HARETU AND THE STABILITY OF THE SOLAR SYSTEM[1]

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Abstract

We discuss the contributions of Spiru Haretu to the problem of the solar system's stability and show their importance relative to the mathematics research of the late 19th century. We also give a brief survey of the subsequent developments and the consequences of Haretu's results.

Keywords: stability, solar system, n-body problem.

Is the solar system stable? Properly speaking, the answer is still unknown, and yet this question has led to very deep results which probably are more important than the answer to the original question.

JÜRGEN MOSER, 1978

1 Introduction

Like Paul Painlevé (1863-1933), Spiru Haretu (1851-1912) is remembered both as a great mathematician and as a politician of strong character. His most important scientific result appeared in his doctoral dissertation, *On the invariance of the planets' major axes*, defended on the morning of January 30, 1878 to a committee formed by Puiseux (president), Briot, and Baillaud, and published by Gauthier-Villars [H,1878]. His main political achievement was that of establishing the Romanian elementary school system, while holding the education portfolio as a liberal member of parliament in several governments. For these and other contributions he is viewed as a prominent figure in the history of Romanian cultural and public life and as a scientist of world fame. A statue representing him stands since 1935 in Bucharest's University Square. A lunar crater bears his name.

But while his political achievements present only national interest, the result of his 1878 dissertation is a landmark in the development of mathematics. At a time when very few doctoral degrees were awarded, Haretu became the first Romanian and foreigner to obtain a Ph.D. in mathematics at a Parisian institution. He was immediately offered a university position in France, which he turned down to return to his native land. Romania had recently declared its independence from the Turkish Empire and was still involved in a liberation war. Haretu decided that he could contribute to the well being of his country, so he went back home and eventually received a professorship at the University of Bucharest.

Before entering politics, Haretu developed several courses at his young university and continued to do research. In a pioneering book entitled *Social Mechanics* [H,1910], he used mathematical tools to study social phenomena. But none of his new works reached the level of importance and fame of his doctoral achievement, which we will analyse in what follows. Let us start with a brief review of the results obtained on the stability of the solar system before Haretu entered the scene of French and world mathematics.

2 The stability problem

The notion of stability has many mathematical connotations, most of which developed from a question regarding the solar system. Once the Newtonian theory of gravitation made clear that planets moved on approximately elliptical orbits having the sun in their common focus, it was natural to ask whether this property would be forever preserved. The attempts of understanding the n-body problem of celestial mechanics led to the growth of mathematics and eventually to power-series solutions of the differential equations describing it. These were obtained through the work of Sundman [S,1912], for $n = 3$, and Wang [W,1991] for any $n \geq 3$. But because of their very slow convergence, these solutions are of little practical interest and add nothing to the understanding of the stability problem (see [D,1996] for more details).

In Newton's mind the solar system was unstable. He thought that gravitation alone could not account for its stability and that the intervention of divine forces was necessary for preserving the regular planetary motion around the sun. In the General Scholium of *Principia*'s volume two, *The system of the world* [N,1966], p. 544, Newton wrote:

> ...it is not to be conceived that mere mechanical causes could give birth to so many regular [planetary] motions, since the comets range over all parts of the heavens in very eccentric orbits...This most beautiful system of the sun, planets, and comets could only proceed from the counsel and dominion of an intelligent and powerful Being. And if the fixed stars are the centres of other like systems, these, being formed by the like wise counsel, must be all subject to the dominion of One.

Newton, however, never approached the stability question with mathematical tools. His main goal was that of explaining the motion of the moon. So his belief in the instability of the solar system was based only on intuition.

Already before the mid 18th century, research on the n-body problem split in two directions: one regarded properties holding in a general gravitational context, whereas the other considered approximations for as-long-as-possible time intervals in specific systems, in particular the solar one. The second direction grew into the theory of perturbations, which is fundamental in the study of the stability question.

The idea of using perturbation theory to approach stability works as follows. Consider an n-body problem in which one mass (the sun) is much larger than all the others (the planets). If the small masses were all zero, their orbits around the sun would be ellipses with a common focus. Using the method of variation of parameters (introduced by Euler and perfected by Lagrange), we can look at the elliptical elements of the orbit of a planet as being perturbed by the other planets and then find relatively simple formulae that express this variation in terms of a perturbative function R. For $R = 0$ we recover the elliptical orbits.

Taking the first power-series approximation of R, integration of the resulting equations of motion yields for the elliptical elements first-order perturbation terms relative to the masses. If we substitute these values back into the equations, take a second-order approximation of R, and integrate again, we obtain second-order perturbation terms in the masses for the elliptical elements. In principle, we can continue this process to the desired degree of approximation. However, the computations soon grow so laborious and complicated that only those mastering great technical virtuosity can overcome the difficulties.

In this chain of computations, two kind of terms can show up: those derived from summands in R, which do not contain time explicitly, and those obtained by integration, which may depend on time. The latter terms are called *secular*, and their effect on the elliptic elements can be significant in the long run. Their presence raises a serious doubt on the stability of the solar system. Their effect could be cancelled out but it could also add up and thus create instability.

