}$$

$$r_{34} > \min\{r_{14}, r_{24}\}, \qquad \text{i.e.,} \quad \frac{1}{r_{34}} < \max\left\{\frac{1}{r_{14}}, \frac{1}{r_{24}}\right\}. \tag{45}$$

With (41) these four inequalities can be put in a useful form. We begin with the first inequality. There are two possibilities, namely, $\frac{1}{r_{14}} \geq \frac{1}{r_{13}}$ and $\frac{1}{r_{14}} \leq \frac{1}{r_{13}}$. If $\frac{1}{r_{14}} \geq \frac{1}{r_{13}}$, by inequality (42), we have $\frac{1}{r_{12}} < \frac{1}{r_{14}}$. In this case, we claim that $\frac{1}{r_{13}} > \frac{1}{r_{23}}$. If not, by inequality (44), we have $\frac{1}{r_{34}} < \frac{1}{r_{23}}$. Then the two inequalities $\frac{1}{r_{12}} < \frac{1}{r_{14}}$ and $\frac{1}{r_{34}} < \frac{1}{r_{23}}$ lead to $\frac{1}{r_{12}r_{34}} < \frac{1}{r_{23}r_{14}}$, which contradicts with Equation (41). Therefore, in the case $\frac{1}{r_{14}} \geq \frac{1}{r_{13}}$, we also have $\frac{1}{r_{13}} > \frac{1}{r_{23}}$, i.e., $\frac{1}{r_{14}} \geq \frac{1}{r_{13}} > \frac{1}{r_{23}}$. Similarly, the other case $\frac{1}{r_{14}} \leq \frac{1}{r_{13}}$ leads to $\frac{1}{r_{13}} \geq \frac{1}{r_{14}} > \frac{1}{r_{24}}$.

Using a similar argument on all the four inequalities, we obtain the following 8 inequalities:

$$\frac{1}{r_{14}} \geq \frac{1}{r_{13}} > \frac{1}{r_{23}} \quad \text{or} \quad \frac{1}{r_{13}} \geq \frac{1}{r_{14}} > \frac{1}{r_{24}}, \tag{46}$$

$$\frac{1}{r_{24}} \geq \frac{1}{r_{23}} > \frac{1}{r_{13}} \quad \text{or} \quad \frac{1}{r_{23}} \geq \frac{1}{r_{24}} > \frac{1}{r_{14}}, \tag{47}$$

$$\frac{1}{r_{13}} \geq \frac{1}{r_{23}} > \frac{1}{r_{24}} \quad \text{or} \quad \frac{1}{r_{23}} \geq \frac{1}{r_{13}} > \frac{1}{r_{14}}, \tag{48}$$

$$\frac{1}{r_{14}} \geq \frac{1}{r_{24}} > \frac{1}{r_{23}} \quad \text{or} \quad \frac{1}{r_{24}} \geq \frac{1}{r_{14}} > \frac{1}{r_{13}}. \tag{49}$$

Denote the left one of the i-th inequality by $(i+)$ and the right one by $(i-)$. Then we have 16 possibilities ((46)$*$, (47)$*$, (48)$*$, (49)$*$), where $*$ is $+$ or $-$. However, none of them is consistent. If we take (46)+ and (47)+, then we get $\frac{1}{r_{13}} > \frac{1}{r_{23}}$ and $\frac{1}{r_{13}} < \frac{1}{r_{23}}$, which is a contradiction. If we take (46)+ and (47)−, then we get $\frac{1}{r_{14}} \geq \frac{1}{r_{13}} > \frac{1}{r_{23}} \geq \frac{1}{r_{24}} > \frac{1}{r_{14}}$, which is a contradiction. If we take (46)− and (47)−, then we get $\frac{1}{r_{14}} > \frac{1}{r_{24}}$ and $\frac{1}{r_{14}} < \frac{1}{r_{24}}$, which is a contradiction. If we take (46)− and (47)+, then we get $\frac{1}{r_{13}} \geq \frac{1}{r_{14}} > \frac{1}{r_{24}} \geq \frac{1}{r_{23}} > \frac{1}{r_{13}}$, which is a contradiction.

