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A Century-long Loop*

The first goal of this work . . . is to . . . deal with a branch of geometry called Analysis Situs, *which describes the relative position of points, lines, and surfaces, with no regard for their size.*

Henri Poincaré, in *Analysis Situs*, 1895

Most mathematical theories are like unfaithful offspring: they forget their origins. But some remember them. In 1892, while pursuing his studies on the 3-body problem, Henri Poincaré laid the foundations of algebraic topology. The new field flourished, finding applications in many branches of mathematics. A hundred years later its tools were used to answer Poincaré's initial question.

This is a tale about a theory that glimpsed back to its roots, solving one of the problems that had created it. But before telling this story, let me generalize on how mathematical theories arise.

The Birth of New Theories

Two basic mechanisms—one internal, the other external—govern the development of mathematics. The internal one acts when posing a purely mathematical problem. This leads to theorems, new concepts, generalizations, and higher levels of abstraction. Galois theory, for example, grew as a result of repeated attempts to find a formula for the solutions of a polynomial equation of any degree. The external mechanism is triggered by other fields. Some of the questions they pose stimulate the rise of new mathematical branches. In this sense, Newton founded the theory of differential equations while trying to explain the motion of the moon.

Most theories grow under the influence of both internal and external factors, thus closely relating the evolution of pure and applied mathematics. A new field is usually born under the weight of one or more questions asked at the right time, which are intriguing enough to rouse attention and to keep the interest alive. But even at maturity, a theory may be unable to solve the original problems. Most researchers of the new domain are unaware of or uninterested in the initial setting, whereas those who still seek answers are usually overwhelmed by the growth of the new field. If enough time elapses, changes in fashion may discard the original statements or, in rare cases, promote them to the rank of famous conjectures.

So in many cases the birth of new mathematical branches is a *deflective* phenomenon: when unable to solve a problem, mathematicians create a theory in order to answer the initial question. Often the question fades out of the collective memory and the new field takes on different paths (see Figure 1).[1] Algebraic topology followed the same rule. But at least one of its original questions stayed alive due to the growing interest in the qualitative theory of dynamical systems.

Poincaré's Question

Soon after finishing his doctoral degree (1878), Poincaré started working on the 3-body problem. At that time he had

*To the memory of Aristide Halanay (1924–1997) of the University of Bucharest, founding editor of the *Journal of Differential Equations.*

[1]This is not the paradigm-type evolution described by Thomas Kuhn [K,1970]. Unlike scientific theories, which replace each other, mathematical fields live together and aim to make connections. To relate Kuhn's model to this one would take another paper.

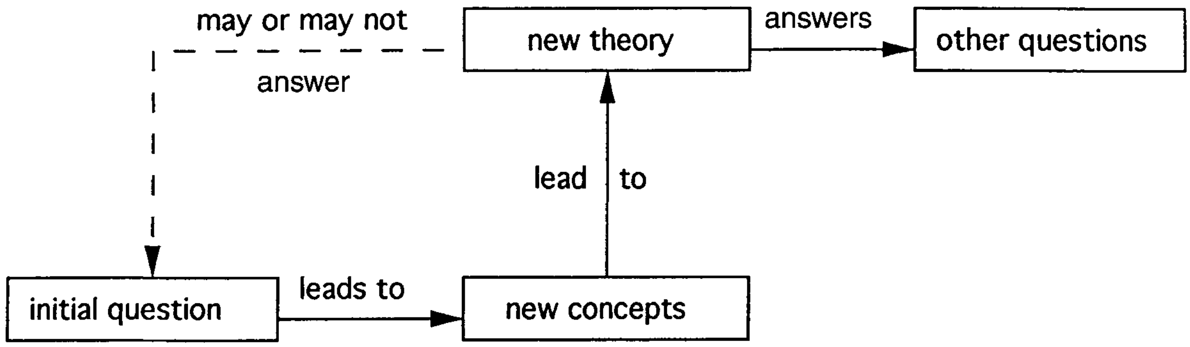

Figure 1. Deflective development: an original question leads to new concepts and to a new theory, which answers other questions but may not answer the original one.

