Relative Equilibria of the Curved N-Body Problem

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Preface

The guiding light of this monograph is a question easy to understand but difficult to answer: *What is the shape of the universe?* In other words, how do we measure the shortest distance between two points of the physical space? Should we follow a straight line, as on a flat table, fly along a great circle of a sphere, as between Paris and New York, or take some other course, and if so, what would that path look like? If we accept the model proposed here, which assumes that a Newtonian gravitational law extended to a universe of constant curvature is a good approximation of the physical reality (and we will later outline a few arguments in favor of this approach), then we can hint at a potential proof to the above question for distances comparable to those of our solar system. More precisely, this monograph provides a first step towards showing that, for distances of the order of 10 AU, space is Euclidean. Even if rigorously proved, this conclusion won't surprise astronomers, who accept the small-scale flatness of the universe due to the many observational confirmations they have. But the analysis of some recent spaceship orbits raises questions either about the geometry of space or our understanding of gravitation, [26].

Figure 1: Ernst Christian Friedrich Schering (1824-1897) was a professor at Georg-August University in Gttingen and a reluctant editor of Gauss's papers.

However, we cannot emphasize enough that the main goal of this monograph is mathematical. We aim to shed some light on the dynamics of N point masses that move in spaces of nonzero constant curvature according to an attraction law which extends classical Newtonian gravitation beyond $\mathbb{R}^3$. This natural generalization employs the cotangent potential, first introduced in 1870 by Ernst Schering, who obtained its analytic expression following the geometric approach of János Bolyai and Nikolai Lobachevsky for a 2-body problem in hyperbolic space, [87], [8], [70]. As Newton's idea of gravitation was to use a force inversely proportional to the area of a sphere of radius equal in length to the Euclidean distance between the bodies, Bolyai and Lobachevsky thought of a similar definition in terms of the hyperbolic distance in hyperbolic space. Our generalization of the cotangent potential to any number N of bodies led us to the recent discovery of some interesting properties, [35], [36]. These new results reveal certain connections among at least five branches of mathematics: classical dynamics, non-Euclidean geometry, geometric topology, Lie groups, and the theory of polytopes. But how does the astronomical aspect mentioned above relate to these mathematical endeavors?

Figure 2: Carl Friedrich Gauss (1777-1855), dubbed *Princeps Mathematicorum*, was arguably the greatest mathematician of all times.

To answer this question, let us get into some history. It appears that, sometime between 1818 and 1820, Carl Friedrich Gauss was the first ever to wonder about the shape of the physical space, [51]. Before him, the straight line was assumed to be the shortest distance between two points. What prompted Gauss to rebuff the general belief and conduct experiments to find the answer was his research into a question mathematicians had asked since ancient times[1]: *Does Euclid's fifth postulate follow*

[1]The first to doubt the self-evidence of Euclid's fifth postulate seems to have been the Greek neoplatonist philosopher Proclus Lycaeus (412-487), [99].

from the other axioms of classical geometry? Unlike what most of his contemporaries believed, Gauss reached the conclusion that it was independent, as Euclid had implied when introducing this axiom. But Gauss never made his ideas public, for he feared "the scream of the Boeotians," as he wrote to Friedrich Bessel in 1829, before learning about the independent advances of Bolyai (1823) and Lobachevsky (1826), [11], [8], [51]. The two young mathematicians had made explicit statements about the independence of the fifth postulate, based on their abundant evidence that a new geometry seemed to arise when this axiom was negated.

Gauss believed in a classical universe. He accepted Newton's mechanical model of a 3-dimensional space and an independent 1-dimensional time. In his view, gravitation was a universal force that acts through space. Like Bolyai and Lobachevsky, he understood that there is a strong connection between geometry and physics, a fact Poincaré took later to a different level in his philosophical essays about mathematics and science, [83]. Then Einstein brought matter, space, and time together in his general relativity, where gravitation became a geometric property of the 4-dimensional space-time, [41]. His model involves an expanding universe in which the geodesics may be straight lines, great circles of great spheres, or hyperbolic lines. More recently, string theory introduced new models of the universe with ten or eleven dimensions, [5].

This monograph is independent of the development of physics since Einstein because the property investigated here is valid in any model invented so far. We are interested only in the shortest distance between two points, an issue that does not interfere with an expanding universe. So we can restrict our study to Newton's approach, leaving open the possibility that space could be elliptic, flat, or hyperbolic.

There is another way to look at the problem. Rejecting Gauss's point of view, Poincaré thought that the laws of physics should be always expressed in terms of Euclidean geometry because this setting is simpler than any other geometry that could describe the surrounding reality. To him, if we found out someday that light travels along great circles instead of straight lines, we should accept this physical property and express all the laws of physics accordingly, within the Euclidean space, [83]. Under this assumption, however, the real difficulties are not overcome, but only pushed somewhere else.

By measuring the sum of the angles of topographic triangles, a result that could in principle decide the shape of the physical space, Gauss implicitly assumed that light obeys the geometry of the universe by travelling along geodesics. But his experiment failed to provide an answer because, if space is not flat, the difference between his topographical readings and the physical reality proved to be below the measurement errors that occur for triangles some 10 km wide.

The connection between the N-body problem in spaces of constant curvature and the shape of the physical space becomes now apparent. Instead of measuring angles of triangles, we can observe celestial motions. If some of the latter correspond only to orbits found in Euclidean space, and are proved mathematically not to exist in other spaces of constant curvature, then space must be flat, at least within the range of these orbits. As mentioned earlier, we will make a first step towards proving this statement for distances comparable to those of our solar system, i.e. of the order of 10 AU. Nevertheless, more work is necessary to achieve this goal.

This probable result, however, seems to be the limitation of the method, in the sense that it has little chance to apply to much larger distances. As Gauss was stuck with measuring angles of triangles on Earth, since he could not travel in space, and we cannot apply his approach to some triangle of stars because such cosmic objects are beyond our reach, the method of observing celestial motions has its limitations too. To become specific within a certain geometry alone, celestial orbits appear to be complicated enough only within the range of a planetary system. Galaxies and clusters merely move away from each other, so their simple dynamics could be found in physical spaces of any shape. The current astronomical methods have so far failed to detect whether these objects have more complicated orbits than radial motion.

But another intriguing aspect of the model given by the cotangent potential is that it includes the main features of a standard relativistic Big-Bang system, in which particles eject from a singularity. Under the classical assumption that space exists *a priori* instead of being created during the expansion of matter, these conclusions can be summarized in the table below (explained in Section 3.10):

Geometry	**Volume**	**Fate** [b]
elliptic, $\mathbb{S}^3$	finite	eventual collapse
Euclidean, $\mathbb{R}^3$	finite or infinite	eternal expansion or eventual collapse
hyperbolic, $\mathbb{H}^3$	finite or infinite	eternal expansion or eventual collapse [t]

This monograph presents some results we obtained since 2008. Finding a suitable framework, deriving the equations of motion that naturally extend the Newtonian N-body problem to curved space, and trying to understand some of the properties these equations possess, proved to be a highly gratifying experience. We hope that the reader will enjoy this intellectual adventure as much as we did.

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Victoria, B.C., Canada, 24 April 2012

Chapter 1

Introduction

In this introductory chapter, we provide the motivation that led us to this research, define the problem, explain its importance, outline its history, and present the structure of the monograph.

1.1 Motivation

In the early 1820s, with his recently invented heliotrope[1] (see Figure 1.1), Carl Friedrich Gauss, later dubbed *Princeps Mathematicorum*, allegedly tried to determine the nature of the physical space within the framework of classical mechanics, under the assumption that space and time exist a priori and are independent of each other, [75], [49]. He measured the angles of a triangle formed by three hills near Göttingen (Inselsberg, Brocken, and Hoher Hagen) and computed their sum, S, hoping to learn whether space is hyperbolic ($S < \pi$ radians) or elliptic ($S > \pi$ radians), [99]. But the results of his measurements did not deviate form π radians beyond the unavoidable measurement errors, so his experiment was inconclusive. Since we cannot reach distant stars to measure all the angles of some cosmic triangle, Gauss's method is of no practical use for astronomic distances either.[2]

[1]The heliotrope, which Gauss invented in 1820, is an instrument that uses a mirror to reflect sunlight in order to mark the positions of the land surveyors (see "The Gentleman's magazine," Volume 92, Part 2, July 1822, p. 358).

[2]There are other methods, such as cosmic crystallography and the Boomerang experiment, which show strong evidence that if the universe is not flat, its deviation from zero curvature must be very small, [98], [6]. But, in spite of their merits, these attempts rely on certain physical models, principles, and interpretations we do not need in the framework of celestial mechanics. Moreover, they do not yet provide a final answer to the question posed in the Preface.

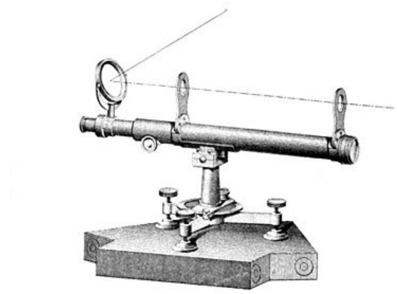

Figure 1.1: A drawing of a heliotrope designed by William Würdemann, a German-American optical instrument maker of the 19th century.