Thus there is no non-singular configuration of four particles on $\mathbb{S}^1$ such that Equation (41) is satisfied. This implies that there is no special central configuration of four particles on $\mathbb{S}^1$. □

To prove our next proposition, we will rely on the following linear algebra result.

Lemma 1. Let $\{\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n\}$ *be a collection of vectors in* $\mathbb{R}^n$ *with rank n. Then*

$$D_0\mathbf{v}_0 - D_1\mathbf{v}_1 + \cdots + (-1)^n D_n\mathbf{v}_n = 0,$$

where $D_k = \det(\mathbf{v}_0, \ldots, \mathbf{v}_{k-1}, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_n)$.

Proof. Without loss of generality, we may assume that $D_0 \neq 0$. Then we can use Cramer's rule to solve the linear system

$$(\mathbf{v}_1, \ldots, \mathbf{v}_n)\mathbf{x} = -\mathbf{v}_0, \mathbf{x} = (x_1, x_2, \ldots, x_n)^T.$$

For $k = 1, \ldots, n$, we get

$$\begin{aligned} x_k &= \frac{\det(\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}, -\mathbf{v}_0, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_n)}{\det(\mathbf{v}_1, \ldots, \mathbf{v}_n)} \\ &= (-1)^k \frac{\det(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{k-1}, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_n)}{D_0} = (-1)^k \frac{D_k}{D_0}. \end{aligned}$$

Then $\sum_{k=0}^{n}(-1)^k D_k \mathbf{v}_k = 0$, so the proof is complete. □

Proposition 9. Let $\mathbf{q}$ *be a tetrahedron configuration of four masses* m_0, m_1, m_2, m_3, *on* $\mathbb{S}^2_{xyz}$ *of the form*

$$\mathbf{q}_0 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \mathbf{q}_1 = \begin{bmatrix}x_1\\y_1\\0\end{bmatrix}, \mathbf{q}_2 = \begin{bmatrix}x_2\\y_2\\z_2\end{bmatrix}, \mathbf{q}_3 = \begin{bmatrix}x_3\\y_3\\z_3\end{bmatrix}.$$

Then $\mathbf{q}$ *is a special central configuration if and only if the following three conditions are satisfied:*

(1) $\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ *are not all on the same hemisphere;*
(2) $\sin d_{01} \sin d_{23} = \sin d_{02} \sin d_{13} = \sin d_{03} \sin d_{12}$;
(3) $m_0 = -m_3 \frac{D_0 \sin^3 d_{01}}{D_3 \sin^3 d_{13}}, m_1 = m_3 \frac{D_1 \sin^3 d_{01}}{D_3 \sin^3 d_{03}}$, *and* $m_2 = -m_3 \frac{D_2 \sin^3 d_{02}}{D_3 \sin^3 d_{03}}$, *where* $D_0 = \det(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3), D_1 = \det(\mathbf{q}_0, \mathbf{q}_2, \mathbf{q}_3), D_2 = \det(\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_3)$, and $D_3 = \det(\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2)$.

Proof. Suppose $\mathbf{q}$ is a tetrahedron special central configuration. Then the four masses are not all on one hemisphere by the discussion at the beginning of this section, and $\mathbf{F}_i = \nabla_{\mathbf{q}_i} U = 0$ for $i = 0, 1, 2, 3$. Consider the z components of $\mathbf{F}_0$ and $\mathbf{F}_1$,

$$\frac{m_2 z_2}{\sin^3 d_{02}} + \frac{m_3 z_3}{\sin^3 d_{03}} = 0, \quad \frac{m_2 z_2}{\sin^3 d_{12}} + \frac{m_3 z_3}{\sin^3 d_{13}} = 0.$$