no idea that his results in celestial mechanics would launch such a brilliant career. In 1889 he was awarded the prestigious prize of King Oscar II of Sweden and Norway (see [BG,1997] and DH,1996]). Poincaré published the prize paper a year later and developed it during the next decade into the 3-volume masterpiece *Les Méthodes nouvelles de la Mécanique céleste.*

The ideas he promoted in the theory of differential equations and in celestial mechanics were revolutionary. Instead of seeking particular solutions or attempting to reduce the order of the system (methods that applied only to a few classes of problems), he developed a qualitative point of view, following the Swiss mathematician Charles Sturm, who had started on this path in 1836. For a given system of differential equations,

$$\mathbf{x}' = f(\mathbf{x}),$$

Poincaré considered the n-dimensional set of the variable $\mathbf{x}$, called *phase space,* and viewed the solutions of the system as curves in this space. His goal was to offer a global geometric description of the solution curves. He explained this in his address to the Fourth International Congress of Mathematicians held in 1908 in Rome:

> *In the past an equation was only considered solved when one had expressed the solution with the aid of a finite number of known functions; but this is hardly possible one time in a hundred. What we can always do, or rather what we should always try to do, is to solve the qualitative problem so to speak, that is, to find the general form of the curve representing the unknown function.*

But applying this strategy to the 3-body problem was no easy task. The equations describing the gravitational motion of 3 point masses m_1, m_2, m_3 in physical space are

$$\begin{cases} \mathbf{q}_i' = m_i^{-1}\mathbf{p}_i \\ \mathbf{p}_i' = \frac{\partial U(\mathbf{q})}{\partial \mathbf{q}_i}, \end{cases} \qquad i = 1, 2, 3,$$

where $\mathbf{q}_i = (q_i^1, q_i^2, q_i^3)$ and $\mathbf{p}_i = (p_i^1, p_i^2, p_i^3)$, $i = 1, 2, 3$, represent the *position vectors* and the *momenta* (i.e., mass $\times$ velocity), respectively;

$$U(\mathbf{q}) = G\left(\frac{m_1m_2}{|\mathbf{q}_1 - \mathbf{q}_2|} + \frac{m_2m_3}{|\mathbf{q}_2 - \mathbf{q}_3|} + \frac{m_3m_1}{|\mathbf{q}_3 - \mathbf{q}_1|}\right)$$

is the *potential function* (the negative of the *potential energy*); $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$ is the *configuration* of the particle system; and G is the gravitational constant.

The 18-dimensional phase space can be first reduced to a 12-dimensional one. More precisely, the equations of motion remain unchanged if one shifts the origin of the reference frame to the center of mass of the particle system, a change that can be expressed by 6 scalar equations derived from *first integrals* (i.e., functions that are constant along solutions):

$$m_1\mathbf{q}_1 + m_2\mathbf{q}_2 + m_3\mathbf{q}_3 = \mathbf{0} \quad \text{and} \quad \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = \mathbf{0}.$$

The reduction can be carried further by using other first integrals. The *energy integral*

$$T(\mathbf{p}) - U(\mathbf{q}) = h, \tag{1}$$

where $T(\mathbf{p}) = \frac{1}{2}(m_1^{-1}|\mathbf{p}_1|^2 + m_2^{-1}|\mathbf{p}_2|^2 + m_3^{-1}|\mathbf{p}_3|^2)$ is the *kinetic energy,* $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ is the *momentum,* and h a real constant, foliates the 12-dimensional phase space into 11-dimensional "slices," which can be subsequently foliated using the 3 *angular-momentum integrals*

$$\mathbf{q}_1 \times \mathbf{p}_1 + \mathbf{q}_2 \times \mathbf{p}_2 + \mathbf{q}_3 \times \mathbf{p}_3 = \mathbf{c}, \tag{2}$$

where $\mathbf{c}$ is a constant vector. This means that the initial 18-dimensional space is partitioned into infinitely many 8-dimensional so-called *integral manifolds,* $\mathbf{M} = M(h, \mathbf{c})$. Since the equations of motion are invariant under rotations, the study can be further reduced to the 7-dimensional components $\mathbf{M}_7 = \mathbf{M}/SO_2$ ($\mathbf{M}$ factorized to rotations), called *reduced integral manifolds.* If we consider the planar 3-body problem instead of the spatial one, the integral manifold corresponding to $\mathbf{M}$, say $\mathbf{m}$, is 6-dimensional; the one corresponding to $\mathbf{M}_7$ is 5-dimensional; let us denote it by $\mathbf{m}_5$.

