

# CENTRAL CONFIGURATIONS OF THE CURVED $N$ -BODY PROBLEM

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**ABSTRACT.** We consider the  $N$ -body problem of celestial mechanics in spaces of nonzero constant curvature. Using the concept of effective potential, we define the moment of inertia for systems moving on spheres and hyperbolic spheres and show that we can recover the classical definition in the Euclidean case. After proving some criteria for the existence of relative equilibria, we find a natural way to define the concept of central configuration in curved spaces using the moment of inertia, and show that our definition is formally similar to the one that governs the classical problem. We prove that, for any given point masses on spheres and hyperbolic spheres, central configurations always exist. We end with results concerning the number of central configurations that lie on the same geodesic, thus extending the celebrated theorem of Moulton to hyperbolic spheres and pointing out that it has no straightforward generalization to spheres, where the count gets complicated even for two bodies.

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## 1. INTRODUCTION

Central configurations for the  $N$ -body problem of celestial mechanics were introduced by Pierre-Simon Laplace in 1789 in connection with the discovery of Eulerian and Lagrangian orbits and via Kepler's laws in the flat case, [36], [27], [35]. But a first systematic study of this concept appeared only in 1900, when Otto Dziobek published a fundamental paper on central configurations, [25]. Research in this direction has continued ever since, showing over the past decades that central configurations are essential for understanding the  $N$ -body problem. Although breakthroughs are rare in this difficult area of mathematics, some recent progress has been made on the Wintner-Smale conjecture, which we will discuss later.

**1.1. Motivation.** In 1772 Joseph Louis Lagrange found the equilateral solutions of the 3-body problem and rediscovered the collinear orbits, whose existence Leonhard Euler had proved a decade earlier. These motions, called homographic because they stay similar to themselves for all time, can be decomposed into homothetic and relative equilibrium solutions. The former are dilations and/or contractions of the particle

system without rotation, whereas the latter are rotations without dilations or contractions, such that the mutual distances remain constant during the motion. Starting from the homothetic Lagrangian orbits, Laplace noticed that it may be simpler to seek the configurations that remain similar to themselves, which we now call central configurations, instead of looking for the homographic solutions to the differential equations, [63]. Central configurations do not involve the time variable and are given by the system

$$\nabla U(\mathbf{q}) = \lambda \nabla I(\mathbf{q}),$$

where  $\mathbf{q}$  provides the positions of the bodies,  $U$  is the force function (the negative of the potential),  $I$  is the moment of inertia as defined in (2) below,  $\lambda$  is a constant, and  $\nabla$  denotes the gradient. In this case, central configurations provide classes of relative equilibrium, homothetic, and homographic orbits by reducing the dynamical question of finding solutions of a differential equation to solving algebraic systems.

**1.2. Importance.** Research done since 1900 has shown that the concept of central configuration opens a path towards understanding the  $N$ -body problem. Not only that it offers a method for finding periodic solutions, but it appears in various other circumstances. For instance, when three or more bodies tend to a simultaneous collision, or when the system becomes unbounded and the mutual distances between bodies tend to infinity, the system tends asymptotically to a central configuration, [52], [53].

But finding central configurations is far from easy. Basic questions related to them are often difficult. The Wintner-Smale conjecture, for instance, became notorious after Steven Smale placed it sixth on his list of open problems for the 21st century, [57]. The problem asks whether, for given  $N$  positive masses, the number of planar central configurations is finite or not. So far, the conjecture was solved only for  $N = 3, 4$ , and  $5$ , see [44] and [2]. In all these cases the answer is that the set of central configurations is finite. But it is possible that for more than five bodies this set is infinite. If so, it may be countable or contain a continuum, as it actually happens when some masses are negative or charges are embedded in the system, [3], [51].

**1.3. Our goal.** We consider here the motion of  $N$  point masses in spaces of constant Gaussian curvature  $\kappa \neq 0$ , namely spheres for  $\kappa > 0$  and hyperbolic spheres for  $\kappa < 0$  in a non-relativistic context. This problem stems from the work of János Bolyai and Nikolai Lobachevsky, done in the 1830s, who independently had the idea of generalizing celestial mechanics to hyperbolic space, being among the first to understand that the laws of physics are related to the geometry of the universe, [6], [39]. The problem was further pursued by Ernest Schering, [54], [55], Wilhelm Killing, [32], Heinrich Liebmann [37], [38], and others. More recent work in this direction appears in [33], [56], [7-23], [28], [42], [43], [46], [49], [60-62], [64], [65]. A history of the problem and reasons why it is worth studying can be found in [9].

In this paper we extend the concept of central configuration to the  $N$ -body problem in spaces of constant Gaussian curvature. Our idea was to find a formal definition that resembles the classical one. To achieve this goal we had to formulate first the correct definition of the moment of inertia for 3-spheres and hyperbolic 3-spheres, such

that it agrees with the standard definition known in the Euclidean space. This step proved more difficult than we expected, also because of a terminological mixup that had occurred in the past few decades in the literature pertaining to the Newtonian  $N$ -body problem. A main obstacle was that, in the 3-dimensional case, the definition of the moment of inertia we considered suitable for our purposes did not match the one in the Euclidean space when the curvature takes the value zero. But in the end we found a way out with the help of the concept of locked inertia tensor used in geometric mechanics and thus clarified the semantic confusion that had occurred in recent years.

We also wanted to develop some criteria for the existence of central configurations and apply them towards finding new classes of such mathematical objects. The reward was higher than expected when we understood that any central configuration on a 3-sphere delivers two classes of relative equilibria, whereas any central configuration on hyperbolic 3-spheres provides three such classes. Unlike in the Euclidean case, however, central configurations do not lead to homothetic orbits, in general. The loss of this property is not only because spheres and hyperbolic spheres are not vector spaces, so the concept of similarity doesn't make much sense, but also for dynamical reasons. In Euclidean space, bodies released from a central configuration with zero initial velocities collide simultaneously. While this happens in some highly symmetric cases in curved space as well, it doesn't happen in general. For example, for fixed points on spheres, which are central configurations, the bodies don't move at all.

We also include in this first paper on central configurations of the curved  $N$ -body problem a complete proof that for any masses on spheres and hyperbolic spheres central configurations exist. Finally, we add some results on the number of geodesic central configurations, in the spirit of the classical theorem proved by Forest Ray Moulton in the classical case, [47].

## 2. THE MOMENT OF INERTIA IN EUCLIDEAN SPACE

In this section we discuss the moment of inertia in Euclidean space, aiming to find later a proper definition of this concept for an  $N$ -body system in spaces of constant curvature. At this stage we do not need any equations of motion, since the moment of inertia does not depend on them. The reason for dealing with this issue here is related to the fact that we will use this concept later in the definition of central configurations.

**2.1. The physical concept.** The moment of inertia first appeared in one of Euler's works of 1765, [26], p. 166. The term apparently made it into dictionaries sometime between 1820 and 1830, [24]. In the spirit of Euler, we can define this concept as follows.

**Definition 1.** *The moment of inertia is the sum of the products of the mass and the square of the perpendicular distance to the axis of rotation of each particle in a body rotating about an axis.*

According to the above definition, given for a rigid body, the moment of inertia  $I$  for a system of  $N$  positive point masses,  $m_1, \dots, m_N$ , relative to the  $z$ -axis in some

$xyz$ -coordinate system of the Euclidean space  $\mathbb{R}^3$ , must be of the form

$$(1) \quad I = \sum_{i=1}^N m_i (x_i^2 + y_i^2),$$

where the position of the body  $m_i$  is given by the vector  $\mathbf{q}_i = (x_i, y_i, z_i)$ . The moment of inertia has the same expression (1) if we restrict the motion to the plane  $\mathbb{R}^2$  and assume that the rotation takes place about the origin of some  $xy$ -coordinate system, where the position vector for the body  $m_i$  is now  $\mathbf{q}_i = (x_i, y_i)$ .

In celestial mechanics, as long as the motion is restricted to  $\mathbb{R}^2$ , the moment of inertia is taken as in (1) or, sometimes, as half this quantity. We will soon clarify the reason for which some authors introduce the factor  $\frac{1}{2}$ , but it is more important for now to note that in celestial mechanics the moment of inertia is taken in  $\mathbb{R}^3$  as

$$(2) \quad I = \sum_{i=1}^n m_i (x_i^2 + y_i^2 + z_i^2)$$

or as half this quantity. The usual physical interpretation of formula (2) given in the field is that the moment of inertia provides a crude measure for the distribution of the bodies in space, with  $I = 0$  at total collision and  $I$  large if at least one body is far away from the others. So not only that there is no match between Definition 1 and formula (2), but the celestial mechanics literature never hints at any connection between the moment of inertia thus defined and the rotation of the bodies about an axis.

We thought that we might find a reason for this mismatch in the original works where formula (2) appeared. The moment of inertia for the classical  $N$ -body problem has been historically known for its presence in the Lagrange-Jacobi equation,

$$\ddot{I} = (2\alpha + 4)U + 4h,$$

where  $I$  is defined as in (2),  $U$  is the force function (i.e. the negative of the potential energy),

$$U: \mathbb{R}^{3N} \rightarrow (0, \infty), \quad U(\mathbf{q}_1, \dots, \mathbf{q}_N) = \sum_{1 \leq i < j \leq N} \frac{m_i m_j}{|\mathbf{q}_i - \mathbf{q}_j|^\alpha},$$

$h$  is the energy constant, and  $\alpha > 0$  is also a constant. The physical units are chosen such that the gravitation constant is 1. Since the right hand-side of the Lagrange-Jacobi formula has a factor of 2, some researchers in celestial mechanics prefer to introduce the factor  $\frac{1}{2}$  in the definition of  $I$ , but this detail is irrelevant. So a good place to start our attempt at answering the above question was the first work that contained the Lagrange-Jacobi equation.

**2.2. Jacobi's approach.** In the winter semester of 1842-43 at the University of Königsberg in East Prussia, Carl Gustav Jacobi gave a lecture series on the  $N$ -body problem, which was very well received, so he published it as a book entitled *Vorlesungen über Dynamik* in 1848, [30]. On page 22, the Lagrange-Jacobi equation appears for the first time as  $\sum m_i r_i^2$ , where  $r_i^2 = x_i^2 + y_i^2 + z_i^2$ . Jacobi never attached a name to this sum, as he did for other important concepts, such as kinetic energy, which he called “lebendige

Kraft” (living force). Between pages 22 and 24 he referred to  $\sum m_i r_i^2$  as “Ausdruck” (expression), “Summe” (sum), or “Grösse” (quantity), but never hinted that it has anything to do with the moment of inertia defined in physics. Recall that this concept had been defined in 1765 and was already in dictionaries around 1830, so Jacobi should have been aware of it by the time of his lectures.

In the first paragraph on page 24, he mentioned that, at the origin of the coordinate system,  $\sum m_i r_i^2$  reaches its minimum value and, when  $\sum m_i r_i^2$  is constant, the bodies can be thought of lying on the same sphere. So he formulated there our current physical interpretation of the moment of inertia in celestial mechanics as a crude measure of the bodies’ distribution in space. And this is all he wrote relative to  $\sum m_i r_i^2$ . It is thus reasonable to think that he made no connection between this expression and the rotation of the bodies about a fixed axis.

**2.3. Wintner’s terminology.** A century later, Aurel Wintner published the first edition of his influential book on the analytical foundations of celestial mechanics, updated in a second edition that appeared in 1947, [63]. On page 234, the quantity  $J = \sum m_i \xi_i^2$  was introduced (with  $\xi_i$  having the same meaning as Jacobi’s  $r_i$  mentioned above), which finally bears a name; he called it the polar inertia momentum. In modern parlance, the *polar moment of inertia*, or the *polar moment of area*, is a quantity used to predict an object’s resistance to torsion. Physicists warn, however, that the polar moment of inertia should not be confused with the moment of inertia, which characterizes an object’s angular acceleration due to torque. So though related, the concepts of torque and torsion mean different things.

**2.4. Recent developments.** Since the publication of Wintner’s book, researchers in celestial mechanics got apparently mixed up in terminology. Though the two physical concepts are identical in the classical  $N$ -body problem as long as  $I$  is defined in the plane  $\mathbb{R}^2$ , in  $\mathbb{R}^3$  we must distinguish between the polar moment of inertia, (2), and the moment of inertia, (1). This remark is important to us for reasons related to the definition we will give for central configurations in spaces of constant curvature and to the fact that we can recover the classical definition when the curvature tends to zero.

In spite of a misleading terminology, the polar moment of inertia was understood in terms of a rotation when considered in the context of relative equilibria (orbits that maintain constant mutual distances between the bodies all along the motion) defined by central configurations, as we will explain later. But the central configurations leading to relative equilibria must be planar, (see [63], p. 287). As there are no spatial relative equilibria in  $\mathbb{R}^3$ , the mixup between concepts was harmless. In the next section, we will provide and justify the correct definition of the moment of inertia for spheres and hyperbolic spheres, and later find another way to back up our findings.

### 3. THE MOMENT OF INERTIA IN SPACES OF CONSTANT CURVATURE

In this section we first introduce the definition of relative equilibria in the context of mechanical systems with symmetry. We use the language of geometric mechanics, [1, 40, 41, 57] to show that finding relative equilibria of mechanical systems in spheres

and hyperbolic spheres is equivalent to finding the critical points of the corresponding effective potentials. The effective potentials corresponding to different relative equilibria have the same form, a fact that motivates our definition of the moment of inertia in spheres and hyperbolic spheres. Throughout this paper, vectors are all column vectors, but written as row vectors in the text, and the masses  $m_1, \dots, m_N$  are all positive.

**3.1. Relative Equilibria.** We begin with some definitions for general mechanical systems. The first is from [57], whereas the second is from [59].

**Definition 2.** *A mechanical system with symmetry consists of a 4-tuple  $(M, K, V, G)$  where  $M$  is a manifold,  $K$  is the kinetic energy,  $V$  the potential energy, and  $G$  a Lie group acting on  $M$  preserving  $K$  and  $V$  with all data smooth.*

For each  $\xi$  in the Lie algebra  $\mathfrak{g}$  of  $G$ , there is a vector field  $\xi_M$  on  $M$  given by

$$\xi_M(\mathbf{q}) := \left. \frac{d}{dt} \right|_{t=0} (\exp(\xi t)\mathbf{q}).$$

We denote by  $\xi_M(\mathbf{q})$  the vector at  $\mathbf{q} \in M$ , and by  $\exp(\xi t)\mathbf{q}$  the action of  $\exp(\xi t)$  on  $\mathbf{q}$ .

**Definition 3.** *A solution of the mechanical system with symmetry  $(M, K, V, G)$  is called a relative equilibrium if it is also an integral curve of the vector field  $\xi_M$ . In other words, a relative equilibrium is a solution in the form of  $\exp(\xi t)\mathbf{q}$ . The curve  $\exp(\xi t) \in G$  is called a 1-parameter subgroup of  $G$ .*

Let us return to mechanical systems in spheres and hyperbolic spheres. We embed them in either the standard Euclidean space,  $\mathbb{R}^4$ , or the Minkowski space,  $\mathbb{R}^{3,1}$ . More precisely, for vectors  $\mathbf{q}_1 = (x_1, y_1, z_1, w_1)$  and  $\mathbf{q}_2 = (x_2, y_2, z_2, w_2)$  in  $\mathbb{R}^4$  or  $\mathbb{R}^{3,1}$ , the inner products are given by

$$\mathbf{q}_1 \cdot \mathbf{q}_2 = x_1 x_2 + y_1 y_2 + z_1 z_2 + \sigma w_1 w_2,$$

where  $\sigma = 1$  for  $\mathbb{R}^4$  and  $\sigma = -1$  for  $\mathbb{R}^{3,1}$ . Then the family of manifolds is

$$\mathbb{M}_\kappa^3 := \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + \sigma w^2 = \kappa, \kappa \neq 0\},$$

with  $w > 0$  for  $\kappa < 0$ . For  $\kappa > 0$ , the manifolds are 3-spheres, which we denote by  $\mathbb{S}_\kappa^3$ , whereas for  $\kappa < 0$ , the manifolds are hyperbolic 3-spheres, which we denote by  $\mathbb{H}_\kappa^3$ .

Given positive masses  $m_1, \dots, m_N$ , whose positions are described by the configuration  $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N) \in (\mathbb{M}_\kappa^3)^N$ ,  $\mathbf{q}_i = (x_i, y_i, z_i, w_i)$ ,  $i = \overline{1, N}$ , we define the singularity set

$$\Delta = \cup_{1 \leq i < j \leq N} \{\mathbf{q} \in (\mathbb{M}_\kappa^3)^N; \mathbf{q}_i = \pm \mathbf{q}_j\}.$$

We take the kinetic energy as

$$K(\dot{\mathbf{q}}) = \sum_{1 \leq i \leq N} \frac{1}{2} m_i \dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i.$$

We assume that the potential energy is invariant under the  $O(4)$  ( $O(3, 1)$ ) action. For instance,  $V = \sum_{1 \leq i < j \leq N} m_i m_j f(d_{ij})$ , where  $d_{ij}$  is the distance between  $\mathbf{q}_i$  and  $\mathbf{q}_j$ .

Now let us consider the relative equilibrium of such a mechanical system that consists of the 4-tuple

$$(3) \quad ((\mathbb{M}_\kappa^3)^N \setminus \Delta, K(\dot{\mathbf{q}}), V(\mathbf{q}), O(4)(O(3,1))).$$

Recall that a  $4 \times 4$  matrix  $A$  is in  $O(4)$  if it keeps the inner product in the 4-dimensional Euclidean space, that is, if

$$\mathbf{A}\mathbf{u} \cdot \mathbf{A}\mathbf{v} = \mathbf{u} \cdot \mathbf{v}, \text{ for any } \mathbf{u}, \mathbf{v} \in \mathbb{R}^4.$$

$O(4)$  is a matrix Lie group that has two components. The component containing the identity matrix  $I$ , i.e. the matrices with determinant one, is denoted by  $SO(4)$ . The tangent space at  $I$ , the Lie algebra of  $O(4)$ , is a 6-dimensional linear space and is denoted by  $\mathfrak{so}(4)$ . A  $4 \times 4$  matrix  $X$  is in  $\mathfrak{so}(4)$  if  $X^T = -X$ .

Recall that a  $4 \times 4$  matrix  $A$  is in  $O(3,1)$  if it keeps the inner product in the 4-dimensional Minkowski space, that is, if

$$\mathbf{A}\mathbf{u} \cdot \mathbf{A}\mathbf{v} = \mathbf{u} \cdot \mathbf{v}, \text{ for any } \mathbf{u}, \mathbf{v} \in \mathbb{R}^{3,1}.$$

$O(3,1)$  is a matrix Lie group with four components [48]. The two components with determinant one form  $SO(3,1)$ , out of which the component containing  $I$  is denoted by  $SO^+(3,1)$ . The tangent space at  $I$ , the Lie algebra of  $O(3,1)$ , is a 6-dimensional linear space and denoted by  $\mathfrak{so}(3,1)$ . A  $4 \times 4$  matrix  $X$  belongs to  $\mathfrak{so}(3,1)$  if  $\psi X^T \psi = -X$ , where  $\psi = \text{diag}(1, 1, 1, -1)$ .

**Proposition 1.** *A 1-parameter subgroup of  $SO(4)$  is of the form  $PA_{\alpha,\beta}(t)P^{-1}$ , with  $P \in SO(4)$  and*

$$A_{\alpha,\beta}(t) = \begin{bmatrix} \cos \alpha t & -\sin \alpha t & 0 & 0 \\ \sin \alpha t & \cos \alpha t & 0 & 0 \\ 0 & 0 & \cos \beta t & -\sin \beta t \\ 0 & 0 & \sin \beta t & \cos \beta t \end{bmatrix}, \quad \alpha, \beta \in \mathbb{R}.$$

We call these rotations positive elliptic-elliptic if  $\alpha \neq 0$  and  $\beta \neq 0$ , and positive elliptic if only one of them is zero. We call the corresponding relative equilibria *positive elliptic-elliptic relative equilibria* and *positive elliptic relative equilibria*, respectively.

**Proposition 2.** *A 1-parameter subgroup of  $SO^+(3,1)$  is of the form  $PB_{\alpha,\beta}(t)P^{-1}$  or  $PC_\eta(t)P^{-1}$ , with  $P \in SO(3,1)$ , and*

$$B_{\alpha,\beta}(t) = \begin{bmatrix} \cos \alpha t & -\sin \alpha t & 0 & 0 \\ \sin \alpha t & \cos \alpha t & 0 & 0 \\ 0 & 0 & \cosh \beta t & \sinh \beta t \\ 0 & 0 & \sinh \beta t & \cosh \beta t \end{bmatrix}, \quad C_\eta(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\eta t & \eta t \\ 0 & \eta t & 1 - \eta t^2/2 & \eta t^2 \\ 0 & \eta t & -\eta t^2 & 1 + \eta t^2/2 \end{bmatrix},$$

where  $\alpha, \beta, \eta \in \mathbb{R}$ .