Since by assumption they are not on a great circle, y_1, z_2, and z_3 are non-zero. Thus we obtain $\sin^3 d_{03} \sin^3 d_{12} = \sin^3 d_{02} \sin^3 d_{13}$. By symmetry and relabeling of the masses, we also get the relation $\sin^3 d_{01} \sin^3 d_{23} = \sin^3 d_{03} \sin^3 d_{12}$. Therefore

$$\sin d_{03} \sin d_{12} = \sin d_{02} \sin d_{13} = \sin d_{01} \sin d_{23}.$$

For the masses, we look at

$$\mathbf{F}_0 = m_1 \frac{\mathbf{q}_1 - \cos d_{01} \mathbf{q}_0}{\sin^3 d_{01}} + m_2 \frac{\mathbf{q}_2 - \cos d_{02} \mathbf{q}_0}{\sin^3 d_{02}} + m_3 \frac{\mathbf{q}_3 - \cos d_{03} \mathbf{q}_0}{\sin^3 d_{03}},$$

and we have the z component $\frac{m_2 z_2}{\sin^3 d_{02}} + \frac{m_3 z_3}{\sin^3 d_{03}} = 0$, which implies

$$m_2 = -m_3 \frac{\sin^3 d_{02} z_3}{\sin^3 d_{03} z_2} = -m_3 \frac{\sin^3 d_{02} y_1 z_3}{\sin^3 d_{03} y_1 z_2} = -m_3 \frac{D_2 \sin^3 d_{02}}{D_3 \sin^3 d_{03}}. \tag{50}$$

The y component is $\frac{m_1 y_1}{\sin^3 d_{01}} + \frac{m_2 y_2}{\sin^3 d_{02}} + \frac{m_3 y_3}{\sin^3 d_{03}} = 0$, which, after inserting (50), gives

$$m_1 = m_3 \frac{(y_2 z_3 - z_2 y_3) \sin^3 d_{01}}{y_1 z_2 \sin^3 d_{03}} = m_3 \frac{D_1 \sin^3 d_{01}}{D_3 \sin^3 d_{03}}. \tag{51}$$

For m_0, we look at the inner product of $(y_1, -x_1, 0)^T$ with

$$\mathbf{F}_1 = m_0 \frac{\mathbf{q}_0 - \cos d_{01} \mathbf{q}_1}{\sin^3 d_{01}} + m_2 \frac{\mathbf{q}_2 - \cos d_{12} \mathbf{q}_1}{\sin^3 d_{12}} + m_3 \frac{\mathbf{q}_3 - \cos d_{13} \mathbf{q}_1}{\sin^3 d_{13}} = \mathbf{0}$$

to get

$$\begin{aligned}
0 &= \frac{m_0 y_1}{\sin^3 d_{01}} + \frac{m_2 (x_2 y_1 - x_1 y_2)}{\sin^3 d_{12}} + \frac{m_3 (y_1 x_3 - x_1 y_3)}{\sin^3 d_{13}} \\
&= \frac{m_0 y_1}{\sin^3 d_{01}} - \frac{m_3 \sin^3 d_{02}}{\sin^3 d_{03} \sin^3 d_{12}} \frac{z_3 (x_2 y_1 - x_1 y_2)}{z_2} + \frac{m_3 (x_3 y_1 - x_1 y_3)}{\sin^3 d_{13}} \\
&= \frac{m_0 y_1}{\sin^3 d_{01}} - \frac{m_3 (x_2 y_1 z_3 - x_1 y_2 z_3 - y_1 x_3 z_2 + x_1 y_3 z_2)}{z_2 \sin^3 d_{13}}.
\end{aligned}$$

So we have

$$m_0 = m_3 \frac{\sin^3 d_{01}}{\sin^3 d_{13}} \frac{y_1 (x_2 z_3 - x_3 z_2) - x_1 (y_2 z_3 - y_3 z_2)}{y_1 z_2} = -m_3 \frac{D_0 \sin^3 d_{01}}{D_3 \sin^3 d_{13}}. \tag{52}$$

Conversely, suppose that $\mathbf{q}$ is a configuration satisfying the above three conditions. We prove that $\mathbf{q}$ is a special central configuration, i.e., $F_i = \mathbf{0}$ and $m_i > 0$ for $i = 0, 1, 2, 3$.

First, $\mathbf{F}_0 = \mathbf{0}$ since $\nabla_{\mathbf{q}_0} U \cdot \mathbf{q}_0 = 0$, i.e., the x-component of $\mathbf{F}_0$ is zero, and the y and z components are zero by (50) and (51).