But celestial mechanics can help us find a new approach towards establishing the geometric nature of the physical space. The idea we suggest is to use the connection between geometry and dynamics, and reduce the problem to performing astronomical observations instead of taking measurements. To explain this method in more detail, let us recall that, in the 17th century, Isaac Newton derived the equations of motion of the N-body problem in Euclidean space, i.e. obtained the system of differential equations that describe the gravitational dynamics of N point masses, albeit in a geometric form, very different form how we write these equations today. Physicists agreed later that a universe of constant Gaussian curvature is a good approximation of the macrocosmic reality, a hypothesis they still accept. So if we succeed to extend Newton's gravitational law to 3-dimensional spheres and 3-dimensional hyperbolic manifolds, and also prove the existence of solutions that are specific to only one of the negative, zero, or positive constant Gaussian curvature spaces, but not to the other two, then the problem of understanding the geometric nature of the universe reduces to finding, through astronomical observations, some of the orbits proved mathematically to exist.

Therefore obtaining a natural extension of the Newtonian N-body problem to spaces of nonzero constant Gaussian curvature, such that the properties of the original potential are satisfied when taking the Euclidean space as a limit, and studying the system of differential equations thus derived, appears to be a worthy endeavor towards comprehending the geometry of the physical space. Additionally, an investigation of this system when the curvature tends to zero may help us better understand the dynamics of the classical Euclidean case, viewed as a particular problem within a more general mathematical framework. This general methodological approach has been useful in many branches of mathematics.

1.2 The problem

A first difficulty we are facing towards reaching our goal is that of finding a natural extension of the Newtonian N-body problem to spaces of nonzero constant curvature. Since there is no unique way of generalizing the classical equations of motion, to recover them at the limit, i.e. when the curved ambient space flattens out, we seek a potential that satisfies the same basic properties as the Newtonian potential in its simplest possible setting, that of one body moving around a fixed center, a pivotal question in celestial mechanics known as the Kepler problem. Two basic properties characterize it, a mathematical and a dynamical one: the Newtonian potential of the Kepler problem is a harmonic function in 3-dimensional (but not in 2-dimensional) space, i.e. it satisfies Laplace's equation, and generates a central field in which all bounded orbits are closed, a result proved by Joseph Louis Bertrand in 1873, [7].

On one hand, the cotangent potential we define in Chapter 3 approaches the classical Newtonian potential when the curvature tends to zero, whether through positive or negative values. In the Kepler problem, on the other hand, this potential satisfies Bertrand's property and is a 3-dimensional (but not a 2-dimensional) solution of the Laplace-Beltrami equation, the natural generalization of Laplace's equation to Riemannian and pseudo-Riemannian manifolds, which include the spaces of constant curvature $\kappa \neq 0$ we are interested in: the spheres $\mathbb{S}^3_\kappa$, for $\kappa > 0$, and the hyperbolic manifolds $\mathbb{H}^3_\kappa$, for $\kappa < 0$, [61]. For simplicity, we will further call the dynamical problem defined in these spaces: *the curved N-body problem*.

In the Euclidean case, the Kepler problem and the 2-body problem are equivalent. The reason for this likeness is the existence of the center-of-mass and linear-momentum integrals. It can be shown with their help that the behavior of the orbits is identical, whether the origin of the coordinate system is fixed at the center of mass or fixed at one of the two bodies. For nonzero curvature, however, things change. As we will later see, the equations of motion of the curved N-body problem lack the integrals of the center of mass and linear momentum, which prove to characterize only the Euclidean case. Consequently the curved Kepler problem and the curved 2-body problem are not equivalent anymore. It turns out that, as in the Euclidean case, the curved Kepler problem is integrable in the sense of Liouville (i.e. there is a maximal set of Poisson commuting invariants), but, unlike in the Euclidean case, the curved 2-body problem is not integrable, [94]. The curved N-body problem is not integrable for $N \geq 3$, a property that is also true in the Euclidean case. As we will show, however, the loss of symmetries, when passing from flat to curved space, seems to make the curved N-body problem less rich from the dynamical point of view, if compared to its classical counterpart.

As stated in Section 1.1, our main goal is to find solutions that are specific to each of the spaces corresponding to $\kappa < 0$, $\kappa = 0$, and $\kappa > 0$. We did that in some previously published papers, but only in the 2-dimensional case, [28], [37], [35], [36]. In this monograph, we will see that the 3-dimensional curved N-body problem puts into the evidence even more differences between the qualitative behavior of orbits in each of these spaces.

1.3 Importance

We already mentioned that the study of the curved N-body problem may help us better understand the geometry of the universe and put the classical Euclidean case in a broader mathematical perspective. In fact, in [33] and [35], we made a first step towards proving that space is Euclidean for distances of the order of 10 AU. The reason is the existence of Lagrangian orbits of unequal masses (e.g., the equilateral triangles formed by the Sun, Jupiter, and the Trojan/Greek asteroids) in our solar system and the fact that, for nonzero curvature, 2-dimensional Lagrangian solutions exist only if the masses are equal, a fact which makes them unlikely to form in a universe of nonzero constant curvature. We will present and analyze these aspects in detail in the last part of this monograph.

These results, however, only hint at the fact that space is Euclidean for solar-system scales. Indeed, the motions of the Trojan and Greek asteroids are not exactly at the vertices of equilateral triangles but close to them, the physical existence of these orbits following from their stability for a certain range of masses. A better way to approach this issue is through the circular restricted 3-body problem, when the Trojan and Greek asteroids are close to moving on 3-dimensional invariant tori, but the values of the inclinations are large for most of these little planets. Nonetheless, this approach must also take into account the perturbation from the other planets. Finally, we do not yet exclude the existence of Lagrangian quasiperiodic orbits of unequal masses in 3-dimensional curved space, motions that might be hard to distinguish in practice from the orbits we observe in our solar system. So a final answer about the curvature of our local universe still awaits an extensive investigation of the differential equations we will derive here.

Another practical aspect that has not been studied so far has to do with motion in the neighborhood of large celestial bodies, such as the Sun. While for distances of 1 AU or so, space can be assumed to have constant curvature, light rays are known to bend near large cosmic objects. These things are explained in terms of general relativity, but the ideas that led to the derivation of the curved N-body problem may be also used to study this bending with classical tools.

Apart from these philosophical and applicative aspects, the curved N-body problem lays bridges between the theory of dynamical systems and several other branches of mathematics: the geometry and topology of 3-dimensional manifolds of constant curvature, Lie group theory, and the theory of regular polytopes, all of which we use in this monograph, and with differential geometry and Lie algebras, as already shown in [37] and [81].

As the results we prove here point out, many of which are surprising and non-intuitive, the geometry of the space in which the bodies move strongly influences their dynamics, including some important qualitative properties, stability among them. Topological concepts, such as the Hopf fibration and the Hopf link; geometric objects, like the pentatope, the Clifford torus, and the hyperbolic cylinder; or geometric properties, such as the Hegaard splitting of genus 1, become essential tools for understanding the gravitational motion of the bodies.

Since we provide here only a first study in this direction of research, reduced to the simplest orbits the equations of motion have, namely the relative equilibria (and the fixed points in the case of positive curvature), we expect that many more connections between dynamics, geometry, topology, and other branches of mathematics will be discovered in the near future through a deeper exploration of the curved 3-dimensional problem.

1.4 History

The first researchers who took the idea of gravitation beyond the Euclidean space were Nikolai Lobachevsky and János Bolyai, the founders of hyperbolic geometry. In 1835, Lobachevsky proposed a Kepler problem in the 3-dimensional hyperbolic space, $\mathbb{H}^3$, by defining an attractive force proportional to the inverse area of the 2-dimensional sphere of radius equal in length to the distance between bodies, [70]. Independently of him, and at about the same time, Bolyai came up with a similar idea, [8]. Both of them grasped the intimate connection between geometry and physical laws, a relationship that proved very prolific ever since. Gauss also understood that hyperbolic geometry existed independently, but never went as far as Bolyai and Lobachevsky, and there is no historical evidence (until now) that he thought of how to extend Newton's gravitational law beyond the Euclidean space.

These co-discoverers of the first non-Euclidean geometry had no followers in their pre-relativistic attempts until 1860, when Paul Joseph Serret[3] extended the gravi-

[3]Paul Joseph Serret (1827-1898) should not be mixed up with another French mathematician, Joseph Alfred Serret (1819-1885), known for the Frenet-Serret formulas of vector calculus.

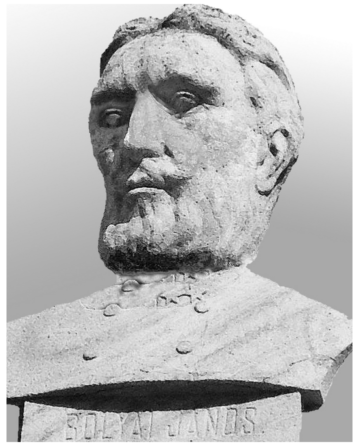

Figure 1.2: János Bolyai (1802-1860) was a Hungarian mathematician born in Transylvania (then in Hungary, now in Romania), where he spent most of his life. This sculpture on the frontispiece of the Culture Palace in Târgu Mureş (known in Hungarian as Maros-Vásárhely) seems to be the only true image left of him, [24].

tational force to the sphere $\mathbb{S}^2$ and solved the corresponding Kepler problem, [90]. Ten years later, Ernst Schering revisited the Bolyai-Lobachevsky law, for which he obtained an analytic expression given by the curved cotangent potential we study in this paper, [87]. Schering also wrote that Lejeune Dirichlet had told some friends to have dealt with the same problem during his last years in Berlin[4], but Dirichlet never published anything in this direction, and we found no evidence of any unpublished manuscripts in which he had studied this particular topic, [88]. In 1873, Rudolph Lipschitz considered the problem in $\mathbb{S}^3$, but defined a potential proportional to $1/\sin\frac{r}{R}$, where r denotes the distance between bodies and R is the curvature radius, [69]. He succeeded to obtain the general solution of the corresponding differential equations only in terms of elliptic functions, a dead end nobody followed. His failure to provide an explicit formula, which could have helped him draw some conclusions about the motion of the bodies, showed the advantage of Schering's approach.