Similarly, the negative elliptic, negative hyperbolic, negative elliptic-hyperbolic and parabolic transformations correspond to  $\alpha \neq 0$  and  $\beta = 0$ ,  $\alpha = 0$  and  $\beta \neq 0$ ,  $\alpha \neq 0$

and  $\beta \neq 0$ , and  $\eta \neq 0$ , respectively. We call the corresponding relative equilibria *negative elliptic relative equilibria*, *negative hyperbolic relative equilibria*, *negative elliptic-hyperbolic relative equilibria* and *parabolic relative equilibria*, respectively.

We can easily check that

$$A_{\alpha,\beta}(t) = \exp(\xi_1 t), \quad B_{\alpha,\beta}(t) = \exp(\xi_2 t), \quad C_\eta(t) = \exp(\xi_3 t),$$

where  $\xi_1 \in \mathfrak{so}(4)$ ,  $\xi_2, \xi_3 \in \mathfrak{so}(3, 1)$ , and

$$\xi_1 = \begin{bmatrix} 0 & -\alpha & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta \\ 0 & 0 & \beta & 0 \end{bmatrix}, \quad \xi_2 = \begin{bmatrix} 0 & -\alpha & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & \beta & 0 \end{bmatrix}, \quad \xi_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\eta & \eta \\ 0 & \eta & 0 & 0 \\ 0 & \eta & 0 & 0 \end{bmatrix}.$$

It is easy to see that for any  $\phi$  in the isometry group,  $\mathbf{q}(t)$  solves the mechanical system (3) if and only if  $\phi\mathbf{q}(t)$  does. Thus we cover all possible relative equilibria for the mechanical system if we define them in terms of the three normal forms of the 1-parameter subgroup. To simplify the notation, we will denote initial positions without any argument and attach the argument  $t$  to functions depending on time.

**Definition 4.** Let  $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N)$  be an initial configuration of the masses  $m_1, \dots, m_N$ ,  $N \geq 2$  in  $(\mathbb{M}_\kappa^3)^N \setminus \Delta$ , where the initial position vectors are  $\mathbf{q}_i = (x_i, y_i, z_i, w_i)$ ,  $i = \overline{1, N}$ . Then a solution of the form

$$\mathbf{q}(t) = Q(t)\mathbf{q} := (Q(t)\mathbf{q}_1, \dots, Q(t)\mathbf{q}_N)$$

of system (3), with  $Q(t)$  being  $A_{\alpha,\beta}(t)$ ,  $B_{\alpha,\beta}(t)$ , or  $C_\eta(t)$ , is called a *relative equilibrium* of the mechanical system (3).

It was proved in [9] that parabolic relative equilibria don't exist for system (3).

**3.2. Effective potentials and moment of inertia.** First recall the following result.

**Theorem 1** (Smale, [59]). Suppose  $(M, K, V, G)$  is a mechanical system with symmetry and  $\xi \in \mathfrak{g}$ . Then  $\exp(\xi t)\mathbf{q}$  is a relative equilibrium if and only if  $\mathbf{q}$  is a critical point of the real valued function on  $M$  which sends  $\mathbf{q}$  into  $V(\mathbf{q}) - K(\xi_M(\mathbf{q}), \xi_M(\mathbf{q}))$ , the effective potential corresponding to  $\xi$ .

**Theorem 2.** Consider system (3). Let  $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N)$ ,  $\mathbf{q}_i = (x_i, y_i, z_i, w_i)$ ,  $i = \overline{1, N}$ , be a configuration in  $(\mathbb{M}_\kappa^3)^N \setminus \Delta$ . In  $(\mathbb{S}_\kappa^3)^N \setminus \Delta$ ,  $\exp(\xi_1 t)\mathbf{q} = A_{\alpha,\beta}(t)\mathbf{q}$  is a relative equilibrium if and only if this configuration satisfies the equation

$$\frac{\alpha^2 - \beta^2}{2} \nabla_{\mathbf{q}_i} \left( \sum_{i=1}^N m_i (x_i^2 + y_i^2) \right) = \nabla_{\mathbf{q}_i} V(\mathbf{q}), \quad i = \overline{1, N}.$$

In  $(\mathbb{H}_\kappa^3)^N \setminus \Delta$ ,  $\exp(\xi_2 t)\mathbf{q} = B_{\alpha,\beta}(t)\mathbf{q}$  is a relative equilibrium if and only if this configuration satisfies the equations

$$\frac{\alpha^2 + \beta^2}{2} \nabla_{\mathbf{q}_i} \left( \sum_{i=1}^N m_i (x_i^2 + y_i^2) \right) = \nabla_{\mathbf{q}_i} V(\mathbf{q}), \quad i = \overline{1, N}.$$

*Proof.* The action of  $\exp(\xi_i t)$  is  $\exp(\xi_i t)\mathbf{q} = (\exp(\xi_i t)\mathbf{q}_1, \dots, \exp(\xi_i t)\mathbf{q}_N)$ . Thus the vector fields generated by  $\xi_1$  and  $\xi_2$  on  $(\mathbb{S}_\kappa^3)^N$  and  $(\mathbb{H}_\kappa^3)^N$  are  $\xi_1\mathbf{q} = (\xi_1\mathbf{q}_1, \dots, \xi_1\mathbf{q}_N)$  and  $\xi_2\mathbf{q} = (\xi_2\mathbf{q}_1, \dots, \xi_2\mathbf{q}_N)$ , respectively.

Recall that the kinetic energy is  $K(\dot{\mathbf{q}}, \dot{\mathbf{q}}) = \sum_{i=1}^N \frac{1}{2} m_i \dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i$ . In  $\mathbb{S}_\kappa^3$ , using the fact  $\mathbf{q}_i \cdot \mathbf{q}_i = 1$ , we obtain

$$\begin{aligned} K(\xi_1\mathbf{q}, \xi_1\mathbf{q}) &= \sum_{i=1}^N \frac{1}{2} m_i \xi_1\mathbf{q}_i \cdot \xi_1\mathbf{q}_i = \sum_{i=1}^N \frac{1}{2} m_i (-\alpha y_i, \alpha x_i, -\beta w_i, \beta z_i) \cdot (-\alpha y_i, \alpha x_i, -\beta w_i, \beta z_i) \\ &= \sum_{i=1}^N \frac{1}{2} m_i (\alpha^2(x_i^2 + y_i^2) + \beta^2(\kappa - x_i^2 - y_i^2)) = \frac{\alpha^2 - \beta^2}{2} \sum_{i=1}^N m_i(x_i^2 + y_i^2) + \frac{\kappa\beta^2}{2} \sum_{i=1}^N m_i. \end{aligned}$$

In  $\mathbb{H}_\kappa^3$ , by using  $\mathbf{q}_i \cdot \mathbf{q}_i = \kappa$ , we similarly obtain

$$K(\xi_2\mathbf{q}, \xi_2\mathbf{q}) = \frac{\alpha^2 + \beta^2}{2} \sum_{i=1}^N m_i(x_i^2 + y_i^2) - \frac{\kappa\beta^2}{2} \sum_{i=1}^N m_i.$$

Ignoring the constant, the effective potentials with respect to  $\xi_1$  and  $\xi_2$  are

$$V(\mathbf{q}) - \sum_{i=1}^N \frac{m_i}{2} (\alpha^2 - \beta^2)(x_i^2 + y_i^2), \quad V(\mathbf{q}) - \sum_{i=1}^N \frac{m_i}{2} (\alpha^2 + \beta^2)(x_i^2 + y_i^2).$$

So  $\exp(\xi_i t)\mathbf{q}$  is a relative equilibrium if and only if  $\mathbf{q}$  is a critical point of these effective potentials, which is equivalent to the equations. This remark completes the proof.  $\square$

The effective potentials depend on the parameters  $\alpha, \beta$  in such a manner since the spheres are 3-dimensional. Note that the quantity  $\sum_{i=1}^N m_i(x_i^2 + y_i^2)$ , which has the same form as the moment of inertia in  $\mathbb{R}^3$ , see Definition 1, is related to relative equilibria of system (3) in the same way as the moment of inertia is related to relative equilibria of mechanical systems in  $\mathbb{R}^3$ . We can now introduce the natural definition of the moment of inertia for the mechanical systems in  $(\mathbb{M}_\kappa^3)^N \setminus \Delta$ .

**Definition 5.** Consider a mechanical system that is determined by the 4-tuple (3). Assume that their configuration is given by the vectors  $\mathbf{q}_i = (x_i, y_i, z_i, w_i) \in \mathbb{M}_\kappa^3$ ,  $i = \overline{1, N}$ . Then the moment of inertia of the particle system is the function

$$(4) \quad I(\mathbf{q}) := \sum_{i=1}^N m_i(x_i^2 + y_i^2).$$

#### 4. EQUATIONS OF MOTION

In this section we introduce the  $N$ -body problem in spaces of constant nonzero curvature, which we will refer to as the *curved  $N$ -body problem*. It is the study of motion of particle systems under Newton-like attraction. We will call its analogue in Euclidean space the *Newtonian  $N$ -body problem*.

As in [9], we set the curved  $N$ -body problems  $\mathbb{M}_{\pm 1}^3$ , which we will simply denote by  $\mathbb{S}^3$  and  $\mathbb{H}^3$ . For convenience, we will also use the notation

$$\mathbb{M}^3 := \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + \sigma w^2 = \sigma, \text{ with } w > 0 \text{ for } \sigma = -1\}.$$

Given the positive masses  $m_1, \dots, m_N$ , whose positions are described by the configuration  $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N) \in (\mathbb{M}^3)^N$ ,  $\mathbf{q}_i = (x_i, y_i, z_i, w_i)$ ,  $i = \overline{1, N}$ , we define the singularity set

$$\Delta = \cup_{1 \leq i < j \leq N} \{\mathbf{q} \in (\mathbb{M}^3)^N; \mathbf{q}_i \cdot \mathbf{q}_j = \pm \sigma\}.$$

If  $d_{ij}$  is the geodesic distance between the point masses  $m_i$  and  $m_j$ , we define the force function  $U$  ( $-U$  being the potential function) on  $(\mathbb{M}^3)^N \setminus \Delta$  as

$$U(\mathbf{q}) := \sum_{1 \leq i < j \leq N} m_i m_j \text{ctnd}_{ij},$$

where  $\text{ctn}(x)$  stands for  $\cot(x)$  in  $\mathbb{S}^3$  and  $\coth(x)$  in  $\mathbb{H}^3$ . We would like to mention that there are many other choices of the potential, but this potential is coherent with the Newtonian  $N$ -body problem, see [9]. We also introduce two more notations, which unify the trigonometric and hyperbolic functions,

$$\text{sn}(x) = \sin(x) \text{ or } \sinh(x), \quad \text{csn}(x) = \cos(x) \text{ or } \cosh(x).$$

Then the distance  $d_{ij}$  is given by the expression  $d_{ij} := \arccsn(\sigma \mathbf{q}_i \cdot \mathbf{q}_j)$ , where  $\arccsn(x)$  is the inverse function of  $\text{csn}(x)$ . We define the kinetic energy as

$$T(\mathbf{p}) = \sum_{1 \leq i \leq N} m_i \dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i = \sum_{1 \leq i \leq N} m_i^{-1} \mathbf{p}_i \cdot \mathbf{p}_i,$$

where  $\mathbf{p}_i := m_i \dot{\mathbf{q}}_i$  is the momentum of  $m_i$ . We also denote the momentum of the particle system by  $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_N)$ . Then the curved  $N$ -body problem is given by the Hamiltonian system on  $T^*((\mathbb{M}^3)^N \setminus \Delta)$ , with

$$H(\mathbf{q}, \mathbf{p}) := T(\mathbf{q}, \mathbf{p}) - U(\mathbf{q}).$$

Let us derive the equations of motion for the Hamiltonian system in  $\mathbb{S}^3$ . The Hamiltonian is

$$H = \sum_{1 \leq i \leq N} \frac{1}{2} m_i^{-1} \mathbf{p}_i \cdot \mathbf{p}_i - \sum_{1 \leq i < j \leq N} m_i m_j \cot d_{ij}.$$

Here  $U$  is defined on  $(\mathbb{S}^3)^N \setminus \Delta$ , with the set of singularities  $\Delta = \Delta^- \cup \Delta^+$ , where

$$(5) \quad \begin{aligned} \Delta^- &:= \cup_{1 \leq i < j \leq N} \{\mathbf{q} \in (\mathbb{S}^3)^N : \mathbf{q}_i = -\mathbf{q}_j\} \setminus \cup_{1 \leq i < j \leq N} \{\mathbf{q} \in (\mathbb{S}^3)^N : \mathbf{q}_i = \mathbf{q}_j\}, \\ \Delta^+ &:= \cup_{1 \leq i < j \leq N} \{\mathbf{q} \in (\mathbb{S}^3)^N : \mathbf{q}_i = \mathbf{q}_j\} \setminus \cup_{1 \leq i < j \leq N} \{\mathbf{q} \in (\mathbb{S}^3)^N : \mathbf{q}_i = -\mathbf{q}_j\}. \end{aligned}$$

We will call  $\Delta^-$  the antipodal singularity set and  $\Delta^+$  the collision singularity set. Using constrained Hamiltonian dynamics, we get the equations describing the motion of the bodies,

$$\begin{cases} \dot{\mathbf{q}}_i = m_i^{-1} \mathbf{p}_i \\ \dot{\mathbf{p}}_i = \nabla_{\mathbf{q}_i} U - m_i^{-1} (\mathbf{p}_i \cdot \mathbf{p}_i) \mathbf{q}_i = \nabla_{\mathbf{q}_i} U - m_i (\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i) \mathbf{q}_i \\ \mathbf{q}_i \cdot \mathbf{q}_i = 1, \quad \mathbf{p}_i \cdot \mathbf{q}_i = 0, \quad i = \overline{1, N}, \end{cases}$$

where  $\nabla_{\mathbf{q}_i} U$  stands for the gradient of  $U$  on the manifold  $(\mathbb{S}^3)^N$ .  $\nabla_{\mathbf{q}_i} U$  can be interpreted as the attractive force on  $\mathbf{q}_i$  produced by all other particles, and  $-m_i^{-1}(\mathbf{p}_i \cdot \mathbf{p}_i)\mathbf{q}_i$  can be viewed as the constraint force keeping the particles on the sphere. Thus we denote  $\nabla_{\mathbf{q}_i} U$  and  $\nabla_{\mathbf{q}_i} m_i m_j \cot d_{ij}$  by  $\mathbf{F}_i$  and  $\mathbf{F}_{ij}$ , respectively. We have

$$\mathbf{F}_{ij} = \frac{-m_i m_j}{\sin^2 d_{ij}} \nabla_{\mathbf{q}_i} d_{ij} = \frac{-m_i m_j}{\sin^2 d_{ij}} \nabla_{\mathbf{q}_i} \cos^{-1} \mathbf{q}_i \cdot \mathbf{q}_j = \frac{m_i m_j}{\sin^3 d_{ij}} \nabla_{\mathbf{q}_i} \mathbf{q}_i \cdot \mathbf{q}_j.$$

The gradient of  $\mathbf{q}_i \cdot \mathbf{q}_j$  on the manifold  $(\mathbb{S}^3)^N$  can be computed as follows. We extend any function  $f: (\mathbb{S}^3)^N \rightarrow \mathbb{R}$  to the ambient space  $\bar{f}: (\mathbb{R}^4)^N \rightarrow \mathbb{R}$ ,

$$\bar{f}(\mathbf{q}) = f\left(\frac{\mathbf{q}_1}{\sqrt{\mathbf{q}_1 \cdot \mathbf{q}_1}}, \dots, \frac{\mathbf{q}_N}{\sqrt{\mathbf{q}_N \cdot \mathbf{q}_N}}\right).$$

Then  $\bar{f}(\lambda \mathbf{q}) = \bar{f}(\mathbf{q})$  for  $\lambda > 0$ , i.e.,  $\bar{f}$  is a homogeneous function of degree zero. Let  $\tilde{\nabla}$  be the gradient in the ambient space and  $\frac{\partial}{\partial n_i}$  the unit normal vector of the  $i$ -th unit sphere. Since  $\frac{\partial \bar{f}}{\partial r_i} = 0$ , we obtain  $(\tilde{\nabla}_{\mathbf{q}_i} \bar{f})|_{(\mathbb{S}^3)^N} = \nabla_{\mathbf{q}_i} f + \frac{\partial \bar{f}}{\partial r_i} \frac{\partial}{\partial n_i} = \nabla_{\mathbf{q}_i} f$ . Thus

$$\begin{aligned} \mathbf{F}_{ij} &= \frac{m_i m_j}{\sin^3 d_{ij}} \tilde{\nabla}_{\mathbf{q}_i} \frac{\mathbf{q}_i \cdot \mathbf{q}_j}{\sqrt{\mathbf{q}_i \cdot \mathbf{q}_i} \sqrt{\mathbf{q}_j \cdot \mathbf{q}_j}} = \frac{m_i m_j}{\sin^3 d_{ij}} \frac{\sqrt{\mathbf{q}_i \cdot \mathbf{q}_i} \sqrt{\mathbf{q}_j \cdot \mathbf{q}_j} \mathbf{q}_j - \mathbf{q}_i \cdot \mathbf{q}_j \frac{\sqrt{\mathbf{q}_j \cdot \mathbf{q}_j}}{\sqrt{\mathbf{q}_i \cdot \mathbf{q}_i}} \mathbf{q}_i}{(\sqrt{\mathbf{q}_i \cdot \mathbf{q}_i} \sqrt{\mathbf{q}_j \cdot \mathbf{q}_j})^2} \\ &= \frac{m_i m_j [\mathbf{q}_j - \cos d_{ij} \mathbf{q}_i]}{\sin^3 d_{ij}}. \end{aligned}$$

Thus the equations of motion for the curved  $N$ -body problem in  $\mathbb{S}^3$  are

$$\begin{cases} \dot{\mathbf{q}}_i = m_i^{-1} \mathbf{p}_i \\ \dot{\mathbf{p}}_i = \sum_{j=1, j \neq i}^N \frac{m_i m_j [\mathbf{q}_j - \cos d_{ij} \mathbf{q}_i]}{\sin^3 d_{ij}} - m_i (\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i) \mathbf{q}_i \\ \mathbf{q}_i \cdot \mathbf{q}_i = 1, \quad \mathbf{p}_i \cdot \mathbf{q}_i = 0, \quad i = \overline{1, N}. \end{cases}$$

**Gravitation law in  $\mathbb{S}^3$ .** A mass  $m_2$  at  $\mathbf{q}_2 \in \mathbb{S}^3$  attracts another mass  $m_1$  at  $\mathbf{q}_1 \in \mathbb{S}^3$  ( $\mathbf{q}_1 \neq \pm \mathbf{q}_2$ ) along the minimal geodesic connecting the two points with a force whose magnitude is  $\frac{m_1 m_2}{\sin^2 d_{12}}$ . More precisely,

$$\mathbf{F}_{12} = \frac{m_1 m_2 [\mathbf{q}_2 - \cos d_{12} \mathbf{q}_1]}{\sin^3 d_{12}}.$$

Similarly, we can derive the equations of motion for the Hamiltonian system in  $\mathbb{H}^3$ . The Hamiltonian is

$$H = T(\mathbf{q}, \mathbf{p}) - U(\mathbf{q}) = \sum_{1 \leq i \leq N} \frac{1}{2} m_i^{-1} \mathbf{p}_i \cdot \mathbf{p}_i - \sum_{1 \leq i < j \leq N} m_i m_j \coth d_{ij}.$$

Here  $U$  is defined on  $(\mathbb{H}^3)^N \setminus \Delta$ , and the set of singularities is

$$\Delta := \cup_{1 \leq i < j \leq N} \{\mathbf{q} \in (\mathbb{H}^3)^N : \mathbf{q}_i = \mathbf{q}_j\}.$$

We interpret  $\nabla_{\mathbf{q}_i} U$  and  $\nabla_{\mathbf{q}_i} m_i m_j \coth d_{ij}$  as  $\mathbf{F}_i$  and  $\mathbf{F}_{ij}$  respectively. Similar computations lead to

$$\mathbf{F}_{ij} = \frac{m_i m_j [\mathbf{q}_j - \cosh d_{ij} \mathbf{q}_i]}{\sinh^3 d_{ij}},$$

and the equations of motion for the curved  $N$ -body problem in  $\mathbb{H}^3$  are

$$\begin{cases} \dot{\mathbf{q}}_i = m_i^{-1} \mathbf{p}_i \\ \dot{\mathbf{p}}_i = \sum_{j=1, j \neq i}^N \frac{m_i m_j [\mathbf{q}_j - \cosh d_{ij} \mathbf{q}_i]}{\sinh^3 d_{ij}} + m_i (\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i) \mathbf{q}_i \\ \mathbf{q}_i \cdot \mathbf{q}_i = -1, \quad \mathbf{p}_i \cdot \mathbf{q}_i = 0, \quad i = \overline{1, N}. \end{cases}$$

**Gravitation law in  $\mathbb{H}^3$ .** A mass  $m_2$  at  $\mathbf{q}_2 \in \mathbb{H}^3$  attracts another mass  $m_1$  at  $\mathbf{q}_1 \in \mathbb{H}^3$  ( $\mathbf{q}_1 \neq \mathbf{q}_2$ ) along the minimal geodesic connecting the two points with a force whose magnitude is  $\frac{m_1 m_2}{\sinh^2 d_{12}}$ . More precisely,

$$\mathbf{F}_{12} = \frac{m_1 m_2 [\mathbf{q}_2 - \cosh d_{12} \mathbf{q}_1]}{\sinh^3 d_{12}}.$$

Using the functions  $\text{sn}(x)$  and  $\text{csn}(x)$  introduced earlier, we can blend the two systems of equations into one system in  $(\mathbb{M}^3)^N \setminus \Delta$  [9, 10],

$$(6) \quad \begin{cases} \dot{\mathbf{q}}_i = m_i^{-1} \mathbf{p}_i \\ \dot{\mathbf{p}}_i = \sum_{j=1, j \neq i}^N \frac{m_i m_j [\mathbf{q}_j - \text{csn} d_{ij} \mathbf{q}_i]}{\text{sn}^3 d_{ij}} - \sigma m_i (\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i) \mathbf{q}_i \\ \mathbf{q}_i \cdot \mathbf{q}_i = \sigma, \quad \mathbf{p}_i \cdot \mathbf{q}_i = 0, \quad i = \overline{1, N}. \end{cases}$$

**Remark 1.** If we derive the equation of motion in  $\mathbb{M}_\kappa^3$ , we would see that the gravitational law is

$$\mathbf{F}_{12} = \frac{m_1 m_2 |\kappa|^{\frac{3}{2}} [\mathbf{q}_2 - \text{csn} |\kappa|^{\frac{1}{2}} d_\kappa(\mathbf{q}_1, \mathbf{q}_2) \mathbf{q}_1]}{\text{sn}^3 \left( |\kappa|^{\frac{1}{2}} d_\kappa(\mathbf{q}_1, \mathbf{q}_2) \right)},$$

[9, page 29], where  $d_\kappa(\mathbf{q}_1, \mathbf{q}_2)$  is the distance between the two particles in  $\mathbb{M}_\kappa^3$ . Formally, it tends to the gravitational law in  $\mathbb{R}^3$  when  $\kappa \rightarrow 0$ , which again shows that the potential is coherent with the Newtonian potential.