For $i = 1, 2, 3$, $\mathbf{F}_i = \mathbf{0}$ if and only if $\mathbf{F}_i \cdot \mathbf{v}_{ij} = 0, j = 1, 2, 3$, where $\mathbf{v}_{i1} = \mathbf{q}_i$, and $\{\mathbf{v}_{i1}, \mathbf{v}_{i2}, \mathbf{v}_{i3}\}$ is an orthonormal basis of $\mathbb{R}^3$. First, $\mathbf{F}_i \cdot \mathbf{v}_{i1} = \nabla_{\mathbf{q}_i} U \cdot \mathbf{q}_i = 0$. For $i = 1, j = 2, 3$, we have

$$\begin{aligned}
\mathbf{F}_1 \cdot \mathbf{v}_{1j} &= \left(m_0 \frac{\mathbf{q}_0 - \cos d_{01} \mathbf{q}_1}{\sin^3 d_{01}} + m_2 \frac{\mathbf{q}_2 - \cos d_{12} \mathbf{q}_1}{\sin^3 d_{12}} + m_3 \frac{\mathbf{q}_3 - \cos d_{13} \mathbf{q}_1}{\sin^3 d_{13}} \right) \cdot \mathbf{v}_{1j} \\
&= \left(\frac{m_0 \mathbf{q}_0}{\sin^3 d_{01}} + \frac{m_2 \mathbf{q}_2}{\sin^3 d_{12}} + \frac{m_3 \mathbf{q}_3}{\sin^3 d_{13}} \right) \cdot \mathbf{v}_{1j} \\
&= \left(-m_3 \frac{D_0 \mathbf{q}_0}{D_3 \sin^3 d_{13}} - m_3 \frac{D_2 \sin^3 d_{02} \mathbf{q}_2}{D_3 \sin^3 d_{03} \sin^3 d_{12}} + \frac{m_3 \mathbf{q}_3}{\sin^3 d_{13}} \right) \cdot \mathbf{v}_{1j} \\
&= -\frac{m_3}{D_3 \sin^3 d_{13}} (D_0 \mathbf{q}_0 + D_2 \mathbf{q}_2 - D_3 \mathbf{q}_3) \cdot \mathbf{v}_{1j} = -\frac{m_3}{D_3 \sin^3 d_{13}} D_1 \mathbf{q}_1 \cdot v_{1j} = 0,
\end{aligned}$$

the second last equality following from the previous lemma.

Through similar computations, we can see that for $j = 2, 3$,

$$\begin{aligned}
\mathbf{F}_2 \cdot \mathbf{v}_{2j} &= -\frac{m_3}{D_3 \sin^3 d_{23}} (D_0 \mathbf{q}_0 - D_1 \mathbf{q}_1 - D_3 \mathbf{q}_3) \cdot \mathbf{v}_{2j} = \frac{m_3 D_2}{D_3 \sin^3 d_{23}} \mathbf{q}_2 \cdot \mathbf{v}_{2j} = 0, \\
\mathbf{F}_3 \cdot \mathbf{v}_{3j} &= -\frac{m_3 \sin^3 d_{01}}{D_3 \sin^3 d_{13} \sin^3 d_{03}} (D_0 \mathbf{q}_0 - D_1 \mathbf{q}_1 + D_2 \mathbf{q}_2) \cdot \mathbf{v}_{3j} \\
&= -\frac{m_3 \sin^3 d_{01}}{\sin^3 d_{13} \sin^3 d_{03}} \mathbf{q}_3 \cdot \mathbf{v}_{3j} = 0.
\end{aligned}$$

Therefore $\mathbf{F}_i = \mathbf{0}$ for $i = 0, 1, 2, 3$. To show that the masses are positive, we first show that $D_i \neq 0$, for $i = 0, 1, 2, 3$. If not, then three of the masses lie on a great circle of $\mathbb{S}^2_{xyz}$, so the four masses all lie on one hemisphere.