In 1885, Wilhelm Killing adapted the Bolyai-Lobachevsky gravitational law to $\mathbb{S}^3$ and defined an extension of the Newtonian force given by the inverse area of a 2-dimensional sphere (in the spirit of Schering), for which he proved a generalization of Kepler's three laws, a step which suggested that the cotangent potential was a good way to extend the classical approach, [59]. Then a breakthrough took place. In 1902, Heinrich Liebmann[5] showed that the orbits of the Kepler problem are conics

[4]This must have happened around 1852, as claimed by Rudolph Lipschitz, [68].

[5]Although he signed his papers and books as Heinrich Liebmann, his full name was Karl Otto

Figure 1.3: Nikolai Ivanovich Lobachevsky (1792-1856) was a Russian mathematician at the University of Kazan, where he also served as president. As in the case of Bolyai, his mathematical work received little recognition during his life time.

in $\mathbb{S}^3$ and $\mathbb{H}^3$ and further generalized Kepler's three laws to $\kappa \neq 0$, [65]. One year later, he proved $\mathbb{S}^2$- and $\mathbb{H}^2$-analogues of Bertrand's theorem, which states that for the Kepler problem there exist only two analytic central potentials in the Euclidean space for which all bounded orbits are closed, one of which corresponds to gravitation, [7], [100], [66]. Liebmann also included his results in the last chapter of a book on hyperbolic geometry published in 1905, which saw two more editions, one in 1912 and the other in 1923, [67]. Intriguing enough, in the third edition he replaced the constant-curvature approach with relativistic considerations.

Liebmann's change of mind about the importance of the constant-curvature approach may explain why this direction of research was ignored in the decades immediately following the birth of special and general relativity. The reason for this neglect was probably connected to the idea that general relativity could allow the study of 2-body problems on manifolds with variable Gaussian curvature, so the constant-curvature case appeared to be outdated. Indeed, although the most important subsequent success of relativity was in cosmology and related fields, there were attempts to discretize Einstein's equations and define a gravitational N-body problem. Remarkable in this direction were the contributions of Jean Chazy, [15],

Heinrich Liebmann (1874-1939). He did most of his work in Heidelberg, where he became briefly the university's president, before the Nazis forced him to retire. A remembrance colloquium was held in his honor at the University of Heidelberg in 2008. On this occasion, Liebmann's son, Karl-Otto Liebmann, also and academic, donated to the Faculty of Mathematics an oil portrait of his father, painted by Adelheid Liebmann, [91]. This author had the opportunity to see that portrait on a visit to Heidelberg in 2009, when Willi Jäger, his former Ph.D. supervisor, showed it to him.

Figure 1.4: Karl Otto Heinrich Liebmann (1874-1939) was a professor at Ruprecht-Karls University in Heidelberg, where he also served as president.

Tullio Levi-Civita, [62], [63], Arthur Eddington, [40], Albert Einstein, Leopold Infeld[6], Banesh Hoffmann, [42], and Vladimir Fock, [46]. Subsequent efforts led to refined post-Newtonian approximations (see, e.g., [20], [21], [22]), which prove very useful in practice, from understanding the motion of artificial satellites—a field with applications in geodesy and geophysics—to using the GPS, [23]. But the equations of the N-body problem derived from relativity are highly complicated even for $N = 2$ and are not prone to analytical studies similar to the ones done in the classical case. This is probably the reason why the need for some simpler equations revived the research on the motion of 2 bodies in spaces of constant curvature.

In 1940, Erwin Schrödinger developed a quantum-mechanical analogue of the Kepler problem in $\mathbb{S}^2$, [89]. He used the same cotangent potential, which he deemed to be the natural extension of Newton's law to the sphere[7]. Further results in this direction were obtained by Leopold Infeld, [54], [97]. In 1945, Infeld and his student Alfred Schild extended this problem to spaces of constant negative curvature using the hyperbolic-cotangent potential, [56]. A comprehensive list of the contributions mentioned above is given in [92], except for Serret's book, [90], which was inadvertently omitted. An extensive bibliography of works on dynamical problems in spaces of constant curvature published before 2006 appears in [94] and [95].

The Russian school of celestial mechanics led by Valeri Kozlov also studied the curved 2-body problem given by the cotangent potential and, starting with the 1990s,

[6]A vivid description of the collaboration between Einstein and Infeld appears in Infeld's autobiographical book [55].

[7]"The correct form of [the] potential (corresponding to $1/r$ of the flat space) is known to be $\cot\chi$," [89], p. 14.

considered related problems in spaces of constant curvature, [61]. An important contribution to the case $N = 2$ and the Kepler problem belongs to José Cariñena, Manuel Rañada, and Mariano Santander, who provided a unified approach in the framework of differential geometry with the help of intrinsic coordinates, emphasizing the dynamics of the cotangent potential in $\mathbb{S}^2$ and $\mathbb{H}^2$, [13] (see [14] and [50] as well). They also proved that the conic orbits known in Euclidean space extend naturally to spaces of constant curvature, in agreement with the results obtained by Liebmann, [92]. Moreover, the authors used the rich geometry of the hyperbolic plane to search for new orbits, whose existence they either proved or conjectured. The study of their paper made us look for a way to write down the equations of motion for any number $N \geq 2$ of bodies.

Thus inspired, we proposed a new setting, which allowed us an easy derivation of the equations of motion for any $N \geq 2$ in terms of extrinsic coordinates, [35]. The combination of two main ideas helped us achieve this goal: the use of Weierstrass's model of the hyperbolic plane (also known as the Lorentz model on the hyperbolic sphere), which is embedded in the 3-dimensional Minkowski space, and the application of the variational approach of constrained Lagrangian dynamics, [48]. The equations we obtained are valid in any finite dimension. In [35] we explored relative equilibria and solved Saari's conjecture in the collinear case (see also [34], [31]), in [36] and [27] we studied the singularities of the curved N-body problem, in [33] we gave a complete classification of the Lagrangian and Eulerian solutions in the 3-body case, and in [28] we obtained some results about polygonal orbits, including a generalization of the Perko-Walter-Elmabsout theorem to both the case of relative equilibria and when expansion/contraction takes place, [43], [82]. Ernesto Pérez-Chavela and J. Guadalupe Reyes Victoria derived the equations of motion in intrinsic coordinates in the case of positive curvature and showed them to be equivalent with the extrinsic equations, [81]. The analysis of the intrinsic equations in the case of negative curvature was done in [37]. This study is more complicated than the one for positive curvature, involving both the Poincaré disk and the upper-half-plane model. Some preliminary results about relative equilibria in the 3-dimensional case occur in [29], out of which many properties will be presented here.

A study of the stability of Lagrangian relative equilibria of the curved 3-body problem in $\mathbb{S}^2$ appeared in [73]. Regina Martínez and Carles Simó thus discovered two zones on the sphere in which the orbit is stable. This result is surprising because equal-mass Lagrangian orbits are unstable in the Euclidean case. Currently we are performing a study of the stability of tetrahedral orbits for the curved 4-body problem in $\mathbb{S}^2$, [32].

1.5 Structure

In what follows, we will present some results obtained in the 2- and 3-dimensional curved N-body problem in the context of differential equations in extrinsic coordinates, as derived in [35]. We are mainly concerned with understanding the motion of the simplest possible orbits, the relative equilibria, which move like rigid bodies by maintaining constant mutual distances for all time.

The main part of this monograph is structured in five parts. Part I provides the mathematical background and gives a complete derivation of the equations of motion. Part II deals with isometries, whose group representations allow us to meaningfully define relative equilibria for spheres and hyperbolic manifolds. Part III is concerned with finding existence criteria for relative equilibria and describing the qualitative behavior of these orbits. Part IV gives many explicit examples of relative equilibria, which illustrate each type of dynamical behavior previously described. Finally, Part V focuses on the 2-dimensional case, provides a proof that Lagrangian orbits must have equal masses (the result that throws some light on the shape of the physical space), and ends with an extension of Saari's conjecture to spaces of constant curvature, which is proved in the geodesic case.

The parts are divided into chapters, numbered increasingly throughout the text. Each part starts with a preamble that provides a brief idea of what to expect, and the beginning of each chapter outlines the main properties we found. To avoid repetition, we will not describe those results here. The rest of the text can therefore be read linearly or by going first to the preambles of the parts and the introductions to the chapters to get a general idea about what results will be proved.

We finally reached the starting point into a mathematical universe that will reveal connections between several branches of mathematics and bring some insight into the geometry of the physical space. We hope that the above introduction has given the reader enough reasons to explore this new world.