Some researchers studied the curved  $N$ -body problem in  $\mathbb{M}_\kappa^3$  with curvature  $\kappa \neq \pm 1$  [31]. This is not necessary since it has been shown in [9] that there are coordinate and time-rescaling transformations,

$$\mathbf{q}_i = |\kappa|^{-1/2} \mathbf{r}_i, \quad i = \overline{1, N} \quad \text{and} \quad \tau = |\kappa|^{3/4} t,$$

which bring the systems from  $\mathbb{S}_\kappa^3$  and  $\mathbb{H}_\kappa^3$  to systems to  $\mathbb{S}^3$  and  $\mathbb{H}^3$ , respectively.

**4.1. Total angular momentum integrals.** The Hamiltonian function is invariant under the action of  $SO(4)$  and  $SO(3, 1)$  for motions in  $\mathbb{S}^3$  and  $\mathbb{H}^3$ , respectively. These symmetries lead to six integrals, which stand for the generalized version of the usual total angular momentum conservation laws in  $\mathbb{R}^3$ ,

$$\omega_{uv} = \sum_{i=1}^N m_i (\dot{u}_i v_i - u_i \dot{v}_i),$$

where  $u, v$  is any combinations of  $x, y, z, w$ , a fact shown in [9] and [10]. We refer to them as *angular momentum integrals*.

## 5. RELATIVE EQUILIBRIA AND CENTRAL CONFIGURATIONS

We can apply Theorem 2 to derive the criteria for relative equilibria of the curved  $N$ -body problem. They are equivalent with the criteria given in [9] and [10], but differ significantly in form. We then define central configuration of the curved  $N$ -body problem and discuss the relationships between central configurations and solutions of the curved  $N$ -body problem.

**5.1. Criterion for relative equilibria in  $\mathbb{M}^3$ .** Let  $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N)$ ,  $\mathbf{q}_i = (x_i, y_i, z_i, w_i)$ ,  $i = \overline{1, N}$ , be a nonsingular configuration and  $Q(t)\mathbf{q}$  a relative equilibrium, where  $Q(t)$  is  $A_{\alpha, \beta}(t)$  or  $B_{\alpha, \beta}(t)$ . Again, we denote initial positions and velocities without any argument and attach the argument  $t$  to functions depending on time.

We first substitute  $\mathbf{q}_i(t) = Q(t)\mathbf{q}_i$ ,  $i = \overline{1, N}$ , into equations (6) and obtain

$$m_i \ddot{Q}(t)\mathbf{q}_i = \nabla_{\mathbf{q}_i} U(t) - \sigma m_i [\dot{Q}(t)\mathbf{q}_i \cdot \dot{Q}(t)\mathbf{q}_i] Q(t)\mathbf{q}_i, \quad i = \overline{1, N}.$$

Since  $U$  is invariant under the isometry group, it is easy to see that  $Q^{-1}(t)\nabla_{\mathbf{q}_i} U(t) = \nabla_{\mathbf{q}_i} U$ . Multiplying to the left by  $Q^{-1}(t)$  yields

$$(7) \quad m_i Q^{-1}(t) \ddot{Q}(t)\mathbf{q}_i = \nabla_{\mathbf{q}_i} U - \sigma m_i [\dot{Q}(t)\mathbf{q}_i \cdot \dot{Q}(t)\mathbf{q}_i] \mathbf{q}_i.$$

**Theorem 3.** *Let  $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N)$ ,  $\mathbf{q}_i = (x_i, y_i, z_i, w_i)$ ,  $i = \overline{1, N}$ , be a nonsingular configuration in  $\mathbb{S}^3$ . Then  $A_{\alpha, \beta}(t)\mathbf{q}$  is a relative equilibrium if and only if this configuration satisfies the equations*

$$(8) \quad m_i(\beta^2 - \alpha^2) \begin{bmatrix} x_i(w_i^2 + z_i^2) \\ y_i(w_i^2 + z_i^2) \\ -z_i(x_i^2 + y_i^2) \\ -w_i(x_i^2 + y_i^2) \end{bmatrix} = \nabla_{\mathbf{q}_i} U, \quad i = \overline{1, N}.$$

*Proof.* Using the fact that  $A_{\alpha, \beta}(t) = \exp(\boldsymbol{\xi}_1 t)$  and that  $\exp(\boldsymbol{\xi}_1 t)$  and  $\boldsymbol{\xi}_1$  commute, straightforward computations show that

$$\begin{aligned} A_{\alpha, \beta}^{-1}(t) \ddot{A}_{\alpha, \beta}(t) &= \text{diag}(-\alpha^2, -\alpha^2, -\beta^2, -\beta^2), \\ \dot{A}_{\alpha, \beta}(t)\mathbf{q}_i \cdot \dot{A}_{\alpha, \beta}(t)\mathbf{q}_i &= \alpha^2(x_i^2 + y_i^2) + \beta^2(z_i^2 + w_i^2). \end{aligned}$$

Substituting these expressions into equations (7), we obtain that

$$m_i(-\alpha^2 x_i, -\alpha^2 y_i, -\beta^2 z_i, -\beta^2 w_i) = \nabla_{\mathbf{q}_i} U - m_i[\alpha^2(x_i^2 + y_i^2) + \beta^2(z_i^2 + w_i^2)]\mathbf{q}_i, \quad i = \overline{1, N}.$$

Using in the above equations the identity  $\mathbf{q}_i \cdot \mathbf{q}_i = 1$ , we can obtain equations (8), a remark that completes the proof.  $\square$

Similarly, we can prove the following criterion for relative equilibria in  $\mathbb{H}^3$ .

**Theorem 4.** *Let  $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N)$ ,  $\mathbf{q}_i = (x_i, y_i, z_i, w_i)$ ,  $i = \overline{1, N}$ , be a nonsingular configuration in  $\mathbb{H}^3$ . Then  $B_{\alpha, \beta}(t)\mathbf{q}$  is a relative equilibrium if and only if this configuration satisfies the equations*

$$(9) \quad -m_i(\alpha^2 + \beta^2) \begin{bmatrix} x_i(w_i^2 - z_i^2) \\ y_i(w_i^2 - z_i^2) \\ z_i(x_i^2 + y_i^2) \\ w_i(x_i^2 + y_i^2) \end{bmatrix} = \nabla_{\mathbf{q}_i} U, \quad i = \overline{1, N}.$$

Theorem 2 and the above two theorems are equivalent. For example, in  $\mathbb{S}^3$ , define  $f(x, y, z, w) = x^2 + y^2$  as a function from  $\mathbb{S}^3$  to  $\mathbb{R}$ . To find the gradient of  $f$ , we employ the trick used to derive  $\nabla_{\mathbf{q}_i} \mathbf{q}_i \cdot \mathbf{q}_j$  in Section 4. Extend  $f$  to a homogeneous function  $\bar{f}$  of degree zero in the ambient space  $\mathbb{R}^4$ ,

$$\bar{f}(x, y, z, w) := \frac{x^2 + y^2}{x^2 + y^2 + z^2 + w^2}.$$

Let  $\tilde{\nabla}$  be the gradient in the ambient space, and  $\frac{\partial}{\partial n}$  be the unit normal vector of the unit sphere. Since  $\frac{\partial \bar{f}}{\partial r} = 0$ , we obtain  $(\tilde{\nabla} \bar{f})|_{\mathbb{S}^3} = \nabla f + \frac{\partial \bar{f}}{\partial r} \frac{\partial}{\partial n} = \nabla f$ . Thus straightforward computations show that

$$\nabla f(x, y, z, w) = 2(x(w^2 + z^2), y(w^2 + z^2), -z(x^2 + y^2), -w(x^2 + y^2)).$$

Hence we can conclude that

$$\nabla_{\mathbf{q}_i} I(\mathbf{q}) = 2m_i(x_i(w_i^2 + z_i^2), y_i(w_i^2 + z_i^2), -z_i(x_i^2 + y_i^2), -w_i(x_i^2 + y_i^2)).$$

Thus the right hand side of (8) is  $\frac{\beta^2 - \alpha^2}{2} \nabla_{\mathbf{q}_i} \left( \sum_{i=1}^N m_i(x_i^2 + y_i^2) \right)$ . Theorem 2 matches Theorem 3. Similarly, in  $\mathbb{H}^3$ ,

$$\nabla_{\mathbf{q}_i} I(\mathbf{q}) = 2m_i(x_i(w_i^2 - z_i^2), y_i(w_i^2 - z_i^2), z_i(x_i^2 + y_i^2), w_i(x_i^2 + y_i^2)).$$

Thus Theorem 2 also matches Theorem 4.

**5.2. Central configurations and relative equilibria.** We are now motivated to study the equation

$$\nabla_{\mathbf{q}_i} U(\mathbf{q}) = \lambda \nabla_{\mathbf{q}_i} \left[ \sum_{i=1}^N m_i(x_i^2 + y_i^2) \right], \quad i = \overline{1, N}.$$

**Definition 6.** *Assume that the point masses  $m_1, \dots, m_N$  in  $\mathbb{M}^3$  have the nonsingular positions given by the vector  $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N)$ ,  $\mathbf{q}_i = (x_i, y_i, z_i, w_i)$ ,  $i = \overline{1, N}$ . Then  $\mathbf{q}$  is a central configuration of the curved  $N$ -body problem in  $\mathbb{M}^3$  if it satisfies the equations*

$$(10) \quad \nabla_{\mathbf{q}_i} U(\mathbf{q}) = \lambda \nabla_{\mathbf{q}_i} I(\mathbf{q}), \quad i = \overline{1, N},$$

where  $\lambda \in \mathbb{R}$  is a constant and  $I$  is the moment of inertia. We call (10) the central configuration equations.

Explicitly, the central configuration equations (10) are

$$(11) \quad \sum_{j \neq i, j=1}^N \frac{m_j m_i \mathbf{q}_j}{\text{sn}^3 d_{ij}} - \sum_{j \neq i, j=1}^N \frac{m_j m_i \text{csnd}_{ij}}{\text{sn}^3 d_{ij}} \mathbf{q}_i = \lambda \nabla_{\mathbf{q}_i} I, \quad i = \overline{1, N}.$$

**Proposition 3.** *The  $i$ -th equation of the central configuration equations (11) holds if and only if there is a constant  $\theta_i$  such that*

$$(12) \quad \sum_{j \neq i, j=1}^N \frac{m_j m_i \mathbf{q}_j}{\text{sn}^3 d_{ij}} - \theta_i \mathbf{q}_i = \lambda \nabla_{\mathbf{q}_i} I.$$

*Proof.* Multiply (12) by  $\mathbf{q}_i$ . Since  $\mathbf{q}_i \cdot \mathbf{q}_j = \sigma \text{csnd}_{ij}$ ,  $\mathbf{q}_i \cdot \mathbf{q}_i = \sigma$ , and  $\mathbf{q}_i \cdot \nabla_{\mathbf{q}_i} I = 0$ , we obtain  $\theta_i = \sum_{j \neq i, j=1}^N \frac{m_j m_i \text{csnd}_{ij}}{\text{sn}^3 d_{ij}}$ . Thus (12) is equivalent to the  $i$ -th equation of (11).  $\square$

The following class of central configurations exists in  $\mathbb{S}^3$  only [9, 10].

**Definition 7.** *Consider the positive masses  $m_1, \dots, m_N$  in  $\mathbb{S}^3$ . Then a configuration  $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N)$ ,  $\mathbf{q}_i = (x_i, y_i, z_i, w_i)$ ,  $i = \overline{1, N}$ , is called a special central configuration if it is a critical point of the force function  $U$ , i.e.*

$$\nabla_{\mathbf{q}_i} U(\mathbf{q}) = 0, \quad i = \overline{1, N}.$$

*In other words,  $\mathbf{F}_i = 0, i = \overline{1, N}$ . To avoid any confusion, we will call ordinary central configurations those central configurations that are not special.*

Here is one remark on terminology. These special central configurations were introduced in [9, 10] under the name of *fixed points*. Given such a configuration  $\mathbf{q}$ , we see with the help of Theorem 3 that  $A_{0,0}(t)\mathbf{q}$  is an associated relative equilibrium, which is a *fixed-point solution*:  $\mathbf{q}(t) = \mathbf{q}$ ,  $\mathbf{p}(t) = 0$ . This explains the old terminology. Let us introduce some new terminology as well.

**Definition 8.** *A central configuration  $\mathbf{q}$  of the curved  $N$ -body problem is called*

- *a geodesic central configuration if it is lying on a geodesic;*
- *an  $\mathbb{S}^2$  central configuration if it is lying on a great 2-sphere;*
- *an  $\mathbb{H}^2$  central configuration if it is lying on a great hyperbolic 2-sphere;*
- *an  $\mathbb{S}^3$  central configuration if it is not lying on any great 2-sphere;*
- *an  $\mathbb{H}^3$  central configuration if it is not lying on any great hyperbolic 2-sphere.*

Central configurations will play an important role in the study of the curved  $N$ -body problem. They influence the topology of the integral manifolds [40, 59]. Now we discuss the connection between them and the motions of the curved  $N$ -body problem. Let

$$\begin{aligned} \mathbb{S}_{xy}^1 &:= \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 = 1, z = w = 0\}, \\ \mathbb{S}_{zw}^1 &:= \{(x, y, z, w) \in \mathbb{R}^4 \mid z^2 + w^2 = 1, x = y = 0\}, \\ \mathbb{H}_{zw}^1 &:= \{(x, y, z, w) \in \mathbb{R}^4 \mid z^2 - w^2 = -1, x = y = 0\}. \end{aligned}$$

**Lemma 1.** *In  $(\mathbb{S}^3)^N$ , we have that  $\nabla_{\mathbf{q}_i} I = 0$  if and only if  $\mathbf{q}_i \in \mathbb{S}_{xy}^1 \cup \mathbb{S}_{zw}^1$ . Similarly, in  $(\mathbb{H}^3)^N$ , we have that  $\nabla_{\mathbf{q}_i} I = 0$  if and only if  $\mathbf{q}_i \in \mathbb{H}_{zw}^1$ .*

*Proof.* In  $(\mathbb{S}^3)^N$ , recall that

$$\nabla_{\mathbf{q}_i} I = 2m_i(x_i(w_i^2 + z_i^2), y_i(w_i^2 + z_i^2), -z_i(x_i^2 + y_i^2), -w_i(x_i^2 + y_i^2)).$$

On one hand, if  $\nabla_{\mathbf{q}_i} I$  is a zero vector, then

$$(x_i(w_i^2 + z_i^2))^2 + (y_i(w_i^2 + z_i^2))^2 = (x_i^2 + y_i^2)(w_i^2 + z_i^2)^2 = 0,$$

which means that  $\mathbf{q}_i \in \mathbb{S}_{xy}^1$  or  $\mathbb{S}_{zw}^1$ . On the other hand, if  $\mathbf{q}_i \in \mathbb{S}_{xy}^1 \cup \mathbb{S}_{zw}^1$ , then  $\nabla_{\mathbf{q}_i} I = 0$ .

In  $(\mathbb{H}^3)^N$ , recall that

$$\nabla_{\mathbf{q}_i} I = 2m_i(x_i(w_i^2 - z_i^2), y_i(w_i^2 - z_i^2), z_i(x_i^2 + y_i^2), w_i(x_i^2 + y_i^2)).$$

Again, on one hand, if  $\nabla_{\mathbf{q}_i} I$  is a zero vector, then

$$(x_i(w_i^2 - z_i^2))^2 + (y_i(w_i^2 - z_i^2))^2 = (x_i^2 + y_i^2)(w_i^2 - z_i^2)^2 = 0,$$

which means that  $x_i = y_i = 0$ , since  $w_i^2 - z_i^2 = 1 + x_i^2 + y_i^2 \neq 0$ . Thus we obtain that  $\mathbf{q}_i \in \mathbb{H}_{zw}^1$ . On the other hand, if  $\mathbf{q}_i \in \mathbb{H}_{zw}^1$ , then  $\nabla_{\mathbf{q}_i} I = 0$ .  $\square$

**Corollary 1.** *Consider a central configuration  $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N)$ ,  $\mathbf{q}_i = (x_i, y_i, z_i, w_i)$ ,  $i = \overline{1, N}$ , in  $\mathbb{M}^3$ . Let  $\lambda$  be the constant in the central configuration equations*

$$\nabla_{\mathbf{q}_i} U(\mathbf{q}) = \lambda \nabla_{\mathbf{q}_i} I(\mathbf{q}).$$

1. *If  $\mathbf{q}$  is an ordinary central configuration in  $\mathbb{S}^3$ , then it gives rise to a 1-parameter family of relative equilibria:  $A_{\alpha, \beta}(t)\mathbf{q}$  with  $\lambda = \frac{\beta^2 - \alpha^2}{2}$ .*
2. *If  $\mathbf{q}$  is in  $\mathbb{H}^3$ , then it gives rise to a 1-parameter family of relative equilibria:  $B_{\alpha, \beta}(t)\mathbf{q}$  with  $\lambda = -\frac{\alpha^2 + \beta^2}{2}$ .*
3. *If  $\mathbf{q}$  is a special central configuration in  $\mathbb{S}^3$  and not all the particles are on  $\mathbb{S}_{xy}^1 \cup \mathbb{S}_{zw}^1$ , then it gives rise to a 1-parameter family of relative equilibria:  $A_{\alpha, \beta}(t)\mathbf{q}$  with  $0 = \beta^2 - \alpha^2$ .*
4. *If  $\mathbf{q}$  is a special central configuration in  $\mathbb{S}^3$  and all the particles are on  $\mathbb{S}_{xy}^1 \cup \mathbb{S}_{zw}^1$ , then it gives rise to a 2-parameter family of relative equilibria:  $A_{\alpha, \beta}(t)\mathbf{q}$  with  $\alpha, \beta \in \mathbb{R}$ .*

Before proving this corollary, let us make the following remark on terminology. In the literature, the concept of relative equilibrium stands for both central configurations and the rigid motions associated to them [40, 59]. In this paper, however, we use the term relative equilibrium only for the associated motion.