Without loss of generality, assume $D_3 > 0$. Consider the two-dimensional subspace V_{12} determined by $\mathbf{q}_1$, $\mathbf{q}_2$. Since the configuration is not on one hemisphere, V_{12} must separate $\mathbf{q}_0$ and $\mathbf{q}_3$. Then

$$D_3 = \det(\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2) = \det(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_0) > 0 \text{ implies } D_0 = \det(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) < 0.$$

Similarly, the subspace V_{02} separates $\mathbf{q}_1$ and $\mathbf{q}_3$, so

$$\det(\mathbf{q}_0, \mathbf{q}_2, \mathbf{q}_1) = -\det(\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2) = -D_3 < 0 \text{ implies } D_1 = \det(\mathbf{q}_0, \mathbf{q}_2, \mathbf{q}_3) > 0,$$

and the subspace V_{01} separates $\mathbf{q}_2$ and $\mathbf{q}_3$, so

$$\det(\mathbf{q}_0, \mathbf{q}_1 \mathbf{q}_2) = D_3 > 0 \text{ implies } D_2 = \det(\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_3) < 0.$$

Then $m_0 > 0, m_1 > 0, m_2 > 0$ if and only if $m_3 > 0$, so $\mathbf{q}$ is a special central configuration. □

D. Special central configurations for five bodies in $\mathbb{S}^3$

In this section, we generalize the method from the previous proof from tetrahedra on $\mathbb{S}^2_{xyz}$ to pentatopes in $\mathbb{S}^3$ to prove the following result.

Proposition 10. Let $\mathbf{q}$ *be a pentatope configuration for five masses,* m_0, m_1, m_2, m_3, m_4, *in* $\mathbb{S}^3$ *of the form*

$$\mathbf{q}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{q}_1 = \begin{bmatrix} x_1 \\ y_y \\ 0 \\ 0 \end{bmatrix}, \mathbf{q}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ 0 \end{bmatrix}, \mathbf{q}_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \\ w_3 \end{bmatrix}, \mathbf{q}_4 = \begin{bmatrix} x_4 \\ y_4 \\ z_4 \\ w_4 \end{bmatrix}.$$

Then $\mathbf{q}$ is a special central configuration if and only if the following conditions are satisfied:

(1) $\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4$ *are not all in one hemisphere;*
(2) $\frac{\sin d_{01}}{\sin d_{04}} = \frac{\sin d_{12}}{\sin d_{24}} = \frac{\sin d_{13}}{\sin d_{34}}$;
(3) $\frac{\sin d_{02}}{\sin d_{04}} = \frac{\sin d_{12}}{\sin d_{14}} = \frac{\sin d_{23}}{\sin d_{34}}$;
(4) $\frac{\sin d_{03}}{\sin d_{04}} = \frac{\sin d_{13}}{\sin d_{14}} = \frac{\sin d_{23}}{\sin d_{24}}$;
(5) $m_0 = m_4\frac{D_0\sin^3 d_{01}}{D_4\sin^3 d_{14}}$, $m_1 = -m_4\frac{D_1\sin^3 d_{01}}{D_4\sin^3 d_{04}}$, $m_2 = m_4\frac{D_2\sin^3 d_{02}}{D_4\sin^3 d_{04}}$, *and* $m_3 = -m_4\frac{D_3\sin^3 d_{03}}{D_4\sin^3 d_{04}}$, *where*

$$D_0 = \det(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4), D_1 = \det(\mathbf{q}_0, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4), D_2 = \det(\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_3, \mathbf{q}_4),$$

$$D_3 = \det(\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_4), \text{ and } D_4 = \det(\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3).$$

Proof. Suppose $\mathbf{q}$ is a pentatope special central configuration. Then the five masses are not all in one hemisphere by the discussion at the beginning of this section, and we have $\mathbf{F}_i = 0$ for $i = 0, 1, 2, 3, 4$. Consider the w components of $\mathbf{F}_0$, $\mathbf{F}_1$, and $\mathbf{F}_2$,

$$\frac{m_3 w_3}{\sin^3 d_{03}} + \frac{m_4 w_4}{\sin^3 d_{04}} = 0, \frac{m_3 w_3}{\sin^3 d_{13}} + \frac{m_4 w_4}{\sin^3 d_{14}} = 0, \frac{m_3 w_3}{\sin^3 d_{23}} + \frac{m_4 w_4}{\sin^3 d_{34}} = 0.$$