Part I

Background and Equations of Motion

Preamble

The goal of Part I is to lay the mathematical foundations for future developments and to obtain the equations of motion of the curved N-body problem together with their first integrals. We will also identify the singularities of the equations of motion in order to avoid impossible configurations for the relative equilibria we are going to construct in Part V. A basic property we prove, which will simplify our presentation, is that, up to its sign, the curvature can be eliminated from the equations of motion through suitable coordinate and time-rescaling transformations. Consequently our study can be reduced to the unit sphere, $\mathbb{S}^3$, for positive curvature, and the unit hyperbolic sphere, $\mathbb{H}^3$, for negative curvature.

Chapter 2

Preliminary developments

In this chapter we will introduce some concepts needed for the derivation of the equations of motion of the N-body problem in spaces of constant curvature as well as for the study of the relative equilibria, which are special classes of orbits that we will investigate later. We will start by introducing a model for hyperbolic geometry, usually attributed to Hendrik Lorentz, but actually due to Karl Weierstrass.

The 2-sphere is familiar to everybody as a complete and compact surface of constant positive Gaussian curvature, $\kappa = 1/R^2$, where R is the radius. But the understanding of its dual, the hyperbolic sphere, which has constant negative curvature, $\kappa = -1/R^2$, needs more imagination. This object is connected to hyperbolic geometry. The standard models of 2-dimensional hyperbolic geometry are the Poincaré disk and the Poincaré upper-half plane, which are conformal, i.e. maintain the angles existing in the hyperbolic plane, as well as the Klein-Beltrami disk, which is not conformal. But we won't use any of these models here.

In the first part of this chapter, we will introduce a less known model, due to Karl Weierstrass, which geometers call the hyperbolic sphere or, sometimes ambiguously, the pseudosphere, whereas physicists refer to it as the Lorentz model. We will first present the 2-dimensional case, which can be easily extended to 3 dimensions. This model is more natural than the ones previously mentioned in the sense that it is analytically similar to the sphere, and will thus be essential in our endeavors to develop a unified N-body problem in spaces of constant positive and negative curvature. We will then provide a short history of this model, and introduce some geometry concepts that are going to be useful later. In the last part of this chapter, we will define the natural metric of the sphere and of the hyperbolic sphere and unify circular and hyperbolic trigonometry in order to introduce a single potential function for both the positive and the negative curvature case.

2.1 The hyperbolic sphere

Since Weierstrass's model of hyperbolic geometry is not widely known among nonlinear analysts or experts in differential equations and dynamical systems, we will briefly present it here. We first discuss the 2-dimensional case, which we will then extend to 3 dimensions. In its 2-dimensional form, this model appeals for at least two reasons: it allows an obvious comparison with the 2-sphere, both from the geometric and from the algebraic point of view, and emphasizes the difference between the hyperbolic (Bolyai-Lobachevsky) and the Euclidean plane as clearly as the well-known difference between the Euclidean plane and the sphere. From the dynamical point of view, the equations of motion of the curved N-body problem in $\mathbb{S}^3_\kappa$ resemble the equations of motion in $\mathbb{H}^3_\kappa$, with just a few sign changes, as we will show soon. The dynamical consequences, however, are going to be significant, but we will be able to use the resemblances between the sphere and the hyperbolic sphere to study the dynamics of the problem.

The 2-dimensional Weierstrass model is built on the hyperbolic sphere, represented by one of the sheets of the hyperboloid of two sheets,

$$x^2 + y^2 - z^2 = \kappa^{-1},$$

where $\kappa < 0$ represents the curvature of the surface in the 3-dimensional Minkowski space $\mathbb{R}^{2,1} := (\mathbb{R}^3, \boxdot)$, in which

$$\mathbf{a} \boxdot \mathbf{b} := a_x b_x + a_y b_y - a_z b_z,$$

defines the Lorentz inner product of the vectors $\mathbf{a} = (a_x, a_y, a_z)$ and $\mathbf{b} = (b_x, b_y, b_z)$. We pick for our model the sheet $z > 0$ of the hyperboloid of two sheets and identify this connected component with the hyperbolic plane $\mathbb{H}^2_\kappa$. We can think of this surface as being a pseudosphere of imaginary radius iR, a case in which the relationship between radius and curvature is given by $(iR)^2 = \kappa^{-1}$.

A linear transformation $T\colon \mathbb{R}^{2,1} \to \mathbb{R}^{2,1}$ is called orthogonal if it preserves the inner product, i.e.

$$T(\mathbf{a}) \boxdot T(\mathbf{a}) = \mathbf{a} \boxdot \mathbf{a}$$

for any $\mathbf{a} \in \mathbb{R}^{2,1}$. The set of these transformations, together with the Lorentz inner product, forms the orthogonal Lie group $O(\mathbb{R}^{2,1})$, given by matrices of determinant ± 1. Therefore the connected Lie group $SO(\mathbb{R}^{2,1})$ of orthogonal transformations of determinant 1 is a subgroup of $O(\mathbb{R}^{2,1})$. Another subgroup of $O(\mathbb{R}^{2,1})$ is $G(\mathbb{R}^{2,1})$, which is formed by the transformations T that leave $\mathbb{H}^2_\kappa$ invariant. Furthermore, $G(\mathbb{R}^{2,1})$ has the closed Lorentz subgroup, $\mathrm{Lor}(\mathbb{R}^{2,1}) := G(\mathbb{R}^{2,1}) \cap SO(\mathbb{R}^{2,1})$.

An important result, with deep consequences in our paper, is the principal axis theorem for $\mathrm{Lor}(\mathbb{R}^{2,1})$, [38], [52]. To present it, let us define the Lorentzian rotations about an axis as the 1-parameter subgroups of $\mathrm{Lor}(\mathbb{R}^{2,1})$ that leave the axis pointwise fixed. Then the principal axis theorem states that every Lorentzian transformation has one of the matrix representations:

$$A = P \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} P^{-1},$$

$$B = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh s & \sinh s \\ 0 & \sinh s & \cosh s \end{bmatrix} P^{-1},$$

or

$$C = P \begin{bmatrix} 1 & -t & t \\ t & 1 - t^2/2 & t^2/2 \\ t & -t^2/2 & 1 + t^2/2 \end{bmatrix} P^{-1},$$

where $\theta \in [0, 2\pi)$, $s, t \in \mathbb{R}$, and $P \in \mathrm{Lor}(\mathbb{R}^{2,1})$. These transformations are called elliptic, hyperbolic, and parabolic, respectively. They are all isometries, i.e. preserve the distances in $\mathbb{H}^2_\kappa$.

In its standard setting of Einstein's relativity, the Minkowski space has time and space coordinates. In our case, however, all coordinates are spatial. Nevertheless, we will use the standard terminology and say that the elliptic transformations are rotations about a *timelike* axis (the z axis in our case) and act along a circle, like in the spherical case; the hyperbolic rotations are about a *spacelike* axis (the x axis in this context) and act along a hyperbola; and the parabolic transformations are rotations about a *lightlike* (or *null*) axis (represented here by the line $x = 0$, $y = z$) and act along a parabola. This result is analogous to Euler's principal axis theorem for the sphere, which states that any element of the rotation group $SO(3)$ can be written, in some orthonormal basis, as a rotation about the z axis.

The geodesics of $\mathbb{H}^2_\kappa$ are the hyperbolas obtained by intersecting the hyperbolic sphere with planes passing through the origin of the coordinate system. For any two distinct points $\mathbf{a}$ and $\mathbf{b}$ of $\mathbb{H}^2_\kappa$, there is a unique geodesic that connects them, and the distance between these points is given by

$$d(\mathbf{a}, \mathbf{b}) = (-\kappa)^{-1/2} \cosh^{-1}(\kappa \mathbf{a} \boxdot \mathbf{b}). \tag{2.1}$$

In the framework of Weierstrass's model, the parallels' postulate of hyperbolic geometry can be translated as follows. Take a geodesic γ, i.e. a hyperbola obtained by intersecting the hyperbolic sphere with a plane through the origin, O, of the coordinate system. This hyperbola has two asymptotes in its plane: the straight lines a and b, which intersect at O. Take a point, P, on the upper sheet of the hyperboloid but not on the chosen hyperbola. The plane aP produces the geodesic hyperbola α, whereas bP produces β. These two hyperbolas intersect at P. Then α and γ are parallel geodesics meeting at infinity along a, while β and γ are parallel geodesics meeting at infinity along b. All the hyperbolas between α and β (also obtained from planes through O) are non-secant with γ.

Like the Euclidean plane, the abstract Bolyai-Lobachevsky plane has no privileged points or geodesics. But the Weierstrass model, given by the hyperbolic sphere, has some convenient points and geodesics, such as the point $(0, 0, |\kappa|^{-1/2})$, namely the vertex of the sheet $z > 0$ of the hyperboloid, and the geodesics passing through it. The elements of $\mathrm{Lor}(\mathbb{R}^{2,1})$ allow us to move the geodesics of $\mathbb{H}^2_\kappa$ to convenient positions, a property that can be used to simplify certain arguments.

More detailed introductions to the 2-dimensional Weierstrass model can be found in [45] and [84]. The Lorentz group is treated in some detail in [4] and [84], but the principal axis theorems for the Lorentz group contained in [4] fails to include parabolic rotations, and is therefore incomplete.