To apply his program of describing the qualitative behavior of solution-curves in $\mathbf{M}_7$ and $\mathbf{m}_5$, Poincaré had first to understand the shape of these sets, or their *topology,* as we call it today. For this he needed a language, so he searched the literature for appropriate tools. He found something[2] in a posthumously published fragment of a

[2]Other aspects of topology started developing at about the same time (see [Sc,1994] and [Da,1994]), viewing manifolds from a different perspective.

manuscript by Bernhard Riemann [R,1953] and in a paper of Enrico Betti [B,1871]. The two had worked together in Pisa, where Riemann, happy to leave the wet climate of Göttingen, which worsened his tuberculosis, had accepted several long-term invitations of his Italian colleague. But Poincaré found he needed to develop these notions further. Thus algebraic topology was born.

In Search of the Origins

This is the most plausible scenario. Unfortunately there is no clear proof that Poincaré thought exactly this way. No written statement has yet been traced. The American mathematician George David Birkhoff, an expert in Poincaré's work in dynamics, wrote a few decades later ([Bi,1927], p. 288),

> *The manifold* M_7 *has fundamental importance for the problem of three bodies, but, so far as I know, it has nowhere been studied, even with respect to the elementary question of connectivity. The work of Poincaré . . . does not consider* M_7 *in the large.*

This is of course no proof that Poincaré ignored the problem. (Painlevé, for example, attributes to Poincaré what is known today in celestial mechanics as Painlevé's conjecture, though there is no trace of it in Poincaré's written heritage (see [DH,1996]). Most probably Painlevé learned of it during a private discussion.) There is, however, clear evidence that Poincaré connected algebraic topology (or *analysis situs*, as he called it following Riemann [R,1953]) to the 3-body problem. In a 1901 paper, published posthumously in 1921, Poincaré wrote ([P,1921], p. 101),

> *All the various ways in which I have successively engaged myself have led me to* Analysis Situs. *I needed the results of this science to pursue my studies of the curves defined by differential equations and for extending them to higher-order differential equations, in particular to those of the 3-body problem. I needed them for the study of nonuniform functions of two variables. I needed them for the study of periods of multiple integrals and for applying this study to expanding the perturbation function. Finally, I would foresee in* Analysis Situs *a way of approaching an important problem of group theory, the research of discrete groups or of finite groups contained in a given continuous group.*

Poincaré's first contributions to algebraic topology appeared in 1892 in a *Comptes Rendus* note, which he developed in 1895 into a longer article entitled *Analysis Situs*. This happened at a time when he was deeply involved in research in celestial mechanics and especially in the 3-body problem. Five more papers on topology appeared between 1899 and 1904 (see [P,1953]), after the publication of *Les Méthodes nouvelles*.

How could Poincaré connect *analysis situs* to the 3-body problem without thinking of the topological description of $\mathbf{M}_7$ or $\mathbf{m}_5$? He was interested in periodic orbits, and he needed to determine the topology of the space in order to find them. Periodic orbits are crucial for understanding what Poincaré finally aimed at—the geometry of the flow, which he could not possibly study without knowing the shape of the integral manifolds.