Recall that we are assuming the configuration is a pentatope, i.e., $\mathbf{q}$ does not lie on a great 2-sphere, which implies that y_1, z_2, w_3, w_4 are non-zero. Thus we obtain

$$\frac{\sin d_{03}}{\sin d_{04}} = \frac{\sin d_{13}}{\sin d_{14}} = \frac{\sin d_{23}}{\sin d_{24}}.$$

By symmetry and relabeling of the masses, we also obtain the relations

$$\frac{\sin d_{01}}{\sin d_{04}} = \frac{\sin d_{12}}{\sin d_{24}} = \frac{\sin d_{13}}{\sin d_{34}}$$

and

$$\frac{\sin d_{02}}{\sin d_{04}} = \frac{\sin d_{12}}{\sin d_{14}} = \frac{\sin d_{23}}{\sin d_{34}}.$$

If we look at

$$\mathbf{F}_0 = m_1\frac{\mathbf{q}_1 - \cos d_{01}\mathbf{q}_0}{\sin^3 d_{01}} + m_2\frac{\mathbf{q}_2 - \cos d_{02}\mathbf{q}_0}{\sin^3 d_{02}} + m_3\frac{\mathbf{q}_3 - \cos d_{03}\mathbf{q}_0}{\sin^3 d_{03}} + m_4\frac{\mathbf{q}_4 - \cos d_{04}\mathbf{q}_0}{\sin^3 d_{04}},$$

which is $\mathbf{0}$, we see that the w component $\frac{m_3 w_3}{\sin^3 d_{03}} + \frac{m_4 w_4}{\sin^3 d_{04}} = 0$ gives

$$m_3 = -m_4\frac{w_4\sin^3 d_{03}}{w_3\sin^3 d_{04}} = -m_4\frac{y_1 z_2 w_4\sin^3 d_{03}}{y_1 z_2 w_3\sin^3 d_{04}} = -m_4\frac{D_3\sin^3 d_{03}}{D_4\sin^3 d_{04}}. \tag{53}$$

After inserting (53) into the z component $\frac{m_2 z_2}{\sin^3 d_{02}} + \frac{m_3 z_3}{\sin^3 d_{03}} + \frac{m_4 z_4}{\sin^3 d_{04}} = 0$, we have

$$m_2 = m_4\frac{y_1(z_3 w_4 - z_4 w_3)\sin^3 d_{02}}{y_1 z_2 w_3\sin^3 d_{04}} = m_4\frac{D_2\sin^3 d_{02}}{D_4\sin^3 d_{04}}. \tag{54}$$

After inserting (53) and (54) into the y component $\frac{m_1 y_1}{\sin^3 d_{01}} + \frac{m_2 y_2}{\sin^3 d_{02}} + \frac{m_3 y_3}{\sin^3 d_{03}} + \frac{m_4 y_4}{\sin^3 d_{04}} = 0$, we obtain

$$m_1 = -m_4\frac{y_2(z_3 w_4 - z_4 w_3) - z_2(y-3w_4 - y_4 w_3)}{y_1 z_2 w_3}\frac{\sin^3 d_{01}}{\sin^3 d_{04}} = -m_4\frac{D_1\sin^3 d_{01}}{D_4\sin^3 d_{04}}. \tag{55}$$

We obtain m_0 by taking the inner product of $(y_1, -x_1, 0, 0)^T$ with

$$\mathbf{F}_1 = m_0\frac{\mathbf{q}_0 - \cos d_{01}\mathbf{q}_1}{\sin^3 d_{01}} + m_2\frac{\mathbf{q}_2 - \cos d_{12}\mathbf{q}_1}{\sin^3 d_{12}}$$
$$+ m_3\frac{\mathbf{q}_3 - \cos d_{13}\mathbf{q}_1}{\sin^3 d_{13}} + m_4\frac{\mathbf{q}_4 - \cos d_{14}\mathbf{q}_1}{\sin^3 d_{14}} = \mathbf{0}$$