The generalization of the hyperbolic 2-sphere to 3 dimensions is straightforward. Consider first the 4-dimensional Minkowski space $\mathbb{R}^{3,1} = (\mathbb{R}^4, \boxdot)$, where $\boxdot$ is now defined as the Lorentz inner product

$$\mathbf{a} \boxdot \mathbf{b} = a_w b_w + a_x b_x + a_y b_y - a_z b_z,$$

with $\mathbf{a} = (a_w, a_x, a_y, a_z)$ and $\mathbf{b} = (b_w, b_x, b_y, b_z)$ belonging to $\mathbb{R}^{3,1}$. We further embed in this Minkowski space the connected component with $z > 0$ of the 3-dimensional hyperbolic manifold given by the equation

$$w^2 + x^2 + y^2 - z^2 = \kappa^{-1}, \tag{2.2}$$

which models the hyperbolic 3-sphere $\mathbb{H}^3_\kappa$ of constant curvature $\kappa < 0$. The distance is given by the same formula (2.1), where $\mathbf{a}$ and $\mathbf{b}$ are now points in $\mathbb{R}^4$ that lie in the hyperbolic 3-sphere (2.2) with $z > 0$.

The next issue to discuss would be that of Lorentzian transformations in $\mathbb{H}^3_\kappa$, i.e. the elements of the corresponding Lorentz group. But we postpone this topic, to present it in Chapter 4 together with the isometries of the 3-sphere and the group $SO(4)$ that generates them. There are two good reasons for this postponement. First, we would like to obtain the equations of motion of the curved N-body problem as soon as possible and, second, to present the similarities and the differences between these transformations in a larger geometrical context than the one considered here.

Some historical remarks

The idea of an "imaginary sphere," at a time when mathematicians were knocking unsuccessfully at the gates of hyperbolic geometry, seems to have first appeared in 1766 in the work of Johann Heinrich Lambert, [9]. In 1826, Franz Adolph Taurinus referred to a "sphere of imaginary radius" in connection with some trigonometric research, [9], but was clearly unaware of the work of Bolyai and Lobachevsky, who were then trying, independently of each other, to make progress in the new universe of the hyperbolic plane. Neither Lambert, nor Taurinus, seem to have viewed these concepts as we do today.

The first mathematician who mentioned Karl Weierstrass in connection with the hyperbolic sphere was Wilhelm Killing. In a paper published in 1880, [58], he used what he called Weierstrass's coordinates to describe the "exterior hyperbolic plane" as an "ideal region" of the Bolyai-Lobachevsky plane. In 1885, he added that Weierstrass had introduced these coordinates, in combination with "numerous applications," during a seminar held in 1872, but also mentioned that Eugenio Beltrami had previously used some related coordinates, [60], pp. 258-259. We found no other evidence of any written account of the hyperbolic sphere for the Bolyai-Lobachevsky plane prior to the one Killing gave in [60], p. 260. His remarks might have inspired Richard Faber to name this model after Weierstrass and to dedicate a chapter to it in [45], pp. 247-278.

2.2 More geometric background

Since we are interested in the motion of point particles on 3-dimensional manifolds, the natural background structure for the study of the 3-dimensional curved N-body problem is the Euclidean ambient space, $\mathbb{R}^4$, endowed with a specific inner-product, which depends on whether the curvature is positive or negative. For positive constant curvature, $\kappa > 0$, the motion takes place on a 3-sphere embedded in the Euclidean space $\mathbb{R}^4$, endowed with the standard dot product, $\cdot$, i.e. on the manifold

$$\mathbb{S}^3_\kappa = \{(w, x, y, z) \,|\, w^2 + x^2 + y^2 + z^2 = \kappa^{-1}\}.$$

For negative constant curvature, $\kappa < 0$, the motion takes place on the hyperbolic sphere, the manifold introduced in the previous subsection, represented by the upper connected component of a 3-dimensional hyperboloid of two connected components embedded in the Minkowski space $\mathbb{R}^{3,1}$, i.e. on the manifold

$$\mathbb{H}^3_\kappa = \{(w, x, y, z) \,|\, w^2 + x^2 + y^2 - z^2 = \kappa^{-1},\ z > 0\},$$

where $\mathbb{R}^{3,1}$ is $\mathbb{R}^4$ endowed with the Lorentz inner product, $\boxdot$. Generically, we will denote these manifolds by

$$\mathbb{M}^3_\kappa = \{(w,x,y,z) \in \mathbb{R}^4 \,|\, w^2 + x^2 + y^2 + \sigma z^2 = \kappa^{-1}, \text{ with } z > 0 \text{ for } \kappa < 0\},$$

where σ is the signum function,

$$\sigma = \begin{cases} +1, & \text{for} \quad \kappa > 0 \\ -1, & \text{for} \quad \kappa < 0. \end{cases} \tag{2.3}$$

In Section 3.7, we will show that, using suitable coordinate and time-rescaling transformations, we can reduce the mathematical study of the equations of motion to the unit sphere,

$$\mathbb{S}^3 = \{(w,x,y,z) \,|\, w^2 + x^2 + y^2 + z^2 = 1\},$$

for positive curvature, and to the unit hyperbolic sphere,

$$\mathbb{H}^3 = \{(w,x,y,z) \,|\, w^2 + x^2 + y^2 - z^2 = -1\},$$

for negative curvature. Generically, we will denote these manifolds by

$$\mathbb{M}^3 = \{(w,x,y,z) \in \mathbb{R}^4 \mid w^2 + x^2 + y^2 + \sigma z^2 = \sigma, \text{ with } z > 0 \text{ for } \kappa < 0\}.$$

Given the 4-dimensional vectors

$$\mathbf{a} = (a_w, a_x, a_y, a_z) \quad \text{and} \quad \mathbf{b} = (b_w, b_x, b_y, b_z),$$

we define their inner product as

$$\mathbf{a} \odot \mathbf{b} := a_w b_w + a_x b_x + a_y b_y + \sigma a_z b_z, \tag{2.4}$$

so $\mathbb{M}^3_\kappa$ is endowed with the operation $\odot$, meaning $\cdot$ for $\kappa > 0$ and $\boxdot$ for $\kappa < 0$.

If R is the radius of the sphere $\mathbb{S}^3_\kappa$, then the relationship between $\kappa > 0$ and R is $\kappa^{-1} = R^2$. As we already mentioned, to have an analogue interpretation in the case of negative curvature, $\mathbb{H}^3_\kappa$ can be viewed as a hyperbolic 3-sphere of imaginary radius iR, such that the relationship between $\kappa < 0$ and iR is $\kappa^{-1} = (iR)^2$.

Let us further define some concepts that will be useful later. Since we are going to work with them only in the context of $\mathbb{S}^3$ and $\mathbb{H}^3$, i.e. for $\kappa = 1$ and $\kappa = -1$, respectively, we will introduce them relative to these manifolds.

Definition 1. *A great sphere of $\mathbb{S}^3$ is a 2-sphere of radius 1.*

Great spheres of $\mathbb{S}^3$ are obtained by intersecting $\mathbb{S}^3$ with hyperplanes of $\mathbb{R}^4$ that pass through the origin of the coordinate system. Examples of great spheres are:

$$\mathbf{S}_w^2 = \{(w, x, y, z) \,|\, x^2 + y^2 + z^2 = 1,\ w = 0\} \tag{2.5}$$

for all possible relabelings of the variables.

Definition 2. *A great circle of a great sphere of* $\mathbb{S}^3$ *is a* 1*-sphere of radius* 1*.*

Definition 2 implies that the curvature of a great circle is the same as the curvature of $\mathbb{S}^3$. Notice that this is not true in general, i.e. for spheres of curvature κ. Then the curvature of a great circle is $\kappa = 1/R$, where R is the radius of $\mathbb{S}_\kappa^3$ and of the great circle.

A great circle can be obtained by intersecting a great sphere with a plane passing through the origin of the coordinate system. Examples of great circles in $\mathbb{S}^3$ are:

$$\mathbf{S}_{wx}^1 = \{(w, x, y, z) \,|\, y^2 + z^2 = 1,\ w = x = 0\} \tag{2.6}$$

for all possible relabelings of the variables. Notice that $\mathbf{S}_{wx}^1$ is a great circle for both the great spheres $\mathbf{S}_w^2$ and $\mathbf{S}_x^2$, whereas $\mathbf{S}_{yz}^1$ is a great circle for the great spheres $\mathbf{S}_y^2$ and $\mathbf{S}_z^2$. Similar remarks can be made about any of the above great circles.

Definition 3. *Two great circles, C_1 and C_2, of two distinct great spheres of* $\mathbb{S}^3$ *are called complementary if there is a coordinate system $wxyz$ such that either of the following two conditions is satisfied:*

$$C_1 = \mathbf{S}_{wx}^1 \quad \text{and} \quad C_2 = \mathbf{S}_{yz}^1, \tag{2.7}$$

$$C_1 = \mathbf{S}_{wy}^1 \quad \text{and} \quad C_2 = \mathbf{S}_{xz}^1. \tag{2.8}$$

The conditions (2.7) and (2.8) for C_1 and C_2 exhaust all possibilities. Indeed, the representation

$$C_1 = \mathbf{S}_{wz}^1 \quad \text{and} \quad C_2 = \mathbf{S}_{xy}^1,$$

for instance, is the same as (2.7) after we perform a circular permutation of the coordinates w, x, y, z. For simplicity, and without loss of generality, we will always use representation (2.7).