Properly speaking, approximate results of the kind described above can neither rigorously prove nor disprove stability, but they bring arguments favouring one belief or the other. Lacking the insight of qualitative methods, the pioneers of calculus had to content themselves with such estimates, hoping to show that the results they obtained were true for any degree of approximation.

Euler was the first to apply the theory of perturbation to the motion of planets. His memoirs on the perturbations of Jupiter and Saturn won the prizes of the French Academy in 1748 and 1752. His works opened the way for the research of Lagrange and Laplace on the stability of the solar system.

In 1773 Laplace showed that in the first power-series approximation of the eccentricities, the major axes of the planets have no secular terms. He thus concluded that, within the limits of this approximation, the solar system is stable. This result incited Lagrange to search deeper into the question. His research from 1774 and 1776 proved that for all order approximations of the eccentricities, for all order approximations of the sine of the angle of the mutual inclinations, and for perturbations of the first order with respect to the masses, secular terms do not show up in the major axes of the planets. This result brought even stronger evidence in favour of stability.

In a memoir presented to the Institut de France on June 20, 1808 (published in 1809 in Journal de l'École Polytechnique, XV-e Cahier, 1809, p. 1-56), Poisson improved the result of Lagrange by showing that the major axes of the planets have no secular terms in the perturbations of the second order with respect to the masses. The computations of Poisson were considerably simplified by Liouville and Puiseux in 1841 and then by Tisserand in 1876, the latter using Jacobi's method of eliminating the nodes, which can reduce by two the dimension of the system of differential equations describing the n-body problem. This was the state of the art of this question at the time Haretu started thinking about it.

On the other hand the interest in the more theoretical but related aspects of solving the n-body problem of celestial mechanics, reached an unprecedented high. In 1858 Dirichlet mentioned in a letter to Kronecker that he had found a method of successively approximating its general solution. Unfortunately, Dirichlet died suddenly without leaving any written evidence to support his statement. This stirred the interest of many mathematicians, Weierstrass among them, who tried to recover the lost method. In a letter to Sonia Kovalevskaya, Weierstrass mentioned that he had found a formal series solution for the n-body problem, but could not prove its convergence. In the context of these interesting developments, Haretu considered working on the stability question for the solar system. He aimed at finding a third-order approximation of the solution, a tour de force that nobody had completed before.

3 Haretu's surprising result

Let us explain Haretu's contribution in a modern language. On the cotangent bundle $T^*\mathbb{R}^{3n}$, define the Hamiltonian of the n-body problem as

$$H(\mathbf{q},\mathbf{p}) = \frac{1}{2}\mathbf{p}^T M^{-1}\mathbf{p} - U(\mathbf{q}),$$

where $\mathbf{q}_i$ is the position vector of the mass m_i, $\mathbf{p}_i = m_i\dot{\mathbf{q}}_i$ is the momentum, $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_n)$, $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_n)$,

$$U(\mathbf{q}) = \sum_{1\leq i<j\leq n} \frac{Gm_i m_j}{|\mathbf{q}_i - \mathbf{q}_j|}$$

is the potential function, where G denotes Newton's gravitational constant, and M represents the matrix having the elements $m_1, m_1, m_1, \ldots, m_n, m_n, m_n$ on the diagonal and 0 elsewhere.

Notice that the Hamiltonian is translation invariant, i.e., $H(\mathbf{q}+\mathbf{x}, \mathbf{p}) = H(\mathbf{q}, \mathbf{p})$ and the linear momentum, $J(\mathbf{q}, \mathbf{p}) = \mathbf{p}_1 + \ldots + \mathbf{p}_n$ is conserved along the flow determined by H.

We further introduce the quotient manifold $J^{-1}(\mu)/\mathbb{R}^3$, where $\mu \in \mathbb{R}^3$ is arbitrary. This manifold, called *reduced phase space*, is symplectic; moreover, H induces a Hamiltonian system on it. In fact this is a general phenomenon, which was fully understood by Marsden and Weinstein a century later [MW,1974]. Using the cotangent-bundle reduction theorem [AM,1978], it follows that $J^{-1}(\mu)/\mathbb{R}^3$ is symplectically diffeomorphic to $T^*\mathbb{R}^{3(n-1)}$, endowed with the canonical symplectic structure. In more concrete terms, this reduction procedure fixes the center of mass and the linear momentum of the system.

In his thesis, Haretu used the previous work of Tisserand to carry out this reduction in a very clever way, such that at the end he obtained the canonical variables on $T^*\mathbb{R}^{3(n-1)}$, explicitly expressed as functions of the old variables; at the same time he wrote down Lagrange's equations in the new variables. Let us emphasize that, in general, the reduction procedure does not produce canonical variables; one needs to carry out the quotienting process carefully in order to directly get canonical variables on the reduced phase space.