*Proof.* The first two claims are obvious by Theorem 2. If  $\mathbf{q}$  is a special central configuration in  $\mathbb{S}^3$ , then by Theorem 2,  $A_{\alpha, \beta}(t)\mathbf{q}$  is an associated relative equilibrium if and only if  $\frac{\beta^2 - \alpha^2}{2} \nabla_{\mathbf{q}_i} I = 0$  for  $i = \overline{1, N}$ .

There are two possibilities: first, if there exists some  $\mathbf{q}_i$  with  $\nabla_{\mathbf{q}_i} I \neq 0$ , i.e., there is some  $\mathbf{q}_i \notin \mathbb{S}_{xy}^1 \cup \mathbb{S}_{zw}^1$ , then  $0 = \beta^2 - \alpha^2$ , i.e., there is a 1-parameter family of relative equilibria associated to the special central configuration  $\mathbf{q}$ :  $A_{\alpha, \beta}(t)\mathbf{q}$  with  $0 = \beta^2 - \alpha^2$ ; second, if  $\nabla_{\mathbf{q}_i} I = 0$  for all  $i$ , that is,  $\mathbf{q}_i \in \mathbb{S}_{xy}^1 \cup \mathbb{S}_{zw}^1$  for all  $i$ , then there is no limitation for  $\alpha, \beta$ , i.e., there is a 2-parameter family of relative equilibria associated to the special central configuration  $\mathbf{q}$ :  $A_{\alpha, \beta}(t)\mathbf{q}$  with  $\alpha, \beta \in \mathbb{R}$ .  $\square$

**Remark 2.** There is a gap in the proof. For a central configuration in  $\mathbb{H}^3$ , we don't have a 1-parameter family of relative equilibria, as claimed, unless we can show that the value of  $\lambda$  is always negative. This fact will be proved in Section 6.

Notice that while 3-dimensional central configurations of the Newtonian  $N$ -body problem do not have associated relative equilibria [63], all central configurations of the curved  $N$ -body problem have associated relative equilibria.

In the class of relative equilibria associated to one central configuration, there are motions of different types. In  $\mathbb{S}^3$ , the relative equilibria can be positive elliptic and positive elliptic-elliptic. In  $\mathbb{H}^3$ , they can be negative elliptic, negative hyperbolic, and negative elliptic-hyperbolic. These solutions can be periodic, quasi-periodic, or different. For an ordinary central configuration in  $\mathbb{S}^3$ , the intersections of the hyperbola  $\lambda = \frac{\beta^2 - \alpha^2}{2}$  and the line  $\beta = k\alpha$ ,  $k$  rational, in the  $\alpha\beta$  plane, gives periodic motions; otherwise, the motions are quasi-periodic. For a special central configuration in  $\mathbb{S}^3$  that not all particles are on  $\mathbb{S}_{xy}^1 \cup \mathbb{S}_{zw}^1$ , the relative equilibria are always periodic. If  $\mathbf{q}$  is on  $\mathbb{S}_{xy}^1 \cup \mathbb{S}_{zw}^1$ , then any points on the line  $\beta = k\alpha$ ,  $k$  rational in the  $\alpha\beta$  plane give periodic motions; otherwise, the motions are quasi-periodic. For an ordinary central configuration in  $\mathbb{H}^3$ , the relative equilibria are periodic if and only if  $\beta = 0$ . Some negative hyperbolic solutions can be mere hyperbolic rotations, which are neither periodic nor quasi-periodic.

However, unlike in the Newtonian  $N$ -body problem, central configurations do not provide us with *homothetic solutions*, which occur only in vector spaces, since they require similarity [63]. Actually, since there is no centre of masses, it makes no sense to talk about homothetic solutions. For a special central configuration, if we set the particles at rest at  $t = 0$ , then we obtain a fixed-point solution.

## 6. CENTRAL CONFIGURATIONS

In this section we prove some basic facts about central configurations. We first write the central configuration equations in another form, then give their physical description, which justifies their name and finally define equivalent classes of central configurations.

**6.1. The central configuration equations.** In the previous section we introduced the central configuration equations in different forms, such as (10), (11), and (12). We now derive another form of the central configuration equations, which will be useful. Define

$$r_i := (x_i^2 + y_i^2)^{1/2}, \quad \rho_i := (\sigma z_i^2 + w_i^2)^{1/2}, \quad i = \overline{1, N}.$$

Then, in  $\mathbb{H}^3$ , we have  $r_i^2 + \sigma \rho_i^2 = \sigma$  and  $r \rho_i^2 > 0$ . Recall that the  $i$ -th equation of (11) is  $\sum_{j \neq i, j=1}^N \frac{m_j m_i \mathbf{q}_j}{\text{sn}^3 d_{ij}} - \sum_{j \neq i, j=1}^N \frac{m_j m_i \text{csn} d_{ij}}{\text{sn}^3 d_{ij}} \mathbf{q}_i = \lambda \nabla_{\mathbf{q}_i} I$ .

**Proposition 4.** *Consider the positive masses  $m_1, \dots, m_N$  on  $\mathbb{S}^3$  at the configuration  $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N)$ , where  $\mathbf{q}_i = (x_i, y_i, w_i)$ . If  $\mathbf{q}_i = (x_i, y_i, z_i, w_i) \notin \mathbb{S}_{xy}^1 \cup \mathbb{S}_{zw}^1$ , then the*

$i$ -th equations of (11) can be written as

$$(13) \quad \begin{cases} \sum_{j=1, j \neq i}^N \frac{m_j(x_i x_j + y_i y_j - r_i^2 \cos d_{ij})}{\sin^3 d_{ij}} = 2\lambda r_i^2 \rho_i^2 \\ \sum_{j=1, j \neq i}^N \frac{m_j(x_i y_j - x_j y_i)}{\sin^3 d_{ij}} = 0 \\ \sum_{j=1, j \neq i}^N \frac{m_j(z_i w_j - z_j w_i)}{\sin^3 d_{ij}} = 0. \end{cases}$$

*Proof.* Since  $\mathbf{q}_i \notin \mathbb{S}_{xy}^1 \cup \mathbb{S}_{zw}^1$ , the following four vectors

$$\mathbf{v}_{i1} = (x_i, y_i, 0, 0), \quad \mathbf{v}_{i2} = (-y_i, x_i, 0, 0), \quad \mathbf{v}_{i3} = (0, 0, z_i, w_i), \quad \mathbf{v}_{i4} = (0, 0, -w_i, z_i),$$

form an orthogonal basis of  $T_{\mathbf{q}_i} \mathbb{R}^4$ . Recall that

$$\nabla_{\mathbf{q}_i} U = \sum_{j=1, j \neq i}^N m_i m_j \frac{\mathbf{q}_j - \cos d_{ij} \mathbf{q}_i}{\sin^3 d_{ij}}, \quad \nabla_{\mathbf{q}_i} I = 2m_i(x_i \rho_i^2, y_i \rho_i^2, -z_i r_i^2, -w_i r_i^2).$$

Then  $\nabla_{\mathbf{q}_i} U = \lambda \nabla_{\mathbf{q}_i} I$  is equivalent to  $\nabla_{\mathbf{q}_i} U \cdot \mathbf{v}_{ik} = \lambda \nabla_{\mathbf{q}_i} I \cdot \mathbf{v}_{ik}, k = 1, 2, 3, 4$ . Thus

$$\begin{aligned} \sum_{j=1, j \neq i}^N \frac{m_i m_j}{\sin^3 d_{ij}} (x_i x_j + y_i y_j - r_i^2 \cos d_{ij}) &= \lambda 2m_i r_i^2 \rho_i^2, & \sum_{j=1, j \neq i}^N \frac{m_i m_j}{\sin^3 d_{ij}} (x_i y_j - y_i x_j) &= 0, \\ \sum_{j=1, j \neq i}^N \frac{m_i m_j}{\sin^3 d_{ij}} (z_i z_j + w_i w_j - \rho_i^2 \cos d_{ij}) &= -\lambda 2m_i r_i^2 \rho_i^2, & \sum_{j=1, j \neq i}^N \frac{m_i m_j}{\sin^3 d_{ij}} (z_i w_j - w_i z_j) &= 0. \end{aligned}$$

Adding the first and the third equation we obtain an identity.  $\square$

Similarly, we can prove the following result.

**Proposition 5.** *Consider the positive masses  $m_1, \dots, m_N$  in  $\mathbb{H}^3$  at the configuration  $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N)$ , where  $\mathbf{q}_i = (x_i, y_i, w_i)$ . If  $\mathbf{q}_i \notin \mathbb{H}_{zw}^1$ , then the  $i$ -th equations of (11) can be written as*

$$\begin{cases} \sum_{j=1, j \neq i}^N \frac{m_j(x_i x_j + y_i y_j - r_i^2 \cosh d_{ij})}{\sinh^3 d_{ij}} = 2\lambda r_i^2 \rho_i^2 \\ \sum_{j=1, j \neq i}^N \frac{m_j(x_i y_j - x_j y_i)}{\sinh^3 d_{ij}} = 0 \\ \sum_{j=1, j \neq i}^N \frac{m_j(z_i w_j - z_j w_i)}{\sinh^3 d_{ij}} = 0. \end{cases}$$

Now we consider the value of  $\lambda$  in the central configuration equations. In this section, let  $M$  be the matrix  $\text{diag}(m_1, m_1, m_1, m_1, \dots, m_N, m_N, m_N, m_N)$ . Introduce a metric in  $(\mathbb{R}^4)^N$   $((\mathbb{R}^{3,1})^N)$ :

$$\langle \mathbf{q}, \mathbf{q} \rangle = \sum_{i=1}^N m_i \mathbf{q}_i \cdot \mathbf{q}_i = \mathbf{q} \cdot M \mathbf{q}.$$

For ordinary central configurations we have

$$\langle M^{-1}\nabla U, M^{-1}\nabla I \rangle = \lambda \langle M^{-1}\nabla I, M^{-1}\nabla I \rangle.$$

**Proposition 6.** *Let  $\mathbf{q}$  be an ordinary central configuration, then the value of  $\lambda$  in the central configuration equation is  $\frac{\langle M^{-1}\nabla U, M^{-1}\nabla I \rangle}{\langle M^{-1}\nabla I, M^{-1}\nabla I \rangle}$ . For central configurations in  $\mathbb{H}^3$ , we have  $\lambda < 0$ .*

*Proof.* Since  $\mathbf{q}$  is an ordinary central configuration,  $\nabla I \neq 0$  and  $\langle M^{-1}\nabla I, M^{-1}\nabla I \rangle \neq 0$ . Thus the value of  $\lambda$  for an ordinary central configuration  $\mathbf{q}$  is  $\frac{\langle M^{-1}\nabla U, M^{-1}\nabla I \rangle}{\langle M^{-1}\nabla I, M^{-1}\nabla I \rangle}$ .

In  $\mathbb{H}^3$ , using the identities  $\cosh d_{ij} = w_i w_j - (x_i x_j + y_i y_j + z_i z_j)$  and  $x_i^2 + y_i^2 + z_i^2 - w_i^2 = -1$ , the denominator is

$$\langle M^{-1}\nabla I, M^{-1}\nabla I \rangle = 4 \sum_{i=1}^N m_i r_i^2 \rho_i^2,$$

and the numerator is

$$\begin{aligned} & \sum_{i=1}^N m_i \left( \sum_{j \neq i}^N m_j \frac{\mathbf{q}_j}{\sinh^3 d_{ij}} \cdot \frac{\nabla_{\mathbf{q}_i} I}{m_i} \right) \\ &= 2 \sum_{1 \leq i < j \leq N} m_i m_j \frac{(x_i x_j + y_i y_j)(\rho_i^2 + \rho_j^2) + (z_i z_j - w_i w_j)(r_i^2 + r_j^2)}{\sinh^3 d_{ij}} \\ &= 2 \sum_{1 \leq i < j \leq N} m_i m_j \frac{2x_i x_j + 2y_i y_j - (r_i^2 + r_j^2) \cosh d_{ij}}{\sinh^3 d_{ij}} \\ &= -2 \sum_{1 \leq i < j \leq N} m_i m_j \frac{(x_i - x_j)^2 + (y_i - y_j)^2 + (r_i^2 + r_j^2)(\cosh d_{ij} - 1)}{\sinh^3 d_{ij}} < 0, \end{aligned}$$

a remark that completes the proof.  $\square$

This proposition fills the gap in the proof of Corollary 1. It also implies that there are no special central configurations in  $\mathbb{H}^3$  and that there is no such central configuration with  $\mathbf{q}_i \in \mathbb{H}_{zw}^1$  for all  $i = \overline{1, N}$ , since in either case we have  $\lambda = 0$ .

In  $\mathbb{S}^3$ , the value of  $\lambda$  could be positive, zero, or negative, see examples in Section 9. The case  $\lambda = 0$  corresponds to special central configurations.

**6.2. A physical description of central configurations.** It turns out that the moment of inertia  $I$  has geometric meaning, a fact that brings some insight into the problem and provides a physical description of central configurations.

**Lemma 2.** *If  $A = (x, y, z, w)$  is a point in  $\mathbb{S}^3$ , then  $z^2 + w^2 = \cos^2 d(A, \mathbb{S}_{zw}^1)$ . If  $A = (x, y, z, w)$  is a point in  $\mathbb{H}^3$ , then  $-z^2 + w^2 = \cosh^2 d(A, \mathbb{H}_{zw}^1)$ , where  $d(A, \mathcal{M}) := \inf_{B \in \mathcal{M}} d(A, B)$ , with  $A, B$  representing points and  $\mathcal{M}$  being a smooth manifold.*

*Proof.* View  $A$  as a vector in  $\mathbb{R}^4$ . Denote by  $\mathbb{R}_A^3$  the 3- (or 2-) dimensional subspace spanned by  $A$ ,  $e_z = (0, 0, 1, 0)$ , and  $e_w = (0, 0, 0, 1)$ . Denote by  $\mathbb{R}_{zw}^2$  the 2-dimensional subspace spanned by  $e_z$  and  $e_w$ .

In  $\mathbb{S}^3$ , the minimal geodesic connecting  $A$  and  $\mathbb{S}_{zw}^1$  is on the great 2-sphere  $\mathbb{S}_A^2 = \mathbb{R}_A^3 \cap \mathbb{S}^3$ . Let  $\theta = d(A, \mathbb{S}_{zw}^1)$ . Then  $A = A_v + A_h \in (\mathbb{R}_{zw}^2)^\perp \oplus \mathbb{R}_{zw}^2$  with  $\|A_v\| = \sin \theta$  and  $\|A_h\| = \cos \theta$ . Hence, we obtain

$$\cos^2 d(A, \mathbb{S}_{zw}^1) = \|A_h\|^2 = \|(A \cdot e_z)e_z + (A \cdot e_w)e_w\|^2 = \|ze_z + we_w\|^2 = z^2 + w^2.$$

In  $\mathbb{H}^3$ , the minimal geodesic connecting  $A$  and  $\mathbb{H}_{zw}^1$  is on the great hyperbolic 2-sphere  $\mathbb{H}_A^2 = \mathbb{R}_A^3 \cap \mathbb{H}^3$ . Let  $\theta = d(A, \mathbb{H}_{zw}^1)$ . Then similarly we have  $A = A_v + A_h \in (\mathbb{R}_{zw}^2)^\perp \oplus \mathbb{R}_{zw}^2$  with  $\|A_v\| = \sinh \theta$  and  $\|A_h\| = \cosh \theta$ . Hence, we obtain

$$\begin{aligned} \cosh^2 d(A, \mathbb{H}_{zw}^1) &= \|A_h\|^2 = \left\| \frac{A \cdot e_z}{e_z \cdot e_z} e_z + \frac{A \cdot e_w}{e_w \cdot e_w} e_w \right\|^2 \\ &= \|ze_z - (-w)e_w\|^2 = |(ze_z + we_w) \cdot (ze_z + we_w)| \\ &= |z^2 - w^2| = -z^2 + w^2. \end{aligned}$$

□

**Theorem 5.** *A nonsingular configuration  $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N)$ ,  $\mathbf{q}_i = (x_i, y_i, z_i, w_i)$ ,  $i = \overline{1, N}$ , in  $\mathbb{M}^3$  is a central configuration if and only if*

$$(14) \quad \begin{aligned} \nabla_{\mathbf{q}_i} U(\mathbf{q}) &= \lambda m_i \sin[2d(\mathbf{q}_i, \mathbb{S}_{zw}^1)] \nabla_{\mathbf{q}_i} d(\mathbf{q}_i, \mathbb{S}_{zw}^1), \quad i = \overline{1, N}, \text{ in } \mathbb{S}^3, \\ \nabla_{\mathbf{q}_i} U(\mathbf{q}) &= \lambda m_i \sinh[2d(\mathbf{q}_i, \mathbb{H}_{zw}^1)] \nabla_{\mathbf{q}_i} d(\mathbf{q}_i, \mathbb{H}_{zw}^1), \quad i = \overline{1, N}, \text{ in } \mathbb{H}^3, \end{aligned}$$

where  $\lambda \in \mathbb{R}$  is a constant.

*Proof.* By Lemma 2, we obtain  $x_i^2 + y_i^2 = \sin^2 d(\mathbf{q}_i, \mathbb{S}_{zw}^1)$  in  $\mathbb{S}^3$  and  $x_i^2 + y_i^2 = \sinh^2 d(\mathbf{q}_i, \mathbb{H}_{zw}^1)$  in  $\mathbb{H}^3$ . Thus

$$I = \sum_{1 \leq i \leq N} m_i \sin^2 d(\mathbf{q}_i, \mathbb{S}_{zw}^1) \text{ in } \mathbb{S}^3, \quad I = \sum_{1 \leq i \leq N} m_i \sinh^2 d(\mathbf{q}_i, \mathbb{H}_{zw}^1) \text{ in } \mathbb{H}^3.$$

Then the central configuration equations (10) can be written as (14). □

By definition, special central configurations are arrangements of the particles for which the forces acting on each particles cancel. By the above theorem, ordinary central configurations are special arrangements of the particles with the property that the gravitational force produced on each particle by all the others particles points towards the geodesic  $\mathbb{S}_{zw}^1$  ( $\mathbb{H}_{zw}^1$ ) and is proportional to  $m_i \sin[2d(\mathbf{q}_i, \mathbb{S}_{zw}^1)]$  ( $m_i \sinh[2d(\mathbf{q}_i, \mathbb{H}_{zw}^1)]$ ).

Define

$$\begin{aligned} \mathbb{H}_{xyw}^2 &:= \{(x, y, z, w) \in \mathbb{R}^4 | x^2 + y^2 - w^2 = -1, z = 0\}, \\ \mathbb{S}_{xyz}^2 &:= \{(x, y, z, w) \in \mathbb{R}^4 | x^2 + y^2 + z^2 = 1, w = 0\}. \end{aligned}$$

Recall that in the Newtonian  $N$ -body problem, central configurations are those arrangements of particles such that all  $\mathbf{F}_i$  are pointing towards the centre of mass [63]. In the curved  $N$ -body problem, instead of a point, all  $\mathbf{F}_i$  are pointing towards a geodesic. Furthermore, it was shown in [65] that all central configurations in  $\mathbb{H}^3$  lie on a

submanifold  $\mathbb{H}_{xyw}^2$  and we will prove that all ordinary  $\mathbb{S}^2$  central configurations lie on a submanifold  $\mathbb{S}_{xyz}^2$ . The intersection of  $\mathbb{H}_{xyw}^2$  and  $\mathbb{H}_{zw}^1$  is  $(0, 0, 0, 1)$ , and the intersections of  $\mathbb{S}_{xyz}^2$  and  $\mathbb{S}_{zw}^1$  are  $(0, 0, \pm 1, 0)$ . It is easy to see that the minimal path connecting  $\mathbf{q}_i$  on  $\mathbb{H}_{xyw}^2$  ( $\mathbb{S}_{xyz}^2$ ) and the geodesic  $\mathbb{H}_{zw}^1$  ( $\mathbb{S}_{zw}^1$ ) lies on the two submanifolds. Thus we can say that for all central configurations in  $\mathbb{H}^3$ , all  $\mathbf{F}_i$  are pointing towards one point; for all ordinary  $\mathbb{S}^2$  central configurations, all  $\mathbf{F}_i$  are pointing towards one of two points. The vector fields  $\nabla(x^2 + y^2)$  on the two submanifolds are sketched in Figure 1.