In topological terms, the complementary circles C_1 and C_2 of $\mathbb{S}^3$ form a Hopf link in a Hopf fibration, which is the map

$$\mathcal{H}\colon \mathbb{S}^3 \to \mathbb{S}^2,\ \mathcal{H}(w, x, y, z) = (w^2 + x^2 - y^2 - z^2, 2(wz + xy), 2(xz - wy))$$

that takes circles of $\mathbb{S}^3$ to points of $\mathbb{S}^2$, [53], [71]. In particular, $\mathcal{H}$ takes $\mathbf{S}^1_{wx}$ to $(1,0,0)$ and $\mathbf{S}^1_{yz}$ to $(-1,0,0)$. Using the stereographic projection, it can be shown that the circles C_1 and C_2 are linked (like any adjacent rings in a chain), hence the name of the pair, [71]. Hopf fibrations have important physical applications in fields such as rigid body mechanics, [72], quantum information theory, [77], and magnetic monopoles, [78]. As we will see later, they are also useful in celestial mechanics via the curved N-body problem.

We will show in the next section that the distance between two points lying on complementary great circles is independent of their position. This remarkable geometric property turns out to be even more surprising from the dynamical point of view. Indeed, given the fact that the distance between 2 complementary great circles is constant, the magnitude of the gravitational interaction (but not the direction of the force) between a body lying on a great circle and a body lying on the complementary great circle is the same, no matter where the bodies are on their respective circles. This simple observation will help us construct some interesting, nonintuitive classes of solutions of the curved N-body problem.

In analogy with great spheres of $\mathbb{S}^3$, we will further define great hyperbolic spheres of $\mathbb{H}^3$.

Definition 4. *A great hyperbolic sphere of* $\mathbb{H}^3$ *is a hyperbolic* 2*-sphere of curvature* -1*.*

Great hyperbolic spheres of $\mathbb{H}^3$ are obtained by intersecting $\mathbb{H}^3$ with hyperplanes of $\mathbb{R}^4$ that pass through the origin of the coordinate system, whenever this intersection is not empty. Examples of great hyperbolic spheres of $\mathbb{H}^3$ are:

$$\mathbf{H}^2_w = \{(w,x,y,z)\,|\,x^2+y^2-z^2=-1,\ w=0\}, \tag{2.9}$$

and the its analogues obtained by replacing w with x or y.

2.3 The metric

A basic preparatory issue lies with introducing the metric used on the manifolds $\mathbb{S}^3_\kappa$ and $\mathbb{H}^3_\kappa$, which, according to the corresponding inner products, we naturally define as

$$d_\kappa(\mathbf{a},\mathbf{b}) := \begin{cases} \kappa^{-1/2}\cos^{-1}(\kappa\mathbf{a}\cdot\mathbf{b}), & \kappa>0 \\ |\mathbf{a}-\mathbf{b}|, & \kappa=0 \\ (-\kappa)^{-1/2}\cosh^{-1}(\kappa\mathbf{a}\boxdot\mathbf{b}), & \kappa<0, \end{cases} \tag{2.10}$$

where the vertical bars denote the standard Euclidean norm.

When $\kappa \to 0$, with either $\kappa > 0$ or $\kappa < 0$, then $R \to \infty$, where R represents the radius of the sphere $\mathbb{S}^3_\kappa$ or the real factor in the expression iR of the imaginary radius of the hyperbolic sphere $\mathbb{H}^3_\kappa$. As $R \to \infty$, both $\mathbb{S}^3_\kappa$ and $\mathbb{H}^3_\kappa$ become $\mathbb{R}^3$, and the vectors $\mathbf{a}$ and $\mathbf{b}$ become parallel, so the distance between them gets to be measured in the Euclidean sense, as indicated in (2.10). Therefore, in a way, d is a continuous function of κ when the manifolds $\mathbb{S}^3_\kappa$ and $\mathbb{H}^3_\kappa$ are pushed away to infinity relative to the origin of the coordinate system.

In terms of intrinsic coordinates, as introduced in [37], [81], the plane does not get pushed to infinity, and the transition from $\mathbb{S}^3_\kappa$ to $\mathbb{R}^3$ to $\mathbb{H}^3_\kappa$ takes place continuously, since all three manifolds have a common point, as Figure 2.1 shows in the 2-dimensional case.

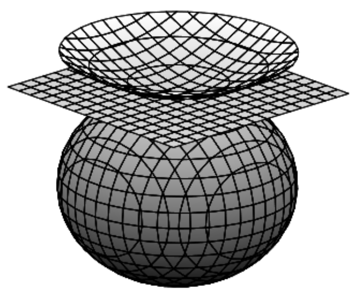

Figure 2.1: The transition from $\mathbb{S}^3_\kappa$ up to (and from $\mathbb{H}^3_\kappa$ down to) $\mathbb{R}^3$, as $\kappa \to 0$, is continuous since these manifolds have a common point, as suggested here in the 2-dimensional case.

In $\mathbb{S}^3_\kappa$, for instance, formula (2.10) is nothing but the well known length of an arc of a circle (the great circle of the sphere that connects the two points): $d_\kappa(\mathbf{a}, \mathbf{b}) = R\alpha$, where $R = \kappa^{-1/2}$ is the radius of the sphere and $\alpha = \alpha(\mathbf{a}, \mathbf{b})$ is the angle from the center of the circle (sphere) that subtends the arc. When $R \to \infty$, we have $\alpha \to 0$, so the limit of $R\alpha$ is undetermined. To see that the distance is a finite number, let us denote by ϵ the length of the chord that subtends the arc. Then, solving the right triangle formed by the vector $\mathbf{a}$, half the chord, and the height of the isosceles triangle formed by $\mathbf{a}, \mathbf{b}$, and the chord, we obtain $\alpha = 2\sin^{-1}[\epsilon/(2R)]$, which means that the length of the arc is

$$d_\kappa(\mathbf{a}, \mathbf{b}) = R\alpha = 2R\sin^{-1}\frac{\epsilon}{2R}.$$

If we assume ϵ constant, then the length of the arc must match the length of the chord when $R \to \infty$. Indeed, using l'Hôpital's rule, we obtain that

$$\lim_{R\to\infty} 2R\sin^{-1}\frac{\epsilon}{2R} = \lim_{R\to\infty}\frac{\epsilon}{\sqrt{1-[\epsilon/(2R)]^2}} = \epsilon.$$

To get more insight into the fact that the metrics in $\mathbb{S}^3_\kappa$ and $\mathbb{H}^3_\kappa$ become the Euclidean metric in $\mathbb{R}^3$ when $\kappa \to 0$, let us use the stereographic projection. Consider the points of coordinates $(w, x, y, z) \in \mathbb{M}^3_\kappa$ and map them to the points of coordinates (W, X, Y) of the 3-dimensional hyperplane $z = 0$ through the bijective transformation

$$W = \frac{Rw}{R-\sigma z}, \quad X = \frac{Rx}{R-\sigma z}, \quad Y = \frac{Ry}{R-\sigma z}, \tag{2.11}$$

which has the inverse

$$w = \frac{2R^2W}{R^2+\sigma W^2+\sigma X^2+\sigma Y^2}, \quad x = \frac{2R^2X}{R^2+\sigma W^2+\sigma X^2+\sigma Y^2},$$
$$y = \frac{2R^2Y}{R^2+\sigma W^2+\sigma X^2+\sigma Y^2}, \quad z = \frac{R(W^2+X^2+Y^2-\sigma R^2)}{R^2+\sigma W^2+\sigma X^2+\sigma Y^2}.$$

From the geometric point of view, the correspondence between a point of $\mathbb{M}^3_\kappa$ and a point of the hyperplane $z = 0$ is made via a straight line through the point $(0, 0, 0, \sigma R)$, called the north pole, for both $\kappa > 0$ and $\kappa < 0$.

For $\kappa > 0$, the projection is the Euclidean space $\mathbb{R}^3$, whereas for $\kappa < 0$ it is the solid Poincaré 3-ball of radius $\kappa^{-1/2}$. The metric in coordinates (W, X, Y) of the hyperplane $z = 0$ is given by

$$ds^2 = \frac{4R^4(dW^2+dX^2+dY^2)}{(R^2+\sigma W^2+\sigma X^2+\sigma Y^2)^2},$$

which can be obtained by substituting the inverse of the stereographic projection into the metric

$$ds^2 = dw^2 + dx^2 + dy^2 + \sigma dz^2.$$

The stereographic projection is conformal (angle preserving), but it is neither isometric (distance preserving) nor area preserving. Therefore we cannot expect to recover the exact Euclidean metric when $\kappa \to 0$, i.e. when $R \to \infty$, but hope, nevertheless, to obtain an expression that resembles it. Indeed, we can divide the numerator and denominator of the right hand side of the above metric by R^4 and write it after simplification as

$$ds^2 = \frac{4(dW^2+dX^2+dY^2)}{(1+\sigma W^2/R^2+\sigma X^2/R^2+\sigma Y^2/R^2)^2}.$$

When $R \to \infty$, we have

$$ds^2 = 4(dW^2 + dX^2 + dY^2),$$

which is the Euclidean metric of $\mathbb{R}^3$ up to a constant factor.

Remark 1. When $\kappa > 0$, we can conclude from (2.10) that if C_1 and C_2 are two complementary great circles, as described in Definition 3, and $\mathbf{a} \in C_1, \mathbf{b} \in C_2$, then the distance between $\mathbf{a}$ and $\mathbf{b}$ is

$$d_\kappa(\mathbf{a}, \mathbf{b}) = \kappa^{-1/2}\pi/2.$$

(In $\mathbb{S}^3$, the distance is $\pi/2$.) This fact shows that two complementary circles are equidistant, a simple property of essential importance for some of the unexpected dynamical orbits we will construct in this monograph.