to get

$$
\begin{aligned}
0 &= \frac{m_0 y_1}{\sin^3 d_{01}} + m_2 \frac{x_2 y_1 - x_1 y_2}{\sin^3 d_{12}} + m_3 \frac{x_3 y_1 - x_1 y_3}{\sin^3 d_{13}} + m_4 \frac{x_4 y_1 - x_1 y_4}{\sin^3 d_{14}} \\
&= \frac{m_0 y_1}{\sin^3 d_{01}} + m_4 \frac{(x_2 y_1 - x_1 y_2)(z_3 w_4 - z_4 w_3)\sin^3 d_{02}}{z_2 w_3 \sin^3 d_{12} \sin^3 d_{04}} \\
&\quad - m_4 \frac{(x_3 y_1 - x_1 y_3) z_2 w_4 \sin^3 d_{03}}{z_2 w_3 \sin^3 d_{13} \sin^3 d_{04}} + m_4 \frac{(x_4 y_1 - x_1 y_4) z_2 w_3}{z_2 w_3 \sin^3 d_{14}} \\
&= \frac{m_0 y_1}{\sin^3 d_{01}} - m_4 \frac{(x_1 y_2 - x_2 y_1)(z_3 w_4 - z_4 w_3) + (x_1 y_4 - x_4 y_1)(w_3 - w_4) z_2}{\sin^3 d_{14}}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
m_0 &= m_4 \frac{(x_1 y_2 - x_2 y_1)(z_3 w_4 - z_4 w_3) + (x_1 y_3 - x_3 y_1)(w_3 - w_4) z_2}{y_1 z_2 w_3} \frac{\sin^3 d_{01}}{\sin^3 d_{14}} \\
&= m_4 \frac{D_0 \sin^3 d_{01}}{D_4 \sin^3 d_{14}}.
\end{aligned} \tag{56}
$$

Conversely, suppose that $\mathbf{q}$ is a configuration which satisfies the above 5 conditions. We now prove that $\mathbf{q}$ is a special central configuration, i.e., $\mathbf{F}_i = \mathbf{0}$, $m_i > 0$ for $i = 0, 1, 2, 3, 4$.

First, $\mathbf{F}_0 = \mathbf{0}$ since $\nabla_{\mathbf{q}_0} U \cdot \mathbf{q}_0 = 0$, i.e., the x-component of $\mathbf{F}_0$ is zero, and the y, z, and w components are zero by (53)–(55).

For $i = 1, 2, 3, 4$, $\mathbf{F}_i = 0$ if and only if $\mathbf{F}_i \cdot \mathbf{v}_{ij} = 0$, $j = 1, 2, 3, 4$, where $\mathbf{v}_{i1} = \mathbf{q}_i$, and $\{\mathbf{v}_{i1}, \mathbf{v}_{i2}, \mathbf{v}_{i3}, \mathbf{v}_{i4}\}$ form an orthonormal basis of $\mathbb{R}^3$. First, $\mathbf{F}_i \cdot \mathbf{v}_{i1} = \nabla_{\mathbf{q}_i} U \cdot \mathbf{q}_i = 0$. For $i = 1$, $j = 2, 3, 4$, we have

$$
\begin{aligned}
\mathbf{F}_1 \cdot \mathbf{v}_{1j} &= \Bigg(m_0 \frac{\mathbf{q}_0 - \cos d_{01} \mathbf{q}_1}{\sin^3 d_{01}} + m_2 \frac{\mathbf{q}_2 - \cos d_{12} \mathbf{q}_1}{\sin^3 d_{12}} \\
&\qquad + m_3 \frac{\mathbf{q}_3 - \cos d_{13} \mathbf{q}_1}{\sin^3 d_{13}} + m_4 \frac{\mathbf{q}_4 - \cos d_{14} \mathbf{q}_1}{\sin^3 d_{14}} \Bigg) \cdot \mathbf{v}_{1j} \\
&= \left(\frac{m_0 \mathbf{q}_0}{\sin^3 d_{01}} + \frac{m_2 \mathbf{q}_2}{\sin^3 d_{12}} + \frac{m_3 \mathbf{q}_3}{\sin^3 d_{13}} + \frac{m_4 \mathbf{q}_4}{\sin^3 d_{14}} \right) \cdot \mathbf{v}_{1j} \\
&= \Bigg(m_4 \frac{D_0 \mathbf{q}_0}{D_4 \sin^3 d_{14}} + m_4 \frac{D_2 \sin^3 d_{02} \mathbf{q}_2}{D_4 \sin^3 d_{04} \sin^3 d_{12}} \\
&\qquad - m_4 \frac{D_3 \sin^3 d_{03} \mathbf{q}_3}{D_4 \sin^3 d_{04} \sin^3 d_{13}} + \frac{m_4 \mathbf{q}_4}{\sin^3 d_{14}} \Bigg) \cdot \mathbf{v}_{1j} \\
&= \frac{m_4}{D_4 \sin^3 d_{14}} (D_0 \mathbf{q}_0 + D_2 \mathbf{q}_2 - D_3 \mathbf{q}_3 + D_4 \mathbf{q}_4) \cdot \mathbf{v}_{1j} \\
&= \frac{m_4 D_1}{D_4 \sin^3 d_{14}} \mathbf{q}_1 \cdot \mathbf{v}_{1j} = 0.
\end{aligned}
$$