Like most of us, Poincaré reached his results from example to theorem, i.e., from the system describing the 3-body problem (on which he worked intensely at the beginning of the 1890s to expand his prize paper into the first two volumes of *Les Méthodes nouvelles*) to the general theory of differential equations. But like most of us too, he presented his results the other way around, considering the 3-body problem as an application of his theory. The previous quotation follows the same pattern of thinking. On the other hand, Poincaré made this statement almost two decades after publishing his first paper on *analysis situs*. Initially he had foreseen applications only "to higher-order differential equations and, in particular, to those of celestial mechanics" (see the Introduction in [P,1895]). These are other arguments favoring the idea that Poincaré thought of the topological description of $\mathbf{M}_7$ and $\mathbf{m}_5$.

But why then did he never state the problem explicitly? Perhaps because the tools he invented were too crude to help him make significant progress, so his interests shifted towards more promising directions. This would be no wonder—the problem *is* very difficult. The first to publish a statement and make a step towards solving it was Birkhoff, who had become famous in 1912 by providing his *fixed-point theorem*[3], thus answering another question unsuccessfully attacked by Poincaré (see [DH,1996]) and also rooted in the 3-body problem.

Though we will never be sure of what was in Poincaré's mind, it seems likely that in developing *analysis situs* he was also targeting the topological description of $\mathbf{M}_7$ and $\mathbf{m}_5$.

From Betti Numbers to Cohomology

The main tools Poincaré used for the topological characterization of a manifold were the *Betti numbers*, which he named after the Italian mathematician Enrico Betti (1823–1892), who had previously introduced certain topological invariants. Poincaré defined the Betti numbers of a manifold in his first paper on *analysis situs*, then reconsidered them in some later articles. In today's terminology, if X is an n-dimensional topological space, the Betti numbers $\beta_0, \beta_1, \ldots, \beta_n$ are defined as: β_0—the number of connected components of X; and β_k $(k \geq 1)$—the number of k-dimensional holes of X (see Figure 2).

In connection with Betti numbers, Poincaré introduced the notion of *homology*, which later on developed into the

[3]At the end of October 1912, Birkhoff presented to the American Mathematical Society a communication entitled "Proof of Poincaré's geometric theorem." The paper derived from this communication appeared a year later ([Bi,1913]).

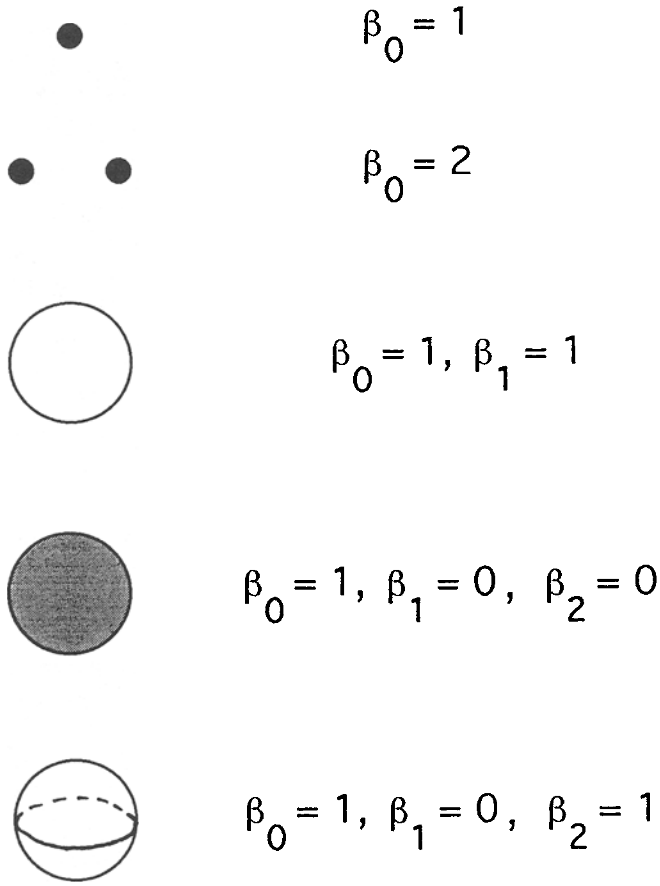

Figure 2. Betti numbers: (a) A point is 0-dimensional and has one component, so $\beta_0 = 1$. (b) The space formed by two points is 0-dimensional and has two components, so $\beta_0 = 2$. (c) A circle is 1-dimensional, has one component and one 1-dimensional hole, so $\beta_0 = \beta_1 = 1$. (d) A disc is 2-dimensional, has one component and no holes, so $\beta_0 = 1$, $\beta_1 = 0$, and $\beta_2 = 0$. (e) A sphere is 2-dimensional, has one component, no 1-dimensional holes, but one 2-dimensional hole, so $\beta_0 = 1$, $\beta_1 = 0$, and $\beta_2 = 1$.