FIGURE 1.  $\nabla(x^2 + y^2)$  on  $\mathbb{S}_{xyz}^2$  and  $\mathbb{H}_{xyw}^2$

**6.3. Equivalent central configurations.** Recall that central configurations in the Newtonian  $N$ -body problem are invariant under translations, rotations, reflections and scaling [63]. In the curved  $N$ -body problem,  $U$  is invariant under the symmetry group  $O(4)$  or  $O(3, 1)$ . We can check by the formula of  $\nabla_{\mathbf{q}_i} U$  that  $\nabla_{\mathbf{q}_i} U|_{\mathbf{q}'=\chi\mathbf{q}} = \chi \nabla_{\mathbf{q}_i} U|_{\mathbf{q}}$  for any  $\chi$  in the symmetry group. Though  $I$  is not invariant under all elements of the symmetry group, it is invariant under a subgroup  $O(2) \times O(2)$  ( $O(2) \times O(1, 1)$ ). Let  $\chi = (\chi_1, \chi_2) \in O(2) \times O(2)$  ( $O(2) \times O(1, 1)$ ). The action is

$$\chi\mathbf{q} = (\chi\mathbf{q}_1, \dots, \chi\mathbf{q}_N), \quad \chi\mathbf{q}_i = (\chi_1(x_i, y_i)^T, \chi_2(z_i, w_i)^T).$$

Then, by using the formula of  $\nabla_{\mathbf{q}_i} I$  or Lemma 2, we can see that  $\nabla_{\mathbf{q}_i} I|_{\mathbf{q}'=\chi\mathbf{q}} = \chi \nabla_{\mathbf{q}_i} I|_{\mathbf{q}}$ .

There is no other obvious transform that keeps the central configuration equations. Also note that  $I$  is not involved in the equation of special central configurations. Thus we introduce the following definition.

**Definition 9.** Let  $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N)$ ,  $\mathbf{q}_i = (x_i, y_i, z_i, w_i)$ ,  $i = \overline{1, N}$ , and  $\mathbf{q}' = (\mathbf{q}'_1, \dots, \mathbf{q}'_N)$ ,  $\mathbf{q}'_i = (x'_i, y'_i, z'_i, w'_i)$ ,  $i = \overline{1, N}$ , be two central configurations in  $\mathbb{M}^3$ .

1. If they are special central configurations in  $\mathbb{S}^3$ , then they are equivalent if there is a  $\chi \in SO(4)$  such that  $\mathbf{q} = \chi\mathbf{q}'$ .
2. If they are ordinary central configurations, then they are equivalent if there is a  $\chi = (\chi_1, \chi_2) \in SO(2) \times SO(2)$  ( $SO(2) \times SO(1, 1)$ ) such that  $\mathbf{q} = \chi\mathbf{q}'$ .

We use  $SO(2) \times SO(2)$  ( $SO(2) \times SO(1, 1)$ ) instead of  $O(2) \times O(2)$  ( $O(2) \times O(1, 1)$ ). We adopt this definition to keep consistency with the critical point characterization of central configurations, which will be introduced in Section 8.

**Example 1** (Lagrangian central configuration on  $\mathbb{S}_{xyz}^2$ ). Let three equal masses  $m_1 = m_2 = m_3 = 1$  be at

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3), \quad \mathbf{q}_j = (x_j, y_j, z_j, w_j), \quad j = 1, 2, 3,$$

$$x_j = \sqrt{1 - c^2} \cos \beta_j, \quad y_j = \sqrt{1 - c^2} \sin \beta_j, \quad z_j = c, \quad w_j = 0, \quad \beta_j = \frac{2\pi(j-1)}{3},$$

where  $c$  could have any value between  $-1$  and  $1$ , see Figure 2. By symmetry, we see that  $\nabla_{\mathbf{q}_i} U$  is pointing towards the north pole if  $c > 0$ , or towards the south pole if  $c < 0$ . Comparing with Figure 1, we get that there must be some constant  $\lambda$  such that  $\nabla_{\mathbf{q}_i} U = \lambda \nabla_{\mathbf{q}_i} I$  for  $1 \leq i \leq 3$ . Note that  $d_{12} = d_{13} = d_{23}$ , which is reminiscent of the 3-body central configuration in the Newtonian  $N$ -body problem found by Lagrange, [63]. We call them Lagrangian central configurations.

By the convention we introduced, rotating the central configurations in the  $xy$ -plane does not lead to new central configurations, and the rotated ones still remain on the original 2-sphere; rotating them in the  $zw$ -plane does not lead to new central configurations either, although they will not remain on the original 2-sphere. Though these central configurations, for different values of  $c$ , are similar in some sense, there does not exist an element in  $SO(2) \times SO(2)$  to relate any two of them. Thus we see that there is a continuum of central configurations.

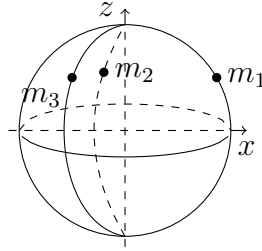


FIGURE 2. Lagrangian central configurations on  $\mathbb{S}_{xyz}^2$

In Section 8, we will see that, for any given masses, there is a continuum of central configurations.

## 7. SOME PROPERTIES OF CENTRAL CONFIGURATIONS

In this section we provide some lemmas and theorems that will be useful in the study of central configurations. We first prove a property that is analogous to the relationship  $\sum_{i=1}^N m_i \mathbf{q}_i = 0$  for central configurations of the Newtonian  $N$ -body problem, [45]. We then focus on lower dimensional ordinary central configurations, namely geodesic central configurations,  $\mathbb{S}^2$  central configurations, and  $\mathbb{H}^2$  central configurations. We show that any geodesic central configuration in  $\mathbb{H}^3$  is equivalent to some central configuration on  $\mathbb{H}_{xw}^1$ . We also show that any  $\mathbb{S}^2$  central configuration in  $\mathbb{S}^3$  can be found on  $\mathbb{S}_{xyz}^2$ , and that any geodesic central configuration in  $\mathbb{S}^3$  is equivalent to some central configuration on  $\mathbb{S}_{xz}^1$ .

**Theorem 6.** *Let  $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N)$ ,  $\mathbf{q}_i = (x_i, y_i, z_i, w_i)$ ,  $i = \overline{1, N}$ , be an ordinary central configuration. Then we have the relationships*

$$(15) \quad \sum_{i=1}^N m_i x_i z_i = \sum_{i=1}^N m_i x_i w_i = \sum_{i=1}^N m_i y_i z_i = \sum_{i=1}^N m_i y_i w_i = 0.$$

*Proof.* We first prove (15) in  $\mathbb{S}^3$ . Let  $\mathbf{v}_{i1} = (z_i, 0, -x_i, 0)$ . Take the inner product of both sides of the  $i$ -th equation of (11) with  $\mathbf{v}_{i1}$ . Since

$$\mathbf{q}_j \cdot \mathbf{v}_{i1} = z_i x_j - x_i z_j, \quad \mathbf{q}_i \cdot \mathbf{v}_{i1} = 0, \quad \nabla_{\mathbf{q}_i} I \cdot \mathbf{v}_{i1} = 2m_i x_i z_i (\mathbf{q}_i \cdot \mathbf{q}_i) = 2m_i x_i z_i,$$

we obtain  $\sum_{j=1, j \neq i}^N \frac{m_i m_j}{\sin^3 d_{ij}} (z_i x_j - x_i z_j) = 2\lambda m_i x_i z_i$ . Summing over all  $i$  leads to

$$2\lambda \sum_{i=1}^N m_i x_i z_i = \sum_{i=1}^N \sum_{j=1, j \neq i}^N \frac{m_i m_j}{\sin^3 d_{ij}} (z_i x_j - x_i z_j) = 0.$$

Since  $\mathbf{q}$  is an ordinary central configuration, we have  $\lambda \neq 0$ , so  $\sum_{i=1}^N m_i x_i z_i = 0$ . The other relationships in  $\mathbb{S}^3$  can be obtained by considering the inner product of (11) with

$$\mathbf{v}_{i2} = (w_i, 0, 0, -x_i), \quad \mathbf{v}_{i3} = (0, z_i, -y_i, 0), \quad \mathbf{v}_{i4} = (0, w_i, 0, -y_i).$$

The relationships in  $\mathbb{H}^3$  can be obtained by considering the inner product of (11) with

$$\mathbf{v}_{i1} = (z_i, 0, -x_i, 0), \quad \mathbf{v}_{i2} = (w_i, 0, 0, x_i), \quad \mathbf{v}_{i3} = (0, z_i, -y_i, 0), \quad \mathbf{v}_{i4} = (0, w_i, 0, y_i),$$

a remark that completes the proof.  $\square$

An obvious application of equations (15) is that of showing with little computational effort why certain configurations are not ordinary central configurations.

Recall that a (hyperbolic) 2-sphere means a sphere (hyperbolic sphere) isometric to the unit sphere (hyperbolic sphere) in  $\mathbb{R}^3$  ( $\mathbb{R}^{2,1}$ ). This is the non-empty intersection of  $\mathbb{M}^3$  with a 3-dimensional linear subspace,  $\{(x, y, z, w) \in \mathbb{R}^4 \mid ax + by + cz + dw = 0\}$ , [5]. Similarly, a geodesic is the non-empty intersection of a (hyperbolic) 2-sphere with a 2-dimensional linear subspace.

**Lemma 3.** *Assume that the intersection of  $\mathbb{S}^3$  ( $\mathbb{H}^3$ ) and the 3-dimensional linear space  $V = \{(x, y, z, w) \in \mathbb{R}^4 \mid ax + by + cz + dw = 0\}$  is non-empty. Let  $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N)$ ,  $N \geq 2$ , be a nonsingular configuration on the (hyperbolic) 2-sphere  $V \cap \mathbb{M}^3$ . If  $\nabla_{\mathbf{q}_i} I$  are not all zero, then  $\nabla_{\mathbf{q}_i} I \in V, i = \overline{1, N}$ , if and only if  $a = b = 0$  or  $c = d = 0$ .*

*Proof.* Recall that

$$\nabla_{\mathbf{q}_i} I = 2m_i (x_i(w_i^2 + \sigma z_i^2), y_i(w_i^2 + \sigma z_i^2), -\sigma z_i(x_i^2 + y_i^2), -\sigma w_i(x_i^2 + y_i^2)).$$

Then  $\nabla_{\mathbf{q}_i} I \in V, i = \overline{1, N}$  if and only if

$$\begin{aligned} 0 &= ax_i(w_i^2 + \sigma z_i^2) + by_i(w_i^2 + \sigma z_i^2) - c\sigma z_i(x_i^2 + y_i^2) - d\sigma w_i(x_i^2 + y_i^2) \\ &= (ax_i + by_i)(w_i^2 + \sigma z_i^2) + \sigma(ax_i + by_i)(x_i^2 + y_i^2) = (ax_i + by_i). \end{aligned}$$

Thus  $cz_i + dw_i = 0$ . Consider the matrix  $A := \begin{bmatrix} a & b & c & d \\ a & b & 0 & 0 \\ 0 & 0 & c & d \end{bmatrix}$ . Then  $(\mathbf{q}_1, \dots, \mathbf{q}_N) \in \ker A$ .

Since  $\mathbf{q}_i$  and  $\mathbf{q}_j$  are linearly independent, we obtain  $\text{rank}(\ker A) \geq 2$ , which implies that  $\text{rank } A = 1$ . Therefore, we have either  $a = b = 0$  or  $c = d = 0$ .  $\square$

Now we turn to central configurations in  $\mathbb{H}^3$ . The following result is from [65].

**Theorem 7.** *Each central configuration in  $\mathbb{H}^3$  is equivalent to some central configuration on  $\mathbb{H}_{xyw}^2$ .*

Thus there are no  $\mathbb{H}^3$  central configurations. However, there exist both special  $\mathbb{S}^3$  central configurations and ordinary  $\mathbb{S}^3$  central configurations, so the set of central configurations in  $\mathbb{S}^3$  is richer and more interesting than in  $\mathbb{H}^3$ , [65].

Define  $\mathbb{H}_{xw}^1 := \{(x, y, z, w) \in \mathbb{H}^3 \mid y = z = 0\}$ .

**Corollary 2.** *Each geodesic central configuration in  $\mathbb{H}^3$  is equivalent to some central configuration on  $\mathbb{H}_{xw}^1$ .*

*Proof.* By Theorem 7, every geodesic central configuration is equivalent to some geodesic central configuration on  $\mathbb{H}_{xyw}^2$ . A geodesic on  $\mathbb{H}_{xyw}^2$  is the non-empty intersection of a 2-dimensional linear space  $V$  and  $\mathbb{H}_{xyw}^2$ . Suppose that  $V$  is defined by  $\{ax + by + dw = 0\}$  and that a central configuration  $\mathbf{q}$  is on  $V \cap \mathbb{H}_{xyw}^2$ . Then  $\nabla_{\mathbf{q}_i} U$  lies in  $V$  for all  $i$ . It implies that each  $\nabla_{\mathbf{q}_i} I$  belongs to  $V$ . As in Lemma 3, we can show that it is sufficient and necessary to require  $d = 0$ .

Then any geodesic central configuration is equivalent to some central configuration on a geodesic  $\{(x, y, w) \in \mathbb{H}_{xyw}^2 \mid ax + by = 0\}$  and there is some element in  $SO(2) \times SO(1, 1)$  that moves the geodesic to  $\mathbb{H}_{xw}^1$ . This remark completes the proof.  $\square$

Now we discuss the  $\mathbb{S}^2$  ordinary central configurations and geodesic ordinary central configurations in  $\mathbb{S}^3$ . Define

$$\mathbb{S}_{xzw}^2 := \{(x, y, z, w) \in \mathbb{S}^3 \mid y = 0\}, \quad \mathbb{S}_{xz}^1 := \{(x, y, z, w) \in \mathbb{S}^3 \mid w = y = 0\}.$$

**Theorem 8.** *Any  $\mathbb{S}^2$  ordinary central configuration is equivalent to some ordinary central configuration on  $\mathbb{S}_{xyz}^2$  or on  $\mathbb{S}_{xzw}^2$ . Furthermore, there is a one-to-one correspondence between the central configurations on  $\mathbb{S}_{xyz}^2$  and the central configurations on  $\mathbb{S}_{xzw}^2$ .*

*Proof.* Lemma 3 implies that any  $\mathbb{S}^2$  ordinary central configuration is either on  $\mathbb{S}^3 \cap \{ax + by = 0\}$  or on  $\mathbb{S}^3 \cap \{cz + dw = 0\}$ . It is easy to see that there is some element in  $SO(2) \times SO(2)$  that would move these 2-spheres to either  $\mathbb{S}_{xyz}^2$  or  $\mathbb{S}_{xzw}^2$ . Thus any  $\mathbb{S}^2$  ordinary central configuration is equivalent to some ordinary central configuration on  $\mathbb{S}_{xyz}^2$  or on  $\mathbb{S}_{xzw}^2$ .

Let  $\mathbf{q}$  be a central configuration on  $\mathbb{S}_{xyz}^2$ , i.e.  $\nabla_{\mathbf{q}_i} U(\mathbf{q}) - \lambda \nabla_{\mathbf{q}_i} I(\mathbf{q}) = 0, i = \overline{1, N}$ . Consider the orthogonal transformation  $\varphi(x_i, y_i, z_i, w_i) = (z_i, w_i, x_i, y_i)$ . Then we have that  $\mathbf{q}' = (\mathbf{q}'_1, \dots, \mathbf{q}'_N) = \varphi \mathbf{q} = (\varphi \mathbf{q}_1, \dots, \varphi \mathbf{q}_N)$  is a configuration on  $\mathbb{S}_{xzw}^2$ . Note that for

$\mathbf{q}'_i = (x'_i, y'_i, z'_i, w'_i) = (z_i, w_i, x_i, y_i)$  we have

$$I(\mathbf{q}') = \sum_{i=1}^N m_i(x'^2_i + y'^2_i) = \sum_{i=1}^N m_i(1 - x^2_i - y^2_i) = \sum_{i=1}^N m_i - I(\mathbf{q}) \text{ and } U(\mathbf{q}') = U(\mathbf{q}).$$

Then  $\nabla U(\mathbf{q}') = \varphi \nabla U(\mathbf{q})$  and  $\nabla I(\mathbf{q}') = -\varphi \nabla I(\mathbf{q})$ . Here  $\nabla U(\mathbf{q}')$  and  $\nabla I(\mathbf{q}')$  mean the gradient of  $U$  and  $I$  at  $\mathbf{q}'$  respectively. Thus  $\mathbf{q}'$  satisfies the central configuration equations  $\nabla_{\mathbf{q}_i} U(\mathbf{q}') + \lambda \nabla_{\mathbf{q}_i} I(\mathbf{q}') = 0, i = \overline{1, N}$ . This remark completes the proof.  $\square$

The proof of the following statement is similar to that of Corollary 2.

**Corollary 3.** *Each ordinary geodesic central configuration in  $\mathbb{S}^3$  is equivalent to some central configuration on  $\mathbb{S}^1_{xz}$ .*

## 8. EXISTENCE OF ORDINARY CENTRAL CONFIGURATIONS

In this section we interpret central configurations as critical points of functions related to  $U$  and prove the existence of ordinary central configurations for any given masses. Then we discuss the Wintner-Smale conjecture for the curved  $N$ -body problem.

**8.1. Central configurations as critical points.** From the first central configuration equations,

$$\nabla_{\mathbf{q}_i} U(\mathbf{q}) - \lambda \nabla_{\mathbf{q}_i} I(\mathbf{q}) = 0,$$

we can derive the following property.

**Proposition 7.** *Central configurations in  $\mathbb{M}^3$  are critical points of the function*

$$U - \lambda I : (\mathbb{M}^3)^N \setminus \Delta \rightarrow \mathbb{R}.$$

*In  $\mathbb{H}^3$ ,  $\lambda$  is a negative constant; in  $\mathbb{S}^3$ ,  $\lambda$  could be any real number, and the case  $\lambda = 0$  corresponds to special central configurations.*

We can also see that an ordinary central configuration is a critical point of the restriction of  $U$  subject to the constraint  $I = \text{constant}$ . From this point of view,  $-\lambda$  is a Lagrange multiplier. More precisely, let us denote

$$S_c := \{\mathbf{q} \in (\mathbb{M}^3)^N \setminus \Delta \mid I(\mathbf{q}) = c\}.$$

**Proposition 8.** *Ordinary central configurations in  $\mathbb{M}^3$  are critical points of  $U|_{S_c}$ , i.e. critical points of*

$$U : S_c \rightarrow \mathbb{R}.$$

Let  $\mathbf{q}$  be an ordinary central configuration and  $\phi$  an element of  $SO(2) \times SO(2)$  or  $SO(2) \times SO(1, 1)$ . Then  $\phi\mathbf{q}$  is also a central configuration. Thus it follows that the critical points of  $U|_{S_c}$  are not isolated, but rather occur as manifolds of critical points. Similarly, these special central configurations are not isolated either. This fact suggests that we can further look at central configurations as critical points of  $U$  subject to a factorization. Note that both  $U$  and  $(\mathbb{M}^3)^N$  are invariant under the isometry group and the set  $S_c$  is invariant under the subgroup  $SO(2) \times SO(2)$  or  $SO(2) \times SO(1, 1)$ . We thus have the following property.

**Proposition 9.** *There is a one-to-one correspondence between the classes of central configurations and the critical points of the force function  $\hat{U}$  induced by  $U$  on the quotient set*

- (1)  $((\mathbb{S}^3)^N \setminus \Delta)/SO(4)$  for special central configurations in  $\mathbb{S}^3$ ,
- (2)  $S_c/(SO(2) \times SO(2))$  for ordinary central configurations in  $\mathbb{S}^3$ , and
- (3)  $S_c/(SO(2) \times SO(1,1))$  for central configurations in  $\mathbb{H}^3$ .

Let  $\mathbf{q}$  in the quotient set be a critical point of  $\hat{U}$ . In the case of special central configuration on  $\mathbb{S}^3$ , the Hessian of  $\hat{U}$  at  $\mathbf{q}$ ,  $D^2\hat{U}(\mathbf{q})$ , is an invariant symmetric bilinear form on  $T_{\mathbf{q}}((\mathbb{S}^3)^N \setminus \Delta)/SO(4)$ . For ordinary central configurations in  $\mathbb{S}^3$  and  $\mathbb{H}^3$ ,  $D^2\hat{U}(\mathbf{q})$  is an invariant symmetric bilinear form on  $T_{\mathbf{q}}\hat{S}_c$ , where  $\hat{S}_c$  is the quotient set in either (2) or (3) of Proposition 9. The index of  $D^2\hat{U}(\mathbf{q})$  is the maximal dimension of a subspace of the tangent space on which this form is negative definite. A critical point  $\mathbf{q}$  of  $\hat{U}$  is degenerate whenever the Hessian has a non-trivial nullspace.

We can now formally introduce the following two concepts.

**Definition 10.** *A central configuration is degenerate (nondegenerate) provided that the corresponding critical point  $\mathbf{q}$  of  $\hat{U}$  is degenerate (nondegenerate).*

**8.2. The structure of  $I^{-1}(c)$ .** Unlike in the Newtonian  $N$ -body problem, where  $I = c > 0$  is always a  $(3N - 1)$ -dimensional ellipsoid, the set  $I^{-1}(c)$  may not be a smooth manifold. To understand the structure of this set, we need the classical Regular Value Theorem, which we further recall for completeness. Let  $\mathcal{M}, \mathcal{N}$  be differentiable manifolds and  $f: \mathcal{M} \rightarrow \mathcal{N}$  a differentiable function. Then  $f$  is called a submersion at  $x \in \mathcal{M}$  if its differential,  $Df_x: T_x\mathcal{M} \rightarrow T_{f(x)}\mathcal{N}$ , is surjective. In this case,  $x$  is called a regular point and  $f(x)$  a regular value. Otherwise,  $x$  is called a critical point and  $f(x)$  a critical value. We can now state the following well known result, [29].