In the second part of his thesis, Haretu used this reduced manifold to obtain the expression of the major axes of the planetary ellipses to first and second order, thus recovering in a different way the results obtained by his predecessors. This part of his work is very elegant and can be read with ease by a modern day mathematician.

But Haretu's most important contribution was in the third and last part of the thesis, in which he used the reduced manifold to discuss the perturbations of the great axes for the planetary ellipses to third-order approximation, going beyond Poisson's work mentioned before. This is a rare computational achievement, in which Haretu overcomes all technical difficulties that lead him to show that secular terms of the form

$$At \cos(\psi + \omega) + Bt^2 \cos(\psi + \omega)$$

appear in some of the summands that give the expression of the planetary axes. Essential in this formula are the factors t and t^2, which could make the solution become unbounded in infinite time. This result did not prove instability, since the effect of these terms may cancel out, but it raised a serious doubt on the stability property and disproved the constancy claim for the major axes of the planets.

It is interesting to note that Poisson had himself discussed the third-order approximation in a paper published in 1816 in the first volume of Mémoires de l'Académie des Sciences, p. 55-67. However, he ignored several aspects of the problem and concluded that no secular terms show up. Moreover, he trusted that this was true for any order of approximation and aimed to find a general proof of this fact. Haretu's result put an end to such attempts.

Laying aside Newton's heuristic belief in the instability of planetary motion, Haretu was the first to dispel the stability myth based on the results of Lagrange, Laplace, Poisson, and others. His thesis stirred high excitement not only in scientific circles, but also in the media. He was hailed in newspaper articles (see [DH,1996]), a rare event for a scientist during the last quarter of the 19th century. In Romania he became a national hero, a feat that also helped him later succeed in public life.

4 Subsequent developments

Haretu's work marked the beginning of the end of an era, that of exclusively quantitative endeavours in mathematics. His achievement was one of the first that pointed out the limits of direct attempts to find explicit solutions for differential equations. It was soon followed by other results of the same kind, as for example those of Heinrich Bruns (see [B,1887]), who showed that the n-body problem cannot have more than 10 linearly independent algebraic first integrals, thus proving the impossibility of reducing the differential equations of motion to an algebraic system.

Already in 1836, the Swiss mathematician Charles Sturm had introduced the qualitative methods in the study of linear ordinary differential equations, but the need for such techniques became obvious only after Haretu, Bruns, and a few others pointed out the short-sightedness of the exact methods.

Henri Poincaré was the one who imposed the qualitative approach not only in the study of linear and nonlinear differential equations, but as a new way of mathematical thinking. In 1908, after almost three decades of qualitative achievements, Poincaré stated at the 4th International Congress of Mathematicians in Rome:

> In the past an equation was only considered solved when one had expressed the solution with the aid of a finite number of known functions; but this is hardly possible one time in a hundred. What we can always do, or rather what we should always try to do, is to solve the *qualitative* problem so to speak, that is to find the general form of the curve representing the unknown function.

Poincaré was well aware of Haretu's work, as he shows in his *Leçons de mécanique céleste*. In fact one of his main scientific goals was that of understanding the stability of the solar system. In 1889 Poincaré received King Oscar's prize for his contributions to the n-body problem. His work contained a multitude of novel ideas, among which a first glimpse of chaotic behaviour in a dynamical system, a property that he discovered while trying to prove a stability result (for more details see [DH,1996] and [BG,1997]). Today, the qualitative theory of dynamical systems is a very intense research area.

In 1958, J. Meffroy reconsidered Haretu's computations and found a simpler way of obtaining the expression of the secular terms [M,1958]. This research, however, did not revive the interest in the quantitative methods, whose limits Haretu had pointed out in 1878. A more detailed discussion about the above subsequent developments appears in the paper by Vasile Mioc and Magda Stavinschi, also included in this volume.

Another direction in understanding stable and chaotic behaviour in the solar system grew during the last few decades after the invention of the modern computer. Researchers like Jack Wisdom, Gerald Sussman, Matthew Holman, Jacques Laskar, Scott Tremaine, Gerlad Quinlan, Myron Lecar, Fred Franklin, Marc Murison, and others investigated the orbital motion of planets and asteroids using numerical methods, and in some cases computed the Liapunov exponents, which can indicate chaotic motion. Their main conclusion is that the solar system is in a state of "relative" stability, in the sense that we should not expect major changes in the motion of the planets for the next few hundred million years. However, the final question of its stability remains still unsolved.

More than a century ago, Poincaré warned us about the true aspect of this problem (see [P,1891]):

> One of the questions with which researchers have been most preoccupied is that of the stability of the solar system. This is, if truth be told, more of a mathematical question than a physical one. Even if one were to discover a general and rigorous proof, one could not conclude that the solar system is eternal. It may, in fact, be subject to forces other than those of Newton.

But even if only of theoretical importance, this question is an early example of collective work in science, of the common effort of several generations of researchers towards reaching a lofty goal. This paper pays an homage to one of those contributors, whose achievement showed that the scientific adventure he and his predecessors had started upon was far from over. His example further illuminates the path for future generations of researchers in this and related fields.

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