Through similar computations, we see that for $j = 2, 3, 4$,

$$
\begin{aligned}
\mathbf{F}_2 \cdot \mathbf{v}_{2j} &= \frac{m_4}{D_4 \sin^3 d_{24}} (D_0 \mathbf{q}_0 - D_1 \mathbf{q}_1 - D_3 \mathbf{q}_3 + D_4 \mathbf{q}_4) \cdot \mathbf{v}_{2j} = \frac{-m_4 D_2}{D_4 \sin^3 d_{24}} \mathbf{q}_2 \cdot \mathbf{v}_{2j} = 0, \\
\mathbf{F}_3 \cdot \mathbf{v}_{3j} &= \frac{m_4}{D_4 \sin^3 d_{34}} (D_0 \mathbf{q}_0 - D_1 \mathbf{q}_1 + D_2 \mathbf{q}_2 + D_4 \mathbf{q}_4) \cdot \mathbf{v}_{3j} = \frac{m_4 D_3}{D_4 \sin^3 d_{23}} \mathbf{q}_3 \cdot \mathbf{v}_{3j} = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{F}_4 \cdot \mathbf{v}_{4j} &= \frac{m_4 \sin^3 d_{01}}{D_4 \sin^3 d_{14} \sin^3 d_{04}} (D_0 \mathbf{q}_0 - D_1 \mathbf{q}_1 + D_2 \mathbf{q}_2 - D_3 \mathbf{q}_3) \cdot \mathbf{v}_{4j} \\
&= \frac{-m_4 \sin^3 d_{01}}{\sin^3 d_{14} \sin^3 d_{04}} \mathbf{q}_3 \cdot \mathbf{v}_{4j} = 0.
\end{aligned}
$$

Therefore $\mathbf{F}_i = 0$ for $i = 0, 1, 2, 3, 4$. To show that the masses are positive, we first show that $D_i \neq 0$ for $i = 0, 1, 2, 3, 4$. If not, then four of the masses lie on a great sphere, so the five masses all lie in one hemisphere.

Without loss of generality, assume $D_4 > 0$. Consider the 3-dimensional subspace V_{123}. Since the configuration is not in one hemisphere, V_{123} must separate $\mathbf{q}_0$ and $\mathbf{q}_4$. Then

$$\det(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_0) = -\det(\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) = -D_4 < 0$$

implies that

$$D_0 = \det(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) > 0.$$

Similarly, the subspace V_{023} separates $\mathbf{q}_1$ and $\mathbf{q}_4$, so

$$\det(\mathbf{q}_0, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_1) = \det(\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) = D_4 > 0$$

implies that

$$D_1 = \det(\mathbf{q}_0, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) < 0.$$

The subspace V_{013} separates $\mathbf{q}_2$ and $\mathbf{q}_4$, so

$$\det(\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_3, \mathbf{q}_2) = -\det(\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) = -D_4 < 0$$

implies that

$$D_2 = \det(\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_3, \mathbf{q}_4) > 0,$$

and the subspace V_{012} separates $\mathbf{q}_3$ and $\mathbf{q}_4$, so

$$\det(\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) = D_4 > 0$$

implies that

$$D_3 = \det(\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_4) < 0.$$

Then $m_0 > 0, m_1 > 0, m_2 > 0, m_3 > 0$ if and only if $m_4 > 0$, so $\mathbf{q}$ is a special central configuration. This remark completes the proof. □

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