Since, to derive the equations of motion, we will apply a variational principle, we need to extend the distance from the 3-dimensional manifolds of constant curvature $\mathbb{S}^3_\kappa$ and $\mathbb{H}^3_\kappa$ to the 4-dimensional ambient space in which they are embedded. We therefore redefine the distance between $\mathbf{a}$ and $\mathbf{b}$ as

$$\bar{d}_\kappa(\mathbf{a}, \mathbf{b}) := \begin{cases} \kappa^{-1/2} \cos^{-1} \frac{\kappa \mathbf{a} \cdot \mathbf{b}}{\sqrt{\kappa \mathbf{a} \cdot \mathbf{a}} \sqrt{\kappa \mathbf{b} \cdot \mathbf{b}}}, & \kappa > 0 \\ |\mathbf{a} - \mathbf{b}|, & \kappa = 0 \\ (-\kappa)^{-1/2} \cosh^{-1} \frac{\kappa \mathbf{a} \boxdot \mathbf{b}}{\sqrt{\kappa \mathbf{a} \boxdot \mathbf{a}} \sqrt{\kappa \mathbf{b} \boxdot \mathbf{b}}}, & \kappa < 0. \end{cases} \tag{2.12}$$

Notice that in $\mathbb{S}^3_\kappa$ we have $\sqrt{\kappa \mathbf{a} \cdot \mathbf{a}} = \sqrt{\kappa \mathbf{b} \cdot \mathbf{b}} = 1$ and in $\mathbb{H}^3_\kappa$ we have $\sqrt{\kappa \mathbf{a} \boxdot \mathbf{a}} = \sqrt{\kappa \mathbf{b} \boxdot \mathbf{b}} = 1$, which means that the new distance, $\bar{d}_\kappa$, reduces to the distance d_κ defined in (2.10), when we restrict $\bar{d}_\kappa$ to the corresponding 3-dimensional manifolds of constant curvature, i.e. $d_\kappa = \bar{d}_\kappa$ in $\mathbb{M}^3_\kappa$.

2.4 Unified trigonometry

Following the work of Cariñena, Rañada, and Santander, [13], we will further define the trigonometric κ-functions, which unify circular and hyperbolic trigonometry. The reason for this step is to obtain the equations of motion of the curved N-body problem in both constant positive and constant negative curvature spaces. We define the κ-sine, sn_κ, as

$$\mathrm{sn}_\kappa(x) := \begin{cases} \kappa^{-1/2} \sin(\kappa^{1/2} x) & \text{if} \quad \kappa > 0 \\ x & \text{if} \quad \kappa = 0 \\ (-\kappa)^{-1/2} \sinh((-\kappa)^{1/2} x) & \text{if} \quad \kappa < 0, \end{cases}$$

the κ-cosine, csn_κ, as

$$\mathrm{csn}_\kappa(x) := \begin{cases} \cos(\kappa^{1/2}x) & \text{if} \quad \kappa > 0 \\ 1 & \text{if} \quad \kappa = 0 \\ \cosh((-\kappa)^{1/2}x) & \text{if} \quad \kappa < 0, \end{cases}$$

as well as the κ-tangent, tn_κ, and κ-cotangent, ctn_κ, as

$$\mathrm{tn}_\kappa(x) := \frac{\mathrm{sn}_\kappa(x)}{\mathrm{csn}_\kappa(x)} \quad \text{and} \quad \mathrm{ctn}_\kappa(x) := \frac{\mathrm{csn}_\kappa(x)}{\mathrm{sn}_\kappa(x)},$$

respectively. The entire trigonometry can be rewritten in this unified context, but the only identity we will further need is the fundamental formula

$$\kappa\, \mathrm{sn}_\kappa^2(x) + \mathrm{csn}_\kappa^2(x) = 1. \tag{2.13}$$

Notice that all the above trigonometric κ-functions are continuous with respect to κ. In the above formulation of the unified trigonometric κ-functions, we assigned no particular meaning to the real parameter κ. In what follows, however, κ will represent the constant curvature of a 3-dimensional manifold. Therefore, with this notation, the distance (2.10) on the manifold $\mathbb{M}_\kappa^3$ can be written as

$$d_\kappa(\mathbf{a}, \mathbf{b}) = \mathrm{csn}_\kappa^{-1}(\kappa\, \mathbf{a} \odot \mathbf{b})$$

for any $\mathbf{a}, \mathbf{b} \in \mathbb{M}_\kappa^3$ and $\kappa \neq 0$. Similarly, we can write that

$$\bar{d}_\kappa(\mathbf{a}, \mathbf{b}) = \mathrm{csn}_\kappa^{-1}\left(\frac{\kappa\, \mathbf{a} \odot \mathbf{b}}{\sqrt{|\mathbf{a} \odot \mathbf{a}|}\sqrt{|\mathbf{b} \odot \mathbf{b}|}}\right)$$

for any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^4$ and $\kappa \neq 0$.

Chapter 3

Equations of motion

The main purpose of this chapter is to derive the equations of motion of the curved N-body problem on the 3-dimensional manifolds $\mathbb{M}^3_\kappa$. To achieve this goal, we will define the curved potential function, which also represents the potential of the particle system, introduce and apply Euler's formula for homogeneous functions to the curved potential function, describe the variational method of constrained Lagrangian dynamics, and write down the Euler-Lagrange equations with constraints. After deriving the equations of motion of the curved N-body problem, we will prove that their study can be reduced, by suitable coordinate and time-rescaling transformations, to the unit manifold $\mathbb{M}^3$. Finally, we will show that the equations of motion can be put in Hamiltonian form and will find their first integrals.

3.1 The potential

Since the classical Newtonian equations of the N-body problem are expressed in terms of a potential function, our next goal is to define such a function that extends to spaces of constant curvature and reduces to the classical Newtonian potential in the Euclidean case, i.e. when $\kappa = 0$.

Consider the point particles (which we will also call point masses or bodies) of masses $m_1, m_2, \ldots, m_N > 0$ in $\mathbb{R}^4$, for $\kappa > 0$, and in $\mathbb{R}^{3,1}$, for $\kappa < 0$, whose positions are given by the vectors $\mathbf{q}_i = (w_i, x_i, y_i, z_i),\ i = 1, 2, \ldots, N$. Let $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N)$ be the configuration of the system and $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N)$, with $\mathbf{p}_i = m_i \dot{\mathbf{q}}_i,\ i = 1, 2, \ldots, N$, the momentum of the system. The gradient operator with respect to the vector $\mathbf{q}_i$ is defined as

$$\widetilde{\nabla}_{\mathbf{q}_i} := (\partial_{w_i}, \partial_{x_i}, \partial_{y_i}, \sigma\partial_{z_i}),\ i = 1, 2, \ldots, N.$$

From now on we will rescale the units such that the gravitational constant G is 1. We thus define the potential of the curved N-body problem, which we will call the curved potential, as the function $-U_\kappa$, where

$$U_\kappa(\mathbf{q}) := \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N} m_i m_j \mathrm{ctn}_\kappa(d_\kappa(\mathbf{q}_i, \mathbf{q}_j))$$

stands for the curved force function. Notice that, for $\kappa = 0$, we have

$$\mathrm{ctn}_0(d_0(\mathbf{q}_i, \mathbf{q}_j)) = |\mathbf{q}_i - \mathbf{q}_j|^{-1},$$

which means that the curved potential becomes the classical Newtonian potential in the Euclidean case, [100]. Moreover, $U_\kappa \to U_0$ as $\kappa \to 0$, whether through positive or negative values of κ because, as shown in Section 2.3, $d_\kappa \to d_0$ as $\kappa \to 0$. It is interesting to notice that U_κ is a homogeneous function of degree 0 for $\kappa \neq 0$, but the Newtonian potential, U_0, defined in the Euclidean space, is a homogeneous function of degree -1. In other words, a bifurcation of U_κ occurs at the transition from $\kappa = 0$ to $\kappa \neq 0$. Such bifurcations are not unusual. For instance, the function $\mathfrak{g}_\alpha\colon (0,\infty) \to \mathbb{R}$ given by $\mathfrak{g}_\alpha(x) = \alpha x^p$, with p a nonzero integer, is homogeneous of degree p for $\alpha \neq 0$, but homogeneous of any degree for $\alpha = 0$.

Now that we defined a potential that satisfies the basic limit condition we required of any extension of the N-body problem beyond the Euclidean space, we emphasize that this function also satisfies the basic properties the classical Newtonian potential fulfills in the case of the Kepler problem, as mentioned in the Introduction: it obeys Bertrand's property, according to which every bounded orbit is closed, and is a solution of the Laplace-Beltrami equation (see Section 3.11), the natural generalization of Laplace's equation to Riemannian and pseudo-Riemannian manifolds. These properties ensure that the cotangent potential provides us with a natural extension of Newton's gravitational law to spaces of constant curvature.