concept of *homology group*. If in its early times algebraic topology paid more attention to *numerical invariants*, as fashion dictated, it soon became clear that a structural description is richer. In fact, today's idea of algebraic topology is to reduce topological problems to algebraic ones, i.e., to obtain information about homeomorphic maps by studying the induced isomorphisms between the corresponding groups.

Thus new concepts appeared, the *cohomology group* among them, which in a certain sense is the "dual" of a homology group. Cohomology groups are not "better" than homology groups (in fact they are less natural, so it took quite a while for this notion to crystalize), but they reveal different topological aspects of the manifolds studied. Also they offer an alternative language for expressing certain topological properties. Similar things can be said about *homotopy groups*, which emerged in the 1930s from the work of Witold Hurewicz and Heinz Hopf.

The growth of algebraic topology has been far from linear, and its history is now a research subject in itself. Jean Dieudonné dedicates an entire volume to the period 1900 to 1960 [D,1989], claiming that the next twenty years would easily fill another volume. Though Poincaré's work on the subject was barely mentioned during the first decades after his death (1912), things changed afterwards. In the preface to his historical account, Dieudonné wrote [D,1989],

> *At first, algebraic topology grew very slowly and did not attract many mathematicians; until 1920 its applications to other parts of mathematics were very scanty (and often shaky). This situation gradually changed with the introduction of more powerful algebraic tools, and Poincaré's vision of the fundamental role topology should play in all mathematical theories began to materialize. Since 1940, the growth of algebraic and differential topology and of its applications has been exponential and shows no signs of slackening.*

The Topology of Integral Manifolds

As I mentioned earlier, the first who explicitly dealt with the topology of $\mathbf{M}_7$ and $\mathbf{m}_5$ was Birkhoff. In 1927 he considered the problem but achieved only little success. In his now famous *Dynamical Systems* ([Bi,1927]), Birkhoff made a few unsatisfactory arguments for the following statement:

Birkhoff's Statement. *For $h < 0$, the topologies of $\mathbf{M}_7$ and $\mathbf{m}_5$ can change only at points that correspond to relative equilibria.*

A *relative equilibrium* is a point $(\mathbf{q}, \mathbf{p})$ in phase space which if taken as an initial condition for the 3-body problem leads to a uniform motion of the bodies on concentric circles. The $\mathbf{q}$-component of a relative equilibrium is called a *central configuration*, and it is always such that the gravitational force has the same direction as the position vector, i.e., $\mathbf{q}'' = k\mathbf{q}$ for some constant $k > 0$. The only possible central configurations for three bodies are the equilateral triangle, and the straight-line position in which the ratio of the distances satisfies a relation depending on the masses (see Figure 3). If released with zero velocity from such a configuration, the bodies move homothetically towards a simultaneous total collapse. Because there are three ways of arranging the masses on a line, the spatial 3-body problem has four central configurations. If the 3-body problem is restricted to a plane, the equilateral triangle has two possible orientations, so the number of central configurations increases to five.