**Regular Value Theorem.** *Let  $f: \mathcal{M} \rightarrow \mathcal{N}$  be a  $C^r$ -map,  $r \geq 1$ . If  $y \in f(\mathcal{M})$  is a regular value, then  $f^{-1}(y)$  is a  $C^r$ -submanifold of  $\mathcal{M}$ .*

If we further regard the moment of inertia as the smooth map

$$I: (\mathbb{M}^3)^N \rightarrow [0, \infty),$$

we have the following properties.

**Lemma 4.** *Assume that the masses  $m_1, \dots, m_N$  are in  $\mathbb{S}^3$ , and consider  $c \geq 0$ , not of the form  $c = \sum_{i=1}^N m_i \mu_i$ , where  $\mu_1, \dots, \mu_N \in \{0, 1\}$ . Then the set  $I^{-1}(c)$  is a smooth manifold.*

*Proof.* Suppose that  $c \geq 0$  is a critical value for  $I$ . This is equivalent to saying that there exists a  $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N)$  such that  $\mathbf{q} \in I^{-1}(c)$  and

$$\nabla_{\mathbf{q}_1} I(\mathbf{q}) = \dots = \nabla_{\mathbf{q}_N} I(\mathbf{q}) = \mathbf{0},$$

which implies that  $\mathbf{q}_i \in \mathbb{S}_{xy}^1 \cup \mathbb{S}_{zw}^1$  by Lemma 1 in Section 6. Then  $x_i^2 + y_i^2 = 0$  or  $1$  and  $c = \sum_{i=1}^N m_i \mu_i$ , where  $\mu_1, \dots, \mu_N \in \{0, 1\}$ , a remark that completes the proof.  $\square$

**Lemma 5.** *Assume that the masses  $m_1, \dots, m_N$  are in  $\mathbb{H}^3$ , and consider  $c \geq 0$ . Then  $I^{-1}(c)$  is always a smooth manifold.*

*Proof.* Suppose that  $c \geq 0$  is a critical value for  $I$ . This is equivalent to saying that there exists a  $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N)$  such that  $\mathbf{q} \in I^{-1}(c)$  and

$$\nabla_{\mathbf{q}_1} I(\mathbf{q}) = \dots = \nabla_{\mathbf{q}_N} I(\mathbf{q}) = \mathbf{0},$$

which implies that  $\mathbf{q}_i \in \mathbb{H}_{zw}^1$  by Lemma 1 in Section 6. Then  $x_i^2 + y_i^2 = 0$  and  $c = 0$ . Moreover,  $I^{-1}(0) = (\mathbb{H}_{zw}^1)^N$ , which is homeomorphic with  $\mathbb{R}^N$ , a remark that completes the proof.  $\square$

**8.3. The existence result.** The characterization of central configurations as critical points provides an easy way to see that ordinary central configurations exist, i.e., that the complicated criteria developed earlier always have solutions for  $\lambda \neq 0$ .

**Theorem 9.** *Assume that the masses  $m_1, \dots, m_N$  are in  $\mathbb{S}^3$  or  $\mathbb{H}^3$ . Then for any positive values these masses take, there is at least one ordinary central configuration in  $\mathbb{S}^3$  and at least one ordinary central configuration in  $\mathbb{H}^3$ .*

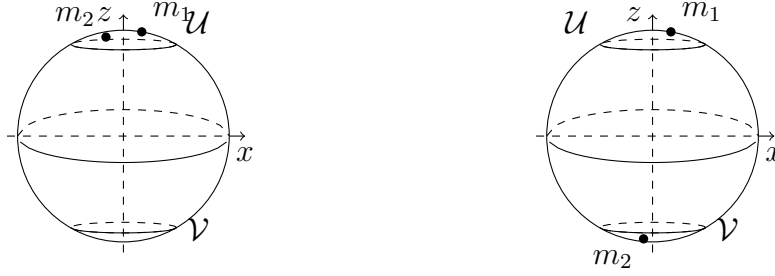
*Proof.* Let us first prove the result in  $\mathbb{H}^3$ . In general, the manifold  $I^{-1}(c)$  is not compact in this case. However, things change if we confine all masses to the hyperbolic circle  $\mathbb{H}_{xw}^1$ , since the set  $I = c > 0$  is homeomorphic to an ellipsoid. Then  $U$  defines a smooth function on the open subset  $S_c$ , and the boundary of  $S_c$  is composed of points in the singularity set. Since the ellipsoid is compact and  $U \rightarrow +\infty$  as  $\mathbf{q}$  approaches the boundary of  $S_c$ , it follows that  $U$  attains a minimum at some non-singular configuration  $\mathbf{q}$ . This will be a critical point of  $U$  on  $S_c$  and hence an ordinary central configuration.

In  $\mathbb{S}^3$ , we need to construct a connected component of  $S_c$  on whose boundary  $U$  approaches  $+\infty$ . Recall that there are two kinds of singularities, collision singularities in  $\Delta^+$  and antipodal singularities in  $\Delta^-$ .  $U$  approaches  $+\infty$  as the configuration approaches  $\Delta^+$ , but approaches  $-\infty$  as the configuration approaches  $\Delta^-$ . Thus we need to construct a connected component of  $S_c$  whose boundary lies only in  $\Delta^+$ .

We confine the particles to  $\mathbb{S}_{xyz}^2$  and order the masses as  $0 < m_1 \leq \dots \leq m_N$ . Let  $0 < c < m_1$ . Then  $S_c$  is a smooth manifold. Let us further choose a configuration  $\mathbf{q} \in S_c$  with all bodies lying near the North Pole  $(0, 0, 1)$ , i.e.,  $z_i > 0$ ,  $i = \overline{1, N}$ . Denote by  $\mathcal{J}$  the connected component of the manifold  $S_c$  that contains the configuration  $\mathbf{q}$ . We claim that the boundary of  $\mathcal{J}$  contains only points from  $\Delta^+$ .

To prove this claim, we define the sets  $\mathcal{U} = \{(x, y, z) \in \mathbb{S}_{xyz}^2 \mid x^2 + y^2 < c/m_1, z > 0\}$  and  $\mathcal{V} = \{(x, y, z) \in \mathbb{S}_{xyz}^2 \mid x^2 + y^2 < c/m_1, z \leq 0\}$ . Since  $I(\mathbf{q}) = \sum_{i=1}^N m_i(x_i^2 + y_i^2) \geq m_1(x_1^2 + y_1^2)$ , it follows that  $x_i^2 + y_i^2 \leq c/m_1$ ,  $i = \overline{1, N}$ , which means that for any configuration  $\mathbf{q} \in \mathcal{J}$  each body lies either in  $\mathcal{U}$  or in  $\mathcal{V}$ .

Let us now suppose that  $\partial\mathcal{J} \cap \Delta^- \neq \emptyset$ . Then there must exist a configuration  $\bar{\mathbf{q}} = (\bar{\mathbf{q}}_1, \dots, \bar{\mathbf{q}}_N) \in \mathcal{J}$  such that one body is in  $\mathcal{U}$  and the another in  $\mathcal{V}$ , say,  $\bar{\mathbf{q}}_1 \in \mathcal{U}$  and  $\bar{\mathbf{q}}_2 \in \mathcal{V}$ . Since  $\mathcal{J}$  is connected, it is also path connected. Then there is a path in  $\mathcal{J}$  connecting  $\mathbf{q}$  and  $\bar{\mathbf{q}}$ , so there is a path that connects  $\mathbf{q}_2 \in \mathcal{U}$  and  $\bar{\mathbf{q}}_2 \in \mathcal{V}$ . But this is impossible since  $\mathcal{U} \cap \mathcal{V} = \emptyset$ . Thus  $\mathcal{J}$  is a connected component of the manifold  $S_c$  whose

FIGURE 3.  $\bar{\mathbf{q}}_1, \bar{\mathbf{q}}_2$  and  $\mathbf{q}_1, \mathbf{q}_2$  on  $\mathbb{S}^2_{xyz}$ 

boundary consists only of points from  $\Delta^+$ . Therefore  $U \rightarrow +\infty$  as  $\mathbf{q}$  approaches  $\partial\mathcal{J}$ . It follows that  $U$  attains a minimum at some configuration  $\mathbf{q}$ , which is then a critical point of  $U$  on  $S_c$ , hence an ordinary central configuration.  $\square$

**8.4. The Wintner-Smale conjecture in spaces of constant curvature.** Recall that three equal masses on  $\mathbb{S}^2_{xyz}$  possess a continuum of central configurations, see Example 1. Notice that these central configurations are on different  $S_c$ . In general, there is no obvious way to relate central configurations in  $S_{c_1}$  and central configurations in  $S_{c_2}$ . Thus we consider them belonging to different classes of central configurations. Notice that the existence proof of ordinary central configurations works for other constant values of  $I$ . Hence there always exist central configurations on  $S_c$  for  $c$  belonging to some open intervals. So we have the following obvious consequence.

**Corollary 4.** *Assume that the masses  $m_1, \dots, m_N$  are in  $\mathbb{S}^3$  or  $\mathbb{H}^2$ . Then for any positive values of these masses, the set of ordinary central configurations has the power of the continuum.*

Recall that the Wintner-Smale problem (Smale's 6th problem) asks whether for some given masses,  $m_1, \dots, m_N > 0$ , the number of classes of planar central configurations for the Newtonian  $N$ -body problem is finite or not. If we extend the problem to the curved  $N$ -body problem in the following way: whether for some given masses,  $m_1, \dots, m_N > 0$ , the number of classes of central configurations for the curved  $N$ -body problem is finite or not, then this extension has an obvious and uninteresting answer. So we modify the problem as follows for ordinary central configurations.

**Question 1.** *In the curved  $N$ -body problem, for given masses  $m_1, \dots, m_N$  and all possible values of  $c$ , is the number of ordinary central configurations on  $S_c$  finite?*

From now on, we say that several masses possess a continuum of ordinary central configurations if the continuum of central configurations is on a certain set  $S_c$ . We will see in Section 10 that even for two equal masses,  $m_1 = m_2 =: m$ , there is a continuum of central configurations on  $S_m$ .

For special central configurations in  $\mathbb{S}^3$ , we also pose a similar question:

**Question 2.** *In the curved  $N$ -body problem in  $\mathbb{S}^3$ , for given masses  $m_1, \dots, m_N$ , is the number of special central configurations finite?*

## 9. EXAMPLES

In this section we produce examples of central configurations of the curved  $N$ -body problem in  $\mathbb{S}^3$  and  $\mathbb{H}^3$  and discuss the associated relative equilibria. Some examples will concern special and ordinary central configurations for  $N = 3$  that lie on the great sphere  $\mathbb{S}_{xyz}^2$  and the great hyperbolic sphere  $\mathbb{H}_{xyw}^2$ . In the Newtonian  $N$ -body problem there are only two classes of central configurations for  $N = 3$ , the Lagrangian (equilateral triangles) and the Eulerian (collinear configurations). For nonzero constant curvature, however, the set of central configurations (and therefore that of relative equilibria) is richer, as we will further show. We also include in this section examples of central configurations for  $N > 3$ . Unless otherwise stated, the relative equilibria associated to all these central configurations were already found in [9] and [10].

The stability question for some relative equilibria of the curved  $N$ -body problem was studied by several authors [15, 22, 42]. In particular, the paper [22] is about the relative equilibria associated to the special central configurations mentioned in Section 9.1.

**9.1. Acute triangle special central configurations on  $\mathbb{S}_{xy}^1$ .** Let us assume that three masses,  $m_1 = \frac{\sin^2 \alpha}{\sin^2 \beta}$ ,  $m_2 = \frac{\sin^2 \alpha}{\sin^2(\alpha+\beta)}$ , and  $m_3 = 1$  form an acute scalene triangle on  $\mathbb{S}_{xy}^1$ . In the complex coordinates of the  $xy$  plane, i.e.,  $q_j = x_j + iy_j \in \mathbb{C}$ , the configuration is given by

$$\mathbf{q}_1 = 1, \quad \mathbf{q}_2 = e^{i\alpha}, \quad \mathbf{q}_3 = e^{i(\alpha+\beta)},$$

for any fixed  $0 < \alpha < \pi, 0 < \beta < \pi, \pi < \alpha + \beta < 2\pi$ , see Figure 4. Then it is easy to verify that  $\nabla_{\mathbf{q}_i} U = 0$  for each  $i = 1, 2, 3$ , [22].

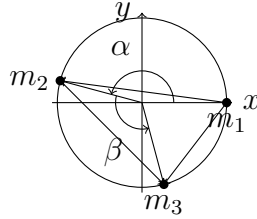


FIGURE 4. An acute triangle special central configuration

Since these special central configurations are confined to  $\mathbb{S}_{xy}^1 \cup \mathbb{S}_{zw}^1$ , they give rise to 2-parameter families of associated relative equilibria:  $A_{\alpha,\beta}(t)\mathbf{q}$ ,  $\alpha, \beta \in \mathbb{R}$ . The rotation in  $zw$ -plane does not affect the configuration, which will stay on  $\mathbb{S}_{xy}^1$ , thus forming a 1-parameter family of associated relative equilibria,  $A_{\alpha,0}(t)\mathbf{q}$ ,  $\alpha \in \mathbb{R}$ .

**9.2. Regular tetrahedron special central configurations on  $\mathbb{S}_{xyz}^2$ .** Let us assume that four masses,  $m_1 = m_2 = m_3 = m_4 = m$ , form a regular tetrahedron on  $\mathbb{S}_{xyz}^2$ , see Figure 5. By symmetry, it is easy to see that this is a special central configuration. Since this special central configuration is not confined to  $\mathbb{S}_{xy}^1 \cup \mathbb{S}_{zw}^1$ , it gives rise to a 1-parameter family of associated relative equilibria,  $A_{\alpha,\pm\alpha}(t)\mathbf{q}$ ,  $\alpha \in \mathbb{R}$ . They are periodic orbits, but the motion is not confined to  $\mathbb{S}_{xyz}^2$ .

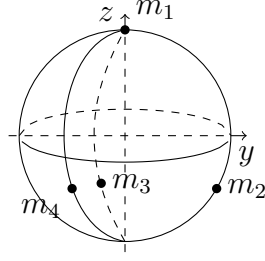


FIGURE 5. Regular tetrahedron special central configuration

**9.3. Regular pentatope special central configurations in  $\mathbb{S}^3$ .** Let us assume that five masses,  $m_1 = m_2 = m_3 = m_4 = m_5 = m$ , form a regular pentatope in  $\mathbb{S}^3$ . By symmetry, it is easy to see that  $\sum_{j=1, j \neq i}^4 \mathbf{F}_{ij} = \nabla_{\mathbf{q}_i} U = 0$  for  $i = 1, 2, 3, 4, 5$ , and thus this is a special central configuration. Since this special central configuration is not confined to  $\mathbb{S}_{xy}^1 \cup \mathbb{S}_{zw}^1$ , it gives rise to a 1-parameter family of associated relative equilibria,  $A_{\alpha, \pm\alpha}(t)\mathbf{q}$ ,  $\alpha \in \mathbb{R}$ , which are periodic orbits.

**9.4. Pair of equilateral triangle special central configuration in  $\mathbb{S}^3$ .** Let us assume that six masses,  $m_1 = m_2 = m_3 = m_4 = m_5 = m_6 = m$ , in  $\mathbb{S}^3$  form two equilateral triangles on complementary great circles:  $\mathbb{S}_{xy}^1$  and  $\mathbb{S}_{zw}^1$ .

To see that  $\sum_{j=1, j \neq i}^6 \mathbf{F}_{ij} = \nabla_{\mathbf{q}_i} U = 0$  for  $i = 1, 2, 3, 4, 5, 6$ , it suffices to check that for  $m_1$ . That is,

$$\mathbf{F}_1 = \mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_{14} + \mathbf{F}_{15} + \mathbf{F}_{16} = 0.$$

By symmetry we obtain that  $\mathbf{F}_{12} + \mathbf{F}_{13} = 0$ . For  $i = 4, 5, 6$ , since  $\mathbf{q}_1 \cdot \mathbf{q}_i = 0$ , we have  $d_{1i} = \pi/2$  and  $\mathbf{F}_{1i} = \frac{m^2(\mathbf{q}_i - \cos d_{1i}\mathbf{q}_1)}{\sin^3 d_{1i}} = m^2\mathbf{q}_i$ . Then

$$\mathbf{F}_{14} + \mathbf{F}_{15} + \mathbf{F}_{16} = m^2(\mathbf{q}_4 + \mathbf{q}_5 + \mathbf{q}_6) = 0,$$

hence this is a special central configuration. Since this special central configuration is on  $\mathbb{S}_{xy}^1 \cup \mathbb{S}_{zw}^1$ , it gives rise to a 2-parameter family of associated relative equilibria,  $A_{\alpha, \beta}(t)\mathbf{q}$ ,  $\alpha, \beta \in \mathbb{R}$ . They are periodic orbits if  $\alpha/\beta$  is rational, but quasi-periodic orbits if  $\alpha/\beta$  is irrational.

**9.5. Lagrangian central configurations in  $\mathbb{S}_{xyz}^2$ .** Let us assume that three equal masses,  $m_1 = m_2 = m_3 = m$ , form an equilateral configuration on  $\mathbb{S}_{xyz}^2$ , parallel with the  $xy$ -plane, so the coordinates are given by

$$\mathbf{q}_1 = (r, 0, z, 0), \quad \mathbf{q}_2 = (-r/2, r\sqrt{3}/2, z, 0), \quad \mathbf{q}_3 = (-r/2, -r\sqrt{3}/2, z, 0),$$

where  $r^2 + z^2 = 1$ ,  $r \in (0, 1)$ , see Figure 2. By symmetry, we notice that  $\mathbf{F}_i$  is pointing towards the North or South poles and that  $|\mathbf{F}_i| = |\mathbf{F}_j|$ . Comparing this with the vector field  $\nabla(x^2 + y^2)$  on  $\mathbb{S}_{xyz}^2$  (see Figure 1), we see that the central configuration equations  $\nabla_{\mathbf{q}_i} U = \lambda \nabla_{\mathbf{q}_i} I$  are satisfied for  $i = 1, 2, 3$ .

To find the value of  $\lambda$ , we use the expression obtained in Proposition 6. For any  $i \neq j$ , we obtain that  $\sin^3 d_{ij} = 3\sqrt{3}r^3 \left(1 - \frac{3r^2}{4}\right)^{3/2}$ ,  $x_i x_j + y_i y_j = -r^2/2$ . Then

$$\begin{aligned} \lambda &= \sum_{1 \leq i \leq N} \sum_{j=1, j \neq i}^N \frac{m_i m_j (2x_i x_j + 2y_i y_j - (r_i^2 + r_j^2) \cos d_{ij})}{\sin^3 d_{ij}} / \left( 2 \sum_{1 \leq i \leq N} m_i r_i^2 \rho_i^2 \right) \\ &= \frac{3m}{2 \sin^3 d} = -\frac{m}{2\sqrt{3}r^3 \left(1 - \frac{3r^2}{4}\right)^{3/2}} < 0. \end{aligned}$$

These central configurations give rise to a 1-parameter families of associated relative equilibria,  $A_{\alpha, \beta}(t)\mathbf{q}$  with  $\lambda = \frac{\beta^2 - \alpha^2}{2}$ .

**9.6. Geodesic central configurations on  $\mathbb{S}_{xz}^1$ .** Let the coordinates of the three bodies of masses  $m_1 = m_2 = m_3 = m$  be given by

$$\mathbf{q}_1 = (0, 0, 1, 0), \quad \mathbf{q}_2 = (r, 0, z, 0), \quad \mathbf{q}_3 = (-r, 0, z, 0),$$

with  $r > 0$ ,  $z \in (-1, 0) \cup (0, 1)$  and  $r^2 + z^2 = 1$ . Given the many zeroes that occur in the above coordinates, it is not difficult to check that the central configuration equations are satisfied for  $i = 1, 2, 3$ .