Let us now focus on the case $\kappa \neq 0$. A straightforward computation, which uses the fundamental trigonometric formula (2.13), shows that

$$U_\kappa(\mathbf{q}) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N} \frac{m_i m_j (\sigma\kappa)^{1/2}\frac{\kappa\mathbf{q}_i\odot\mathbf{q}_j}{\sqrt{\kappa\mathbf{q}_i\odot\mathbf{q}_i}\sqrt{\kappa\mathbf{q}_j\odot\mathbf{q}_j}}}{\sqrt{\sigma - \sigma\left(\frac{\kappa\mathbf{q}_i\odot\mathbf{q}_j}{\sqrt{\kappa\mathbf{q}_i\odot\mathbf{q}_i}\sqrt{\kappa\mathbf{q}_j\odot\mathbf{q}_j}}\right)^2}}, \quad \kappa \neq 0, \tag{3.1}$$

an expression that is equivalent to

$$U_\kappa(\mathbf{q}) = \sum_{1\leq i<j\leq N} \frac{m_i m_j |\kappa|^{1/2}\kappa\mathbf{q}_i\odot\mathbf{q}_j}{[\sigma(\kappa\mathbf{q}_i\odot\mathbf{q}_i)(\kappa\mathbf{q}_j\odot\mathbf{q}_j) - \sigma(\kappa\mathbf{q}_i\odot\mathbf{q}_j)^2]^{1/2}}, \ \kappa \neq 0. \tag{3.2}$$

In fact, we could simplify U_κ even more by recalling that $\kappa\mathbf{q}_i \odot \mathbf{q}_i = 1,\ i = 1, 2, \ldots, N$. But since we still need to compute $\widetilde{\nabla} U_\kappa$, which means differentiating U_κ, we will not make that simplification yet.

3.2 Euler's formula for homogeneous functions

In 1755, Leonhard Euler proved a beautiful formula related to homogeneous functions, [44]. We will further present it and show how it applies to the curved potential.

Definition 5. *A function $F\colon \mathbb{R}^m \to \mathbb{R}$ is called homogeneous of degree $\alpha \in \mathbb{R}$ if for all $\eta \neq 0$ and $\mathbf{q} \in \mathbb{R}^m$, we have*

$$F(\eta\mathbf{q}) = \eta^\alpha F(\mathbf{q}).$$

Euler's formula shows that, for any homogeneous function of degree $\alpha \in \mathbb{R}$, we have

$$\mathbf{q} \cdot \nabla F(\mathbf{q}) = \alpha F(\mathbf{q})$$

for all $\mathbf{q} \in \mathbb{R}^m$.

Notice that $U_\kappa(\eta\mathbf{q}) = U_\kappa(\mathbf{q}) = \eta^0 U_\kappa(\mathbf{q})$ for any $\eta \neq 0$, which means that the curved potential is a homogeneous function of degree zero. With our notations, we have $m = 3N$, therefore Euler's formula can be written as

$$\mathbf{q} \odot \widetilde{\nabla} F(\mathbf{q}) = \alpha F(\mathbf{q}).$$

Since $\alpha = 0$ for U_κ with $\kappa \neq 0$, we conclude that

$$\mathbf{q} \odot \widetilde{\nabla} U_\kappa(\mathbf{q}) = 0. \tag{3.3}$$

We can also write the curved force function as $U_\kappa(\mathbf{q}) = \frac{1}{2}\sum_{i=1}^N U_\kappa^i(\mathbf{q}_i)$, where

$$U_\kappa^i(\mathbf{q}_i) := \sum_{j=1, j\neq i}^{N} \frac{m_i m_j (\sigma\kappa)^{1/2} \frac{\kappa\mathbf{q}_i \odot \mathbf{q}_j}{\sqrt{\kappa\mathbf{q}_i \odot \mathbf{q}_i}\sqrt{\kappa\mathbf{q}_j \odot \mathbf{q}_j}}}{\sqrt{\sigma - \sigma\left(\frac{\kappa\mathbf{q}_i \odot \mathbf{q}_j}{\sqrt{\kappa\mathbf{q}_i \odot \mathbf{q}_i}\sqrt{\kappa\mathbf{q}_j \odot \mathbf{q}_j}}\right)^2}}, \quad i = 1, 2, \ldots, N,$$

are also homogeneous functions of degree 0. Applying Euler's formula for functions $F : \mathbb{R}^3 \to \mathbb{R}$, we obtain that $\mathbf{q}_i \odot \widetilde{\nabla}_{\mathbf{q}_i} U_\kappa^i(\mathbf{q}) = 0$. Then using the identity $\widetilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q}) = \widetilde{\nabla}_{\mathbf{q}_i} U_\kappa^i(\mathbf{q}_i)$, we can conclude that

$$\mathbf{q}_i \odot \widetilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q}) = 0, \quad i = 1, 2, \ldots, N. \tag{3.4}$$

3.3 Constrained Lagrangian dynamics

To obtain the equations of motion of the curved N-body problem, we will use the classical variational theory of constrained Lagrangian dynamics, [48]. According to this theory, let

$$L = T - V$$

be the Lagrangian of a system of N particles constrained to move on a manifold, where T is the kinetic energy and V is the potential energy of the system. If the positions and the velocities of the particles are given by the vectors $\mathbf{q}_i, \dot{\mathbf{q}}_i,\ i = 1, 2, \dots, N$, and the constraints are characterized by the equations $f^i = 0,\ i = 1, 2, \dots, N$, respectively, then the motion is described by the Euler-Lagrange equations with constraints,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}_i}\right) - \frac{\partial L}{\partial \mathbf{q}_i} - \lambda^i(t)\frac{\partial f^i}{\partial \mathbf{q}_i} = \mathbf{0}, \quad i = 1, 2, \dots, N, \tag{3.5}$$

where $\lambda^i,\ i = 1, 2, \dots, N$, are called Lagrange multipliers. For the above equations, the distance is defined in the entire ambient space. Using this classical result, we can now derive the equations of motion of the curved N-body problem.

3.4 Derivation of the equations of motion

In our case, the potential energy V of Section 3.3 is $V = -U_\kappa$, given by the curved force function (3.1), and we define the kinetic energy of the system of particles as

$$T_\kappa(\mathbf{q}, \dot{\mathbf{q}}) := \frac{1}{2}\sum_{i=1}^{N} m_i(\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i)(\kappa \mathbf{q}_i \odot \mathbf{q}_i).$$

The reason for introducing the factors $\kappa\mathbf{q}_i \odot \mathbf{q}_i = 1,\ i = 1, 2, \dots, N$, into the definition of the kinetic energy will become clear in Section 3.6. Then the Lagrangian of the curved N-body system has the form

$$L_\kappa(\mathbf{q}, \dot{\mathbf{q}}) = T_\kappa(\mathbf{q}, \dot{\mathbf{q}}) + U_\kappa(\mathbf{q}).$$

So, according to the theory of constrained Lagrangian dynamics presented in Section 3.3, which requires the use of a distance defined in the ambient space, a condition we satisfied when producing formula (2.12), the equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L_\kappa}{\partial \dot{\mathbf{q}}_i}\right) - \frac{\partial L_\kappa}{\partial \mathbf{q}_i} - \lambda_\kappa^i(t)\frac{\partial f_\kappa^i}{\partial \mathbf{q}_i} = \mathbf{0}, \quad i = 1, 2, \dots, N, \tag{3.6}$$

where $f_\kappa^i = \mathbf{q}_i \odot \mathbf{q}_i - \kappa^{-1}$ is the function that characterizes the constraints $f_\kappa^i = 0,\ i = 1, 2, \dots, N$. Each constraint keeps the body of mass m_i on the surface of constant curvature κ, and λ_κ^i is the Lagrange multiplier corresponding to the same body. Since $\mathbf{q}_i \odot \mathbf{q}_i = \kappa^{-1}$ implies that $\dot{\mathbf{q}}_i \odot \mathbf{q}_i = 0$, it follows that

$$\frac{d}{dt}\left(\frac{\partial L_\kappa}{\partial \dot{\mathbf{q}}_i}\right) = m_i\ddot{\mathbf{q}}_i(\kappa\mathbf{q}_i \odot \mathbf{q}_i) + 2m_i(\kappa\dot{\mathbf{q}}_i \odot \mathbf{q}_i) = m_i\ddot{\mathbf{q}}_i,\ i = 1, 2, \dots, N.$$

This relationship, together with

$$\frac{\partial L_\kappa}{\partial \mathbf{q}_i} = m_i\kappa(\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i)\mathbf{q}_i + \widetilde{\nabla}_{\mathbf{q}_i}U_\kappa(\mathbf{q}),\ i = 1, 2, \dots, N,$$

implies that equations (3.6) are equivalent to

$$m_i\ddot{\mathbf{q}}_i - m_i\kappa(\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i)\mathbf{q}_i - \widetilde{\nabla}_{\mathbf{q}_i}U_\kappa(\mathbf{q}) - 2\lambda_\kappa^i(t)\mathbf{q}_i = \mathbf{0}, \quad i = 1, 2, \dots, N. \tag{3.7}$$

To determine λ_κ^i, notice that $0 = \ddot{f}_\kappa^i = 2\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i + 2(\mathbf{q}_i \odot \ddot{\mathbf{q}}_i)$, so

$$\mathbf{q}_i \odot \ddot{\mathbf{q}}_i = -\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i,\ i = 1, 2, \dots, N. \tag{3.8}$$

Let us also remark that $\odot$-multiplying equations (3.7) by $\mathbf{q}_i$ and using Euler's formula (3.4), we obtain that

$$m_i(\mathbf{q}_i \odot \ddot{\mathbf{q}}_i) - m_i(\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i) - \mathbf{q}_i \odot \widetilde{\nabla}_{\mathbf{q}_i}U_\kappa(\mathbf{q}) = 2\lambda_\kappa^i\mathbf{q}_i \odot \mathbf{q}_i = 2\kappa^{-1}\lambda_\kappa^i,\ i = 1, 2, \dots, N,