The next notable statement was made by the Austrian mathematician Aurel Wintner in a book containing the most significant mathematical results obtained on the n-body problem up to 1941 [Wi,1941]. In his bibliographical notes, Wintner mentions that "... nothing explicit is known as to the topological structure of $\mathbf{M}_7$." Three decades of silence followed until the American mathematicians Stephen Smale [S,1970a], [S,1970b] and Robert Easton [E,1971] came up with important results. Employing Morse theory,

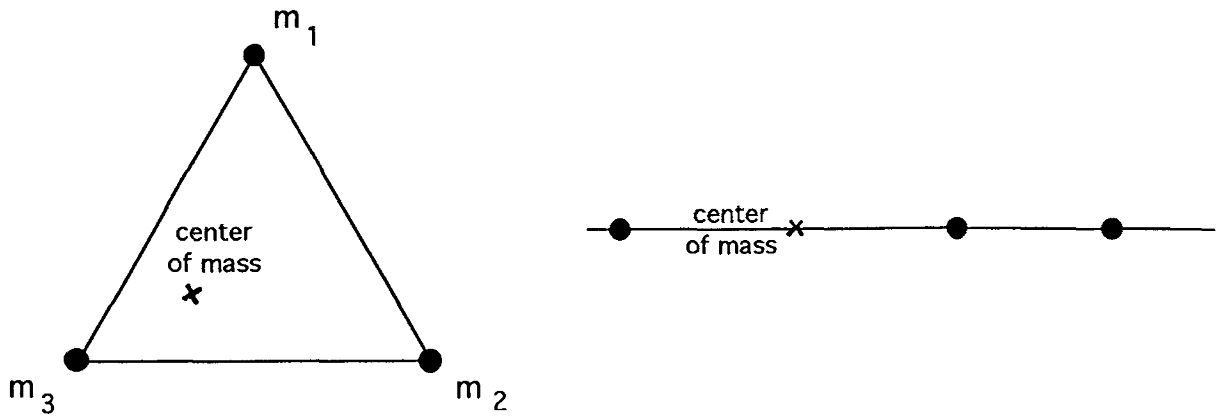

Figure 3. The equilateral and straight-line configurations of the 3-body problem.

Smale found the bifurcation types and proved Birkhoff's statement for $\mathbf{m}_5$. Independently, Easton described the topology of $\mathbf{m}_5$ in case of equal masses in terms of products of intervals, spheres, and tori. At that time Easton had just obtained his Ph.D. degree and was seeking recognition. He had considered the problem without knowing about Birkhoff's and Wintner's remarks, but in the summer of 1970 found out from a famous Brazilian mathematician, Mauricio Peixoto, that Smale had already obtained results in this sense. Anxious, he sent his paper to Smale, who replied with congratulations, saying that these contributions paralleled his.

Using the notes of his mentor Aristide Halanay of the University of Bucharest, who had attended Smale's lectures on the subject, the Romanian mathematician Andrei Iacob rewrote and completed Smale's program [I,1973]. Iacob's results were later included in the second edition of the book by Abraham and Marsden (see [AM,1978], Theorem 10.4.21). A few years later the Chinese mathematician X.Y. Chen published some interesting results on $\mathbf{m}_5$ [Ch,1978]. Rotating configurations into the plane, Chen reduced the spatial problem to the planar one. Unfortunately, due to technical difficulties, Chen missed the complete description of the topology of $\mathbf{M}_7$. Though the planar case could now be considered understood, the spatial one still resisted. But the offensive was strong and all the recent progress was seeding hope.

The attack on the topology of $\mathbf{M}_7$ was launched by the Brazilian mathematician Hildeberto Cabral, who in 1973 published the results of his Ph.D. thesis written in Berkeley under the supervision of Smale. Besides some results on $\mathbf{m}_5$, Cabral characterized $\mathbf{M}_7$ for negative energy and zero angular momentum. Robert Easton made the next step [E,1975]. He extended some of his previous results by projecting the 3-dimensional problem onto the plane. The idea of the projection method had already appeared in Cabral's paper, but the Brazilian mathematician did not pursue it. Easton obtained a series of nice results but unfortunately missed the fact that the topology of $\mathbf{M}_7$ may change not only at central configurations, but also at so-called *critical points at infinity.* These are values of the parameter ν that are not central configurations and appear in connection with the behavior of the energy function restricted to certain level-manifolds of the angular momentum (for a technical definition see [Al,1993], p. 475).