To find the value of  $\lambda$ , we use the expressions obtained in Proposition 6, so we have

$$\begin{aligned} d_{12} = d_{13}, \quad r_1^2 = 0, \quad r_2^2 = r_3^2 = r^2, \quad \sin^3 d_{12} = r^3, \quad \sin^3 d_{23} = 8r^3|z|^3, \\ x_1 x_2 + y_1 y_2 = x_1 x_3 + y_1 y_3 = 0, \quad x_2 x_3 + y_2 y_3 = -r^2. \end{aligned}$$

Then

$$\begin{aligned} \lambda &= \frac{1}{4mr^2z^2} \left( \frac{m^2(-r^2 \cos d_{12})}{\sin^3 d_{12}} + \frac{m^2(-r^2 \cos d_{13})}{\sin^3 d_{13}} + \frac{m^2(-2r^2 - 2r^2 \cos d_{23})}{\sin^3 d_{23}} \right) \\ &= \frac{-m}{2z^2} \left( \frac{\cos d_{12}}{\sin^3 d_{12}} + \frac{1 + \cos d_{23}}{\sin^3 d_{23}} \right) = \frac{-m}{2r^3} \left( \frac{1}{z} + \frac{1}{4|z|^3} \right). \end{aligned}$$

It is easy to see that  $\lambda < 0$  for  $z \in (-1/2, 0) \cup (0, 1)$ ,  $\lambda > 0$  for  $z \in (-1, -1/2)$ , and  $\lambda = 0$  for  $z = -1/2$ , which shows the connection with the special central configuration discussed in the first example.



FIGURE 6. Geodesic central configurations on  $\mathbb{S}_{xz}^1$

All ordinary geodesic central configurations of three masses on  $\mathbb{S}_{xz}^1$  were found in [64]. Some interesting examples were given there, such as the one in which three distinct

masses form an equilateral triangle. In the complex coordinates of the  $xz$  plane, i.e.,  $q_j = x_j + iz_j \in C$ , for instance, one such central configuration is given by

$$\mathbf{q}_1 = e^{i\frac{3\pi}{4}}, \quad \mathbf{q}_2 = e^{i\frac{17\pi}{12}}, \quad \mathbf{q}_3 = e^{i\frac{\pi}{12}},$$

and the masses are  $m_1 = 2, m_2 = 1, m_3 = 3$ .

We could also verify that the central configuration equations are satisfied and  $\lambda = -\frac{8}{3}$ . We can actually find many such examples. For any three unequal masses  $m_1, m_2, m_3$ , we can find  $\lambda \neq 0$  and  $\theta$  such that the equations

$$\sin 2\theta = -\frac{4}{3\lambda}(m_3 - m_2), \quad \cos 2\theta = \frac{4\sqrt{3}}{9\lambda}(2m_1 - m_3 - m_2)$$

hold. Then the configuration

$$\mathbf{q}_1 = e^{i\theta}, \quad \mathbf{q}_2 = e^{i(\theta + \frac{2\pi}{3})}, \quad \mathbf{q}_3 = e^{i(\theta + \frac{4\pi}{3})}$$

is a central configuration.

**9.7. Isosceles central configuration in  $\mathbb{S}_{xyz}^2$ .** Let us assume that three masses,  $m_1 = -2\cos\varphi$ , with  $\varphi \in (\pi/2, \pi)$ ,  $m_2 = m_3 = 1$ , form an isosceles triangle on the sphere  $\mathbb{S}_{xyz}^2$ , parallel with the  $xy$ -plane. The configuration is given by

$$\mathbf{q}_1 = (\sin\theta, 0, \cos\theta, 0), \quad \mathbf{q}_2 = (\sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta, 0),$$

$$\mathbf{q}_3 = (\sin\theta \cos\varphi, -\sin\theta \sin\varphi, \cos\theta, 0),$$

with  $\theta$  chosen such that  $\cos^2\theta = 1 + \frac{2}{(\cos\varphi - 1)(2\cos\varphi + 3)}$ .

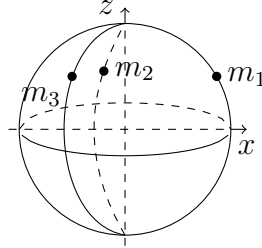


FIGURE 7. Isosceles central configuration on  $\mathbb{S}_{xyz}^2$

By straightforward computations, we can see that the central configuration equations are satisfied, and

$$\lambda = -\frac{2 - 2\cos\varphi}{2\sin^3 d_{12}} = -\frac{2 - 2\cos\varphi}{2\sin^3\theta(1 - \cos\varphi)^{3/2}(1 + \sin^2\theta\cos\varphi + \cos^2\theta)^{3/2}}.$$

The existence of the associated relative equilibria was proved in [13]. Some interesting details concerning this type of central configuration will be given in a future paper.

**9.8. Lagrangian central configurations in  $\mathbb{H}_{xyw}^2$ .** Let us assume that three equal masses,  $m_1 = m_2 = m_3 = m$ , form an equilateral configuration in  $\mathbb{H}_{xyw}^2$ , parallel with the  $xy$ -plane, and the coordinates are given by

$$\mathbf{q}_1 = (r, 0, 0, w), \quad \mathbf{q}_2 = (-r/2, r\sqrt{3}/2, 0, w), \quad \mathbf{q}_3 = (-r/2, -r\sqrt{3}/2, 0, w),$$

where  $r^2 - w^2 = -1$ ,  $w \in (1, +\infty)$ . By symmetry, we notice that  $\mathbf{F}_i$  is pointing towards  $(0, 0, 0, 1)$  and  $|\mathbf{F}_i| = |\mathbf{F}_j|$ . Comparing with the vector field  $\nabla(x^2 + y^2)$  on  $\mathbb{H}_{xyw}^2$  (see Figure 1), we see that the central configuration equations  $\nabla_{\mathbf{q}_i} U = \lambda \nabla_{\mathbf{q}_i} I$  are satisfied for  $i = 1, 2, 3$ .

To find the value of  $\lambda$ , we use the expression obtained in Proposition 6. For any  $i \neq j$ , we obtain that  $\sinh^3 d_{ij} = 3\sqrt{3}r^3 \left(1 + \frac{3r^2}{4}\right)^{3/2}$ ,  $x_i x_j + y_i y_j = -r^2/2$ . Then

$$\begin{aligned} \lambda &= \sum_{1 \leq i \leq N} \sum_{j=1, j \neq i}^N \frac{m_i m_j (2x_i x_j + 2y_i y_j - (r_i^2 + r_j^2) \cosh d_{ij})}{\sinh^3 d_{ij}} / \left( 2 \sum_{1 \leq i \leq N} m_i r_i^2 \rho_i^2 \right) \\ &= \frac{3m}{2 \sinh^3 d} = -\frac{m}{2\sqrt{3}r^3 \left(1 + \frac{3r^2}{4}\right)^{3/2}}. \end{aligned}$$

Each of these central configurations gives rise to 1-parameter family of associated relative equilibria:  $B_{\alpha, \beta}(t)\mathbf{q}$  with  $\lambda = -\frac{\beta^2 + \alpha^2}{2}$ . These orbits are a new discovery that has been missed in previous studies, a fact that shows the power of the central-configuration approach for finding relative equilibria.

Although we build the whole theory of negative-curvature spaces on the hyperbolic sphere model  $\mathbb{H}^3$ , it is convenient to visualize the associated relative equilibria in the Poincaré ball model. Recall that the Poincaré ball model is given by

$$\left( \bar{x}^2 + \bar{y}^2 + \bar{z}^2 < 1, \quad ds^2 = \frac{4(d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2)}{1 - (\bar{x}^2 + \bar{y}^2 + \bar{z}^2)} \right),$$

which can be seen as the perspective projection of the upper 3-dimensional hyperboloid viewed from  $(0, 0, 0, -1)$ . The projection mapping is

$$\bar{x} = \frac{x}{1+w}, \quad \bar{y} = \frac{y}{1+w}, \quad \bar{z} = \frac{z}{1+w}.$$

This projection mapping shows that the isometries of the  $SO(2)$  rotations in the  $xy$ -plane become the rotations in the  $\bar{x}\bar{y}$ -plane, and that the isometries of the  $SO(1, 1)$  rotations in the  $zw$ -plane become action moving points from  $(0, 0, -1)$  to  $(0, 0, 1)$  or in the opposite direction. Thus the relative equilibria  $B_{\alpha, \beta}(t)\mathbf{q}$  in the Poincaré ball model can be viewed as bodies that rotate around the  $\bar{z}$ -axis and move up or down along the projection of the hyperbolic cylinder

$$\mathbf{C}_{r\rho} := \{(x, y, z, w) \in \mathbb{H}^3 \mid x^2 + y^2 = r^2\},$$

a spindle-shaped surface (within the framework of this model) for which the hyperbolic distance from the  $\bar{z}$ -axis is constant (see Figure 8 on the right), hence the name “hyperbolic cylinder” we gave to it in previous studies on relative equilibria, [9], [10].



FIGURE 8. Lagrangian central configurations on  $\mathbb{H}^2_{xyw}$  and the associated relative equilibria in the Poincaré ball.



FIGURE 9. Geodesic central configurations on  $\mathbb{H}^1_{xw}$  and the associated relative equilibria in the Poincaré ball

**9.9. Geodesic central configurations in  $\mathbb{H}^1_{xw}$ .** Let three bodies of masses  $m_1 = m_2 = m_3 = m$  have the coordinates

$$\mathbf{q}_1 = (0, 0, 0, 1), \quad \mathbf{q}_2 = (r, 0, 0, w), \quad \mathbf{q}_3 = (-r, 0, 0, w),$$

with  $r > 0$  and  $r^2 - w^2 = -1$ . It is easy to check that system (11) is satisfied.

To find the value of  $\lambda$ , we use the expression obtained in Proposition 6. We have

$$d_{12} = d_{23}, \quad r_1^2 = 0, \quad r_2^2 = r_3^2 = r^2, \quad \sinh^3 d_{12} = r^3, \quad \sinh^3 d_{23} = 8r^3 w^3,$$

$$x_1 x_2 + y_1 y_2 = x_1 x_3 + y_1 y_3 = 0, \quad x_2 x_3 + y_2 y_3 = -r^2,$$

which yield

$$\begin{aligned} \lambda &= \frac{1}{4mr^2 z^2} \left[ \frac{m^2(-r^2 \cosh d_{12})}{\sinh^3 d_{12}} + \frac{m^2(-r^2 \cosh d_{13})}{\sinh^3 d_{13}} + \frac{m^2(-2r^2 - 2r^2 \cosh d_{23})}{\sinh^3 d_{23}} \right] \\ &= -\frac{m}{2w^2} \left( \frac{\cosh d_{12}}{\sinh^3 d_{12}} + \frac{1 + \cosh d_{23}}{\sinh^3 d_{23}} \right) = -\frac{m}{2r^3} \left( \frac{1}{w} + \frac{1}{4w^3} \right). \end{aligned}$$

As in the last example, we can also represent the associated relative equilibria in the Poincaré ball model, see Figure 9, where the bodies rotate around the  $\bar{z}$ -axis and move up or down, one along the  $\bar{z}$ -axis, and the other two along the projection of the hyperbolic cylinder  $\mathbf{C}_{r\rho}$ , thus maintaining constant mutual distances.

## 10. MOULTON'S THEOREM

In 1910, Forest Ray Moulton sought to extend Euler's results about the collinear central configurations in the Newtonian  $N$ -body problem to any number  $N$  of point masses. He showed that for a given ordering of the bodies on a straight line, there is exactly one class of central configurations, [47]. In this section we are asking whether Moulton's theorem has a natural correspondent in spaces of nonzero constant curvature. As we will further prove, this extension is true on geodesics of  $\mathbb{H}^3$ , but not on geodesics of  $\mathbb{S}^3$ , where even the case  $N = 2$  leads to a complicated count.

Before we get to the curved  $N$ -body problem, let us make some comments about the Euclidean case. The class of central configurations in the above statement of the theorem is meant as the set of central configurations factorized to homotheties. So another equivalent way of stating Moulton's result is to say that, for every ordering of any given masses with  $I(\mathbf{q}) = \text{constant}$ , there is exactly one central configuration. This new formulation is the one we adopt here, since the value of  $I(\mathbf{q})$  could never be the same for central configurations with different sizes, as the definition of equivalent central configurations implies.

**10.1. Geodesic central configurations in  $\mathbb{H}^3$ .** Corollary 2 states that every geodesic central configuration in  $\mathbb{H}^3$  is equivalent to some geodesic central configuration on  $\mathbb{H}_{xw}^1$ . Thus we assume that the point masses  $m_1, \dots, m_N$  lie on  $\mathbb{H}_{xw}^1$ . Expressing the position of each mass  $m_i$  in terms of the oriented hyperbolic distance  $\theta_i \in \mathbb{R}$ ,  $i = \overline{1, N}$ , measured from the vertex,  $(0, 0, 0, 1)$ , we can represent the position vectors and the distances between bodies as

$$\mathbf{q}_i = (\sinh \theta_i, 0, 0, \cosh \theta_i), \quad d_{ij} = |\theta_i - \theta_j|, \quad i, j = \overline{1, N},$$

respectively. Then the force function and the moment of inertia can be written as

$$U(\mathbf{q}) = \sum_{1 \leq i < j \leq N} m_i m_j \coth d_{ij} \quad \text{and} \quad I(\mathbf{q}) = \sum_{i=1}^N m_i \sinh^2 \theta_i.$$

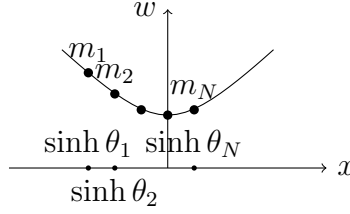
By the critical point characterization of central configurations introduced in Section 8, we only need to find the number of critical points of  $\hat{U}$  on  $\hat{S}_c$  for a constant  $c > 0$ . In this case, we have

$$\hat{S}_c = S_c / SO(2) \times SO(1, 1) = S_c = \{\mathbf{q} \in (\mathbb{H}_{xw}^1)^N \setminus \Delta \mid I(\mathbf{q}) = c\}, \quad \hat{U} = U,$$

where  $\Delta$  denotes the collision set. Equivalently, we only need to find the number of critical points of  $U - \lambda I$  in  $(\mathbb{H}_{xw}^1)^N \setminus \Delta$ , where  $\lambda$  is fixed. We can now state and prove the following result.

**Theorem 10.** *For any given point masses  $m_1, \dots, m_N > 0$  in  $\mathbb{H}^3$  and each  $c > 0$ , there are exactly  $N!/2$  geodesic central configurations with  $I(\mathbf{q}) = c$ , one for each ordering of the masses on the geodesic.*

*Proof.* We follow the idea used to prove the classical theorem of Moulton, [1], [45], and show first that the manifold  $S_c$  contains  $N!$  components, each homeomorphic to

FIGURE 10. A configuration of  $N$ -masses on  $\mathbb{H}_{xw}^1$ 

an  $(N - 1)$ -dimensional disk. We will then prove that the critical points of  $\hat{U}$ , or equivalently, of  $U$ , are local minima on these disks, and finally show that there is just one minimum on each such disk.

To prove that each ordering corresponds to an  $(N - 1)$ -dimensional open disk, it suffices to consider one of the orderings,  $\theta_1 < \dots < \theta_N$ . Denote the corresponding component by  $S'_c$  (see Figure 10). Consider the homomorphism  $\phi : (\mathbb{H}_{xw}^1)^N \rightarrow \mathbb{R}^N$ ,  $\phi(\theta_1, \dots, \theta_N) = (x_1, \dots, x_N)$ , where  $x_i = \sinh \theta_i$ . Then  $S'_c$  is homomorphic to

$$\{(x_1, \dots, x_N) \in \mathbb{R}^N \mid x_1 < \dots < x_N, \sum_{i=1}^N m_i x_i^2 = c\},$$

which is an  $(N - 1)$ -dimensional open disk, [45]. Thus the set  $S_c$  has exactly  $N!$  components, each homomorphic to an  $(N - 1)$ -dimensional open disk. By an argument similar to the one in the proof of Theorem 9, we can establish the existence of a critical point, or a central configuration, on each component. Denote such a critical point by  $\mathbf{q}'$ . We will show that  $\mathbf{q}'$  must be a local minimum of  $U$  in  $S_c$ . For this, we first prove that  $\mathbf{q}'$  is a local minimum of  $U(\mathbf{q}) - \lambda I(\mathbf{q})$  in  $(\mathbb{H}_{xw}^1)^N \setminus \Delta$ , where  $\lambda = \lambda(\mathbf{q}') < 0$  is a constant. To reach this goal, we compute the Hessian of  $U(\mathbf{q}) - \lambda I(\mathbf{q})$  and show that it is positive definite. By straightforward computations, we obtain

$$\begin{aligned} \text{Hess}_{\mathbf{q}} &= D^2 U(\mathbf{q}) - \lambda D^2 I(\mathbf{q}) \\ &= 2 \begin{bmatrix} \sum_{j=1, j \neq 1}^N \frac{m_1 m_j \cosh d_{1j}}{\sinh^3 d_{1j}} & -\frac{m_1 m_2 \cosh d_{12}}{\sinh^3 d_{12}} & \dots & -\frac{m_1 m_N \cosh d_{1N}}{\sinh^3 d_{1N}} \\ -\frac{m_2 m_1 \cosh d_{12}}{\sinh^3 d_{12}} & \sum_{j=1, j \neq 2}^N \frac{m_2 m_j \cosh d_{2j}}{\sinh^3 d_{2j}} & \dots & -\frac{m_2 m_N \cosh d_{2N}}{\sinh^3 d_{2N}} \\ \dots & \dots & \dots & \dots \\ -\frac{m_1 m_N \cosh d_{1N}}{\sinh^3 d_{1N}} & \dots & \dots & \sum_{j=1, j \neq N}^N \frac{m_N m_j \cosh d_{Nj}}{\sinh^3 d_{Nj}} \end{bmatrix} \\ &\quad - 2\lambda \begin{bmatrix} m_1 \cosh 2\theta_1 & 0 & \dots & 0 \\ 0 & m_2 \cosh 2\theta_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & m_N \cosh 2\theta_N \end{bmatrix}. \end{aligned}$$

Notice first that  $-\lambda D^2 I(\mathbf{q})$ , the second term in  $\text{Hess}_{\mathbf{q}}$ , is positive definite. Indeed, the matrix  $D^2 I(\mathbf{q})$  is obviously positive definite, and the coefficient  $-\lambda$  is positive.

For the first term,  $D^2 U$ , let us take any nonzero vector  $\mathbf{v} = (v_1, \dots, v_N)$  in the tangent space  $T_{\mathbf{q}'}((\mathbb{H}_{xw}^1)^N \setminus \Delta)$ . Regarding  $D^2 U$  as a bilinear form, we obtain

$$\begin{aligned} \mathbf{v}^T (D^2 U) \mathbf{v} &= \sum_{i=1}^N \sum_{j=1}^N (D^2 U)_{ij} v_i v_j = 2 \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N \frac{m_i m_j \cosh d_{ij}}{\sinh^3 d_{ij}} v_i^2 \\ &\quad - 2 \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N \frac{m_i m_j \cosh d_{ij}}{\sinh^3 d_{ij}} v_i v_j = \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N \frac{m_i m_j \cosh d_{ij}}{\sinh^3 d_{ij}} (v_i - v_j)^2 \geq 0. \end{aligned}$$

We can conclude that  $\text{Hess}_{\mathbf{q}}(\mathbf{v}, \mathbf{v}) > 0$  for all  $\mathbf{v} \in T_{\mathbf{q}'}(\mathbb{H}_{xw}^1)^N \setminus \Delta$ , so  $\mathbf{q}'$  is a local minimum of  $U(\mathbf{q}) - \lambda I(\mathbf{q})$  on  $(\mathbb{H}_{xw}^1)^N \setminus \Delta$ . Then  $\mathbf{q}'$  is also a local minimum of the new function  $U(\mathbf{q}) - \lambda I(\mathbf{q}) + \lambda c$  restricted to the submanifold  $S_c$ . Note that, on  $S_c$ , this new function becomes  $U$ . Consequently  $\mathbf{q}'$  is a local minimum of  $U$  on  $S_c$ .

We show that such a minimum of  $U$  is unique on each  $(N-1)$ -dimensional open disk. Assume that there are two such minima. Connect these two points with a continuous family of curves. As the two ends are local minima, there must be a local maximum on each curve. Then the minimum of all these maxima must be a saddle point of  $U$ , in contradiction with the positive definiteness of the Hessian.

Note that a  $180^\circ$  rotation in the  $xy$ -plane doesn't change the ordering, which means that we counted each case twice, so there are exactly  $N!/2$  classes of geodesic central configurations, a remark that completes the proof.  $\square$

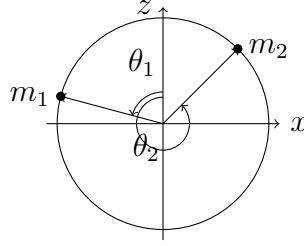
**10.2. Geodesic central configurations in  $\mathbb{S}^3$ .** Unlike in the hyperbolic case, Moulton's theorem has no straightforward generalization to  $\mathbb{S}^3$ . We give an example of geodesic central configurations for two masses to show that the number of central configurations on  $S_c$  depends on the value of  $c$ . This example also provides some degenerate central configurations, as defined in Section 8, and means that the corresponding critical points of  $\hat{U}$  on  $\hat{S}_c$  are degenerate.