To give an idea of what a critical point at infinity means, here is an analogy. Imagine the intersection $L \cap C$ of the curve C in Figure 4 and the line L, which is parallel to the horizontal axis. When the line L moves up and down, the topology of the set $L \cap C$ changes at the finite critical points x_1, x_2, and x_3, but also at $+\infty$, because the curve C is asymptotic to the horizontal axis.

In 1970 Stephen Smale was the first to point out the possible existence of a bifurcation point at infinity. A few years later the Spanish mathematician Carles Simó [Si,1975] proved the existence of three such points at which the topology of $\mathbf{M}_7$ can change. In 1993, in a paper in which he attacked the more difficult problem of understanding the topology of integral manifolds in the n-body problem, the French mathematician Alain Albouy [Al,1993] showed that three was the maximum number of critical points at infinity at which the topology of $\mathbf{M}_7$ can change. All these implied that Birkhoff's statement was false in the 3-dimensional case.

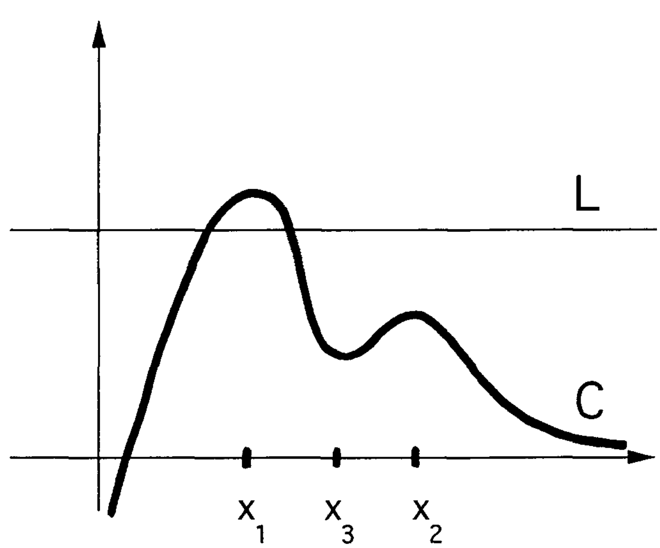

Figure 4. The idea of a critical point at infinity can be seen from the above picture: when the line L moves up and down, the topology of the set $L \cap C$ changes not only at the finite critical points x_1, x_2, and x_3, but at ∞ too.

Other important results were obtained by the American mathematician Donald Saari between 1984 and 1987. Simó had proved the presence of a bifurcation point of the parameter having an interesting property (see [Si,1975]). For values larger than it, there appear restrictions on the orientation of the plane of motion (e.g., the angular-momentum vector cannot lie in this plane), and for values smaller than it there are configurations with unrestricted orientation. Using a clever decomposition of the angular momentum, Saari gave a geometrical description of the integral manifolds in terms of a sphere bundle and completely explained the above restrictions ([Sa,1984], [Sa,1987a], [Sa,1987b]).

The Solution of the Original Problem

The announcement of the complete solution of the problem came in the fall of 1994 at a conference on Hamiltonian Systems and Celestial Mechanics in Cocoyoc, a small town south of Mexico City, in a hacienda founded by Hernando Cortez. The three authors, Chris McCord, Kenneth Meyer, and Quidong Wang of the University of Cincinnati, were all present. McCord, trained as a topologist, had learned about the problem from his colleague Ken Meyer, a leading researcher with many important results in celestial mechanics. Wang, who had just completed his Ph.D. degree with Meyer, was already known for the convergent power series solution he had obtained for the singularity-free n-body problem, a quest generalizing the problem attacked by Poincaré for King Oscar's competition (see [W,1991] and [Di,1992]).

Meyer had first heard of the possibility of bifurcations due to critical points at infinity from Alain Albouy at the 1991 conference on celestial mechanics in Guanajuato. Upon his return to Cincinnati, Meyer asked Wang whether he would like to read Albouy's paper. Wang was already familiar with Chen's and Saari's work and showed immediate interest. Then McCord joined the team.