According to Corollary 3, any geodesic central configuration in  $\mathbb{S}^3$  is equivalent to some geodesic central configuration on  $\mathbb{S}_{xz}^1$ . The example we will exhibit is that of central configurations for two masses on  $\mathbb{S}_{xz}^1$ . Special central configurations cannot exist under these circumstances since any nonsingular configuration would force the two masses to lie inside a semicircle, which turns out to be impossible because such a configuration cannot generate relative equilibria, as proved in [9]. Expressing the positions of  $m_1$  and  $m_2$  in terms of the oriented spherical distance,  $\theta_i \in [0, 2\pi]$ ,  $i = 1, 2$ , measured from  $(0, 0, 1, 0)$  (see Figure 11), we can write the position vectors as

$$\mathbf{q}_1 = (-\sin \theta_1, 0, \cos \theta_1, 0), \quad \mathbf{q}_2 = (-\sin \theta_2, 0, \cos \theta_2, 0), \quad 0 \leq \theta_1 < \theta_2 \leq 2\pi.$$

Then the force function and the moment of inertia have the form

$$U(\mathbf{q}) = m_1 m_2 \cot d_{12} \quad \text{and} \quad I(\mathbf{q}) = m_1 \sin^2 \theta_1 + m_2 \sin^2 \theta_2,$$

FIGURE 11. A configuration of two masses on  $\mathbb{S}_{xz}^1$ 

respectively, where  $d_{12} = \min\{\theta_2 - \theta_1, 2\pi - \theta_2 + \theta_1\}$  is the distance between the bodies. We can also assume, without loss of generality, that  $\theta_1 \in [0, \pi/2]$ . This is all the preparation we need to state and prove the following result.

**Theorem 11.** *Consider two masses  $m_1$  and  $m_2$  on  $\mathbb{S}_{xz}^1$  with positions  $\mathbf{q}_1$  and  $\mathbf{q}_2$  as above. Then these bodies can form a central configuration if and only if*

$$(16) \quad m_1 \sin 2\theta_1 + m_2 \sin 2\theta_2 = 0 \text{ with } \sin 2\theta_1 \neq 0.$$

The number of geodesic central configurations depends on the size  $I(\mathbf{q}) = c$  of each configuration and is given in the table below, where  $M := m_1 + m_2$ . The table on the left is for  $m_1 < m_2$ , whereas the table on the right is for the  $m_1 = m_2 =: m$ .

size: $I(\mathbf{q}) = c$	number	size: $I(\mathbf{q}) = c$	number
$c \in (0, m_1)$	2	$c \in (0, m)$	2
$c \in [m_1, m_2]$	0	$c = m$	$\infty$
$c \in (m_2, M)$	2	$c \in (m, M)$	2

When the masses are equal and  $c = m$ , all central configurations are degenerate critical points of  $U$  on  $S_m$  and the set they form has the power of the continuum.

*Proof.* In this case, the central configuration equations  $\nabla_{\mathbf{q}_i} U = \lambda \nabla_{\mathbf{q}_i} I, i = 1, 2$ , reduce to

$$\frac{\partial U}{\partial \theta_1} = \lambda \frac{\partial I}{\partial \theta_1} \quad \text{and} \quad \frac{\partial U}{\partial \theta_2} = \lambda \frac{\partial I}{\partial \theta_2},$$

which implies that

$$\frac{\pm m_1 m_2}{\sin^2(\theta_2 - \theta_1)} = \lambda m_1 \sin 2\theta_1 \quad \text{and} \quad \frac{\mp m_1 m_2}{\sin^2(\theta_2 - \theta_1)} = \lambda m_2 \sin 2\theta_2,$$

where the signs depend on whether  $d_{12}$  equals  $\theta_2 - \theta_1$  or  $2\pi - \theta_2 + \theta_1$ . From these equations we obtain the condition

$$m_1 \sin 2\theta_1 + m_2 \sin 2\theta_2 = 0 \text{ with } \sin 2\theta_1 \neq 0.$$

This relationship implies that  $\theta_1 \in (0, \frac{\pi}{2})$  and  $\theta_2 \in (\frac{1}{2}\pi, \pi)$  or  $\theta_2 \in (\frac{3}{2}\pi, 2\pi)$ .

To find the number of central configurations on  $S_c$ , we solve the system

$$\begin{cases} m_1 \sin^2 \theta_1 + m_2 \sin^2 \theta_2 = c \\ m_1 \sin 2\theta_1 + m_2 \sin 2\theta_2 = 0, \end{cases}$$



FIGURE 12. The graphs of  $\sin^2 \theta_2 = \frac{c(m_1 - c)}{m_2(M - 2c)}$  for  $m_1 < m_2$  (left) and  $m_1 = m_2 =: m$  (right) in coordinates  $(c, \sin^2 \theta_2)$ .

and obtain

$$\sin^2 \theta_2 = \frac{c(m_1 - c)}{m_2(M - 2c)} \quad \text{and} \quad \sin^2 \theta_1 = \frac{c(m_2 - c)}{m_1(M - 2c)}.$$

Notice that  $\sin 2\theta_i \neq 0$ , so let

$$0 < \frac{c(m_1 - c)}{m_2(M - 2c)} < 1, \quad 0 < \frac{c(m_2 - c)}{m_1(M - 2c)} < 1.$$

We are then led to

$$0 < c < m_1, \quad m_2 < c < M,$$

a fact that can also be seen in the graphs of Figure 12, where a typical function of the form  $\frac{c(m_1 - c)}{m_2(M - 2c)}$  is represented for  $m_1 < m_2$ , on the left, and  $m_1 = m_2$ , on the right.

Thus having  $c$  in this range, we can obtain the values for  $\sin^2 \theta_i < 1, i = 1, 2$ . Using the fact that  $\theta_1 \in (0, \frac{\pi}{2})$  and  $\theta_2 \in (\frac{1}{2}\pi, \pi)$  or  $\theta_2 \in (\frac{3}{2}\pi, 2\pi)$ , we see that there are exactly two central configurations for each  $c$ :

$$(\theta_1, \theta_2) \in \left(0, \frac{\pi}{2}\right) \times \left(\frac{1}{2}\pi, \pi\right) \quad \text{and} \quad (\theta_1, \theta_2 + \pi) \in \left(0, \frac{\pi}{2}\right) \times \left(\frac{3}{2}\pi, 2\pi\right).$$

If  $m_1 = m_2 = m$  and  $I = m$ , we have

$$\hat{S}_m = S_m = \{(\theta_1, \theta_2) \in (0, \pi/2) \times [0, 2\pi] \mid \theta_1 < \theta_2, \sin^2 \theta_1 + \sin^2 \theta_2 = 1\},$$

which implies that

$$\hat{S}_m = \{\theta_1 \in (0, \pi/2), \theta_2 = \theta_1 + \pi/2 \text{ or } \theta_2 = \theta_1 + 3\pi/2\}.$$

Thus  $d_{12} = \pi/2$  and  $\hat{U} = U = m_1 m_2 \cot d_{12} = 0$  on  $S_m$ , which means that all elements of this set are degenerate critical points of  $U$  on  $S_m$ , so they are degenerate central configurations. This remark justifies the values in the above tables and completes the proof.  $\square$

The related problem of finding relative equilibria on  $\mathbb{S}_{xz}^1$  has also been considered by A.A. Kilin, who obtained the same criterion given in the first part of Theorem 11, [31].

**Remark 3.** The complicated count of geodesic central configurations in  $\mathbb{S}^3$  is a consequence of two facts: the boundary of some components in  $S_c$  may contain points in  $\Delta^+$  and  $\Delta^-$ , which can destroy the existence of critical points on those components; and the geodesic central configurations are not necessarily minima of  $U$  on  $S_c$ .

## 11. CONCLUSIONS

So far, the only classes of solutions found for the  $N$ -body problem in spaces of constant curvature were relative equilibria and rotators, the latter allowing dilations and contractions of the configuration, which, of course, fail to maintain similarity, [14], although, very recently, some numerical results point out to the existence of choreographies, including the figure eight solution on the sphere  $\mathbb{S}^2$ , [46]. However, these studies are only at the beginning, and the current paper shows that the approach we took here offers another way to answer some of the natural problems that occur in the qualitative study of the equations of motion and the dynamics of the solutions.

But most questions related to central configurations are far from easy, as it also happens in the Euclidean case. Even finding all the central configurations in the curved 3-body problem, which has been settled in the classical case long time ago, is not trivial in curved space and requires a separate study. As we have already seen, new central configurations, such as the isosceles triangles, or the scalene triangles on the equator of the sphere, none of which have correspondents in the Euclidean case, show up. So far, all these central configurations on  $\mathbb{S}_{xyz}^2$  lie in planes parallel with the  $xy$ -plane, except for the geodesic ones. But at this point we have some indication that most triangular central configurations do not lie in planes parallel with the  $xy$ -plane, and hope to be able to prove this statement in the near future. So even for only three bodies, the set of central configurations of the curved problem is significantly richer than in the Euclidean case, especially in the case of the sphere.

These investigations hint at the rich dynamics of the curved  $N$ -body problem and show that the questions occurring from its study allow us to view the classical case from a new perspective. Having now extended the concept of central configuration to the curved problem, we have a new tool and a new direction of research, which will hopefully shed more light on the equations of motion that govern this mathematical model.

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## REFERENCES

- [1] R. Abraham and J. Marsden, *Foundations of Mechanics*, 2nd ed., Addison-Wesley, 1987.
- [2] A. Albouy and V. Kaloshin, Finiteness of central configurations of five bodies in the plane, *Ann. Math.* **176** (2012), 535–588.
- [3] F. Alfaro and E. Pérez-Chavela, Families of continua of central configurations in charged problems, *Dyn. Cont. Discrete Impuls. Syst. Ser. A Math. Anal.* **9** (2002), 463–465.
- [4] P. de Bernardis et. al., A flat Universe from high-resolution maps of the cosmic microwave background radiation, *Nature* **404**, 6781 (2000), 955–959.
- [5] M. R. Bridson, A. Haefliger, *Metric Spaces of Non-Positive Curvature*. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 319. Springer-Verlag, Berlin, 1999.
- [6] W. Bolyai and J. Bolyai, *Geometrische Untersuchungen*, Teubner, Leipzig-Berlin, 1913.

- [7] F. Diacu, On the singularities of the curved  $N$ -body problem, *Trans. Amer. Math. Soc.* **363**, 4 (2011), 2249–2264.
- [8] F. Diacu, Polygonal homographic orbits of the curved 3-body problem, *Trans. Amer. Math. Soc.* **364** (2012), 2783–2802.
- [9] F. Diacu, *Relative equilibria of the curved  $N$ -body problem*, Atlantis Studies in Dynamical Systems, vol. 1, Atlantis Press, Amsterdam, 2012.
- [10] F. Diacu, Relative equilibria of the 3-dimensional curved  $n$ -body problem, *Memoirs Amer. Math. Soc.* **228**, 1071 (2013).
- [11] F. Diacu, The curved  $N$ -body problem: risks and rewards, *Math. Intelligencer* **35**, 3 (2013), 24–33.
- [12] F. Diacu, The classical  $N$ -body problem in the context of curved space, *Canad. J. Math.* <http://dx.doi.org/10.4153/CJM-2016-041-2>, 21 pages, 2017.
- [13] F. Diacu, Bifurcations of the Lagrangian orbits from the classical to the curved 3-body problem, *J. Math. Phys.* 112701 (2016); <http://doi.org/10.1063/1.4967443>.
- [14] F. Diacu and S. Kordlou, Rotopulsators of the curved  $N$ -body problem, *J. Differential Equations* **255** (2013) 2709–2750.
- [15] F. Diacu, R. Martínez, E. Pérez-Chavela, and C. Simó, On the stability of tetrahedral relative equilibria in the positively curved 4-body problem, *Physica D* **256-7** (2013), 21–35.
- [16] F. Diacu and E. Pérez-Chavela, Homographic solutions of the curved 3-body problem, *J. Differential Equations* **250** (2011), 340–366.
- [17] F. Diacu, E. Pérez-Chavela, and M. Santoprete, Saari’s conjecture for the collinear  $N$ -body problem, *Trans. Amer. Math. Soc.* **357**, 10 (2005), 4215–4223.
- [18] F. Diacu, E. Pérez-Chavela, and M. Santoprete, The  $N$ -body problem in spaces of constant curvature. Part I: Relative equilibria, *J. Nonlinear Sci.* **22**, 2 (2012), 247–266, DOI: 10.1007/s00332-011-9116-z.
- [19] F. Diacu, E. Pérez-Chavela, and M. Santoprete, The  $N$ -body problem in spaces of constant curvature. Part II: Singularities, *J. Nonlinear Sci.* **22**, 2 (2012), 267–275, DOI: 10.1007/s00332-011-9117-y.
- [20] F. Diacu, E. Pérez-Chavela, and J. Guadalupe Reyes Victoria, An intrinsic approach in the curved  $N$ -body problem. The negative curvature case, *J. Differential Equations* **252** (2012), 4529–4562.
- [21] F. Diacu and S. Popa, All Lagrangian relative equilibria have equal masses, *J. Math. Phys.* **55**, 112701 (2014).
- [22] F. Diacu, J.M. Sánchez-Cerritos, and S. Zhu, On the stability of fixed-points in the 3-body problem on  $S^2$ , *J. Dynam. Differential Equations* (to appear).
- [23] F. Diacu and B. Thorn, Rectangular orbits of the curved 4-body problem, *Proc. Amer. Math. Soc.* **143** (2015), 1583–1593.
- [24] *Dictionary.com Unabridged*, <http://dictionary.reference.com/browse/moment+of+inertia> based on the Random House Dictionary, 2015.
- [25] O. Dziobek, Über einen merkwürdigen Fall des Vielkörperproblems, *Astron. Nachr.* **152** (1900), 33–46.
- [26] L. Euler, *Theoria motus corporum solidorum seu rigidorum: Ex primis nostrae cognitionis principiis stabilita et ad omnes motus, qui in huiusmodi corpora cadere possunt, accommodata*, A.F. Röse, Rostock and Greifswald, 1765.
- [27] L. Euler, Considerationes de motu corporum coelestium, *Novi comm. acad. sci. Petropolitanae* **10** (1764), 1766, pp. 544–558 (read at Berlin in april 1762). Also in *Opera Omnia*, S. 2, vol. 25, pp. 246–257 with corrections and comments by M. Schürer.
- [28] L.C. García-Naranjo, J.C. Marrero, E. Pérez-Chavela, M. Rodríguez-Olmos, Classification and stability of relative equilibria for the two-body problem in the hyperbolic space of dimension 2, arXiv:1505.01452.
- [29] M.W. Hirsch, *Differential Topology*, Graduate Texts in Mathematics, vol. 33, Springer Verlag, 1976.

- [30] C.G.J. Jacobi, Vorlesungen über Dynamik, in C.G.J. Jacobis Gesammelte Werke, vol. VIII, Druck und Verlag Von G. Reimer, Berlin, 1884.
- [31] A.A. Kilin, Libration points in spaces  $\mathbf{S}^2$  and  $\mathbf{L}^2$ , *Regul. Chaotic Dyn.* **4**, 1 (1999), 91–103.
- [32] W. Killing, Die Rechnung in den nichteuklidischen Raumformen, *J. Reine Angew. Math.* **89** (1880), 265–287.
- [33] V. V. Kozlov and A. O. Harin, Kepler’s problem in constant curvature spaces, *Celestial Mech. Dynam. Astronom* **54** (1992), 393–399.
- [34] H. Kragh, Is space Flat? Nineteenth century astronomy and non-Euclidean geometry, *J. Astr. Hist. Heritage* **15**, 3 (2012), 149–158.
- [35] J.L. Lagrange, Essai sur le problème des trois corps, 1772, Œuvres tome 6.
- [36] P.S. Laplace, *Oeuvres*, vol. 4, pp. 307–513, vol. 11, pp. 553–558.
- [37] H. Liebmann, Die Kegelschnitte und die Planetenbewegung im nichteuklidischen Raum, *Berichte Königl. Sächsischen Gesell. Wiss., Math. Phys. Klasse* **54** (1902), 393–423.
- [38] H. Liebmann, Über die Zentralbewegung in der nichteuklidische Geometrie, *Berichte Königl. Sächsischen Gesell. Wiss., Math. Phys. Klasse* **55** (1903), 146–153.
- [39] N. I. Lobachevsky, The new foundations of geometry with full theory of parallels [in Russian], 1835–1838, in Collected Works, vol. 2, GITTL, Moscow, 1949.
- [40] J. Marsden, *Lectures on Mechanics*, Cambridge University Press, 2009.
- [41] J. Marsden and T. Ratiu, *Introduction to Mechanics and Symmetry: A Basic Exposition of Classical Mechanical Systems*, Springer Verlag, 1999.
- [42] R. Martínez and C. Simó, On the stability of the Lagrangian homographic solutions in a curved three-body problem on  $\mathbb{S}^2$ , *Discrete Contin. Dyn. Syst. Ser. A* **33** (2013) 1157–1175.
- [43] R. Martínez and C. Simó, Relative equilibria of the restricted 3-body problem in curved spaces, (in preparation).
- [44] R. Moeckel, Finiteness of relative equilibria of the four-body problem, *Invent. Math.* **163** (2006), 289–312.
- [45] R. Moeckel, *Celestial Mechanics—especially central configurations*, unpublished lecture notes: <http://www.math.umn.edu/~rmoeckel/notes/CMNotes.pdf>
- [46] H. Montanelli and N.I. Gushterov, Computing planar and spherical choreographies, *SIAM J. Appl. Dyn. Syst.* (to appear).
- [47] F.R. Moulton, The straight line solutions of  $n$  bodies, *Ann. of Math.* **12**, 1–17.
- [48] G.L. Naber, *The Geometry of Minkowski Spacetime*, Springer-Verlag, 1991.
- [49] E. Pérez-Chavela and J.G. Reyes Victoria, An intrinsic approach in the curved  $N$ -body problem. The positive curvature case, *Trans. Amer. Math. Soc.* **364**, 7 (2012), 3805–3827.
- [50] B. Riemann, Über die Hypothesen welche der Geometrie zu Grunde liegen, *Abhandl. Königl. Ges. Wiss. Göttingen*, **13**, 1854.
- [51] G. Roberts, A Continuum of Relative Equilibria in the Five-Body Problem, *Phys. D* **127** (1999), no. 3–4, 141–145.
- [52] D. Saari, On the role and properties of central configurations, *Celestial Mech.* **21** (1980), 9–20.
- [53] D. Saari, *Collisions, Rings, and Other Newtonian N-Body Problems*, CBMS Regional Conference Series in Mathematics, American Math. Society, 2005.
- [54] E. Schering, Die Schwerkraft im Gaussischen Räume, *Nachr. Königl. Ges. Wiss. Göttingen* **15**, (1870), 311–321.
- [55] E. Schering, Die Schwerkraft in mehrfach ausgedehnten Gaussischen und Riemannschen Räumen. *Nachr. Königl. Ges. Wiss. Göttingen* **6**, (1873), 149–159
- [56] A.V. Shchepetilov, Nonintegrability of the two-body problem in constant curvature spaces, *J. Phys. A: Math. Gen.* **39** (2006), 5787–5806; corrected version at [math.DS/0601382](http://math.DS/0601382).
- [57] S. Smale, Mathematical Problems for the Next Century, *Math. Intelligencer* **20**, 2 (1998) 7–15.
- [58] S. Smale, Topology and mechanics. I, *Invent. Math.* **10**, (1970), 305–331.

- [59] S. Smale, Topology and Mechanics, II. The Planar  $N$ -Body problem, *Invent. Math.* **11** (1970), 45–64.
- [60] P. Tibboel, Polygonal homographic orbits in spaces of constant curvature, *Proc. Amer. Math. Soc.* **141** (2013), 1465–1471.
- [61] P. Tibboel, Existence of a class of rotopulsators, *J. Math. Anal. Appl.* **404** (2013), 185–191.
- [62] P. Tibboel, Existence of a lower bound for the distance between point masses of relative equilibria in spaces of constant curvature, *J. Math. Anal. Appl.* **416** (2014), 205–211.
- [63] A. Wintner, *The Analytical Foundations of Celestial Mechanics*, Princeton University Press, 1947.
- [64] S. Zhu, Eulerian relative equilibria of the curved 3-body problems in  $\mathbb{S}^2$ , *Proc. Amer. Math. Soc.* **142** (2014), 2837–2848.
- [65] S. Zhu, S. Zhao, Three-dimensional central configurations in  $\mathbb{H}^3$  and  $\mathbb{S}^3$ , *J. Math. Phys.* **58** (2017), no. 2, 022901.