This collaboration was a happy one. McCord was a master of algebraic topology. Beyond his erudition in dynamical systems, Meyer had a rich and fruitful research experience and a good feeling for avoiding traps; Wang brought to the team his courage, decisiveness, and enthusiasm. Having together all the ingredients for success at a time when the problem was ready to yield, McCord, Meyer, and Wang provided after many months of intense work a complete topological description of the integral manifolds associated to the 3-body problem. Their 90-page paper appeared in 1998 in the *Memoirs of the American Mathematical Society* [MW,1998] to rave reviews.

The main idea followed by McCord, Meyer, and Wang was to modify the rotation Chen used in [Ch,1978] and thus simplify some of the derived algebraic equations. This allowed them to overcome the difficulties that had stopped Chen. Their work involves several algebraic-topological techniques accessible only to specialists: Gysin and Mayer-Vietoris sequences, results due to Seifert and Van Kampen, Thom classes, bootstrapping, etc. The final result, however, is easy to grasp.

The integral manifolds are analysed with respect to the values of the parameter $\nu = -c^2h$, where h is the energy constant in (1) and c is derived from (2) by taking the reference frame such that the angular-momentum constant $\mathbf{c}$ is of the form $\mathbf{c} = (0, 0, c)$. There are 9 special values for ν at which the topology of integral manifolds may change. We must therefore ask about the topology in each of the ten intervals for ν: $\mathrm{I} = (-\infty, \nu_1)$, $\mathrm{II} = (\nu_1, \nu_2)$, $\mathrm{III} = (\nu_2, \nu_3)$, $\mathrm{IV} = (\nu_3, \nu_4)$, $\mathrm{V} = (\nu_4, \nu_5)$, $\mathrm{VI} = (\nu_5, \nu_6)$, $\mathrm{VII} = (\nu_6, \nu_7)$, $\mathrm{VIII} = (\nu_7, \nu_8)$, $\mathrm{IX} = (\nu_8, \nu_9)$, $\mathrm{X} = (\nu_9, \infty)$. The values ν_1, ν_2, ν_3, ν_4, and ν_5 correspond to critical points at infinity (ν_1 is due to the change from $h > 0$ to $h < 0$, and ν_2, ν_3, and ν_4 were found by Simó), whereas ν_6, ν_7, ν_8, and ν_9 are due to relative equilibria. Apparently the existence of ν_5 contradicts Albouy's finding that there are no other critical points at infinity except the ones of Smale and Simó. But in fact ν_5 is there; it's just that the topology of the integral manifolds remains unchanged at ν_5. This came out from the results of McCord, Meyer, and Wang, who computed the cohomology groups of $\mathbf{M}_7$ in each case. The following table summarizes their conclusions in terms of Betti numbers.

Betti number	β_0	β_1	β_2	β_3	β_4	β_5	β_6	β_7
I	1	0	0	2	0	0	0	0
II	1	0	4	0	5	0	2	0
III	1	0	4	0	3	0	2	0
IV	1	0	4	0	1	0	2	0
V	1	0	4	0	0	1	2	0
VI	1	0	4	0	0	1	2	0
VII	1	1	3	0	0	0	3	0
VIII	1	0	3	0	0	0	2	0
IX	2	0	3	0	0	0	1	0
X	3	0	3	0	0	0	0	0

ν_5, the critical point at infinity, was particularly troublesome for the team. The preliminary computations showed that the topology changed at ν_5, in contradiction with Albouy's previous conclusion. Intrigued, Meyer e-mailed a note to Albouy, who replied that he had three proofs for his result and saw no alternative. Soon Meyer understood that Albouy was right and convinced the others that the mistake must be their own. But it took several months of checking and rechecking their arguments until, to their relief, they found a mere computational error, which when corrected proved that the topology of $\mathbf{M}_7$ was unaffected at ν_5.

The work of McCord, Meyer, and Wang closes a century-long loop, solving a problem to which many others have made direct or indirect contributions. But this is not the end of the journey. New questions regarding the topology of integral manifolds associated to different restricted 3-body problems and to the n-body problem in general can now be attacked with the methods developed in all those years. Moreover, we might be able to understand better the geometry of the flow associated to the 3-body problem—a goal toward which Poincaré strived his entire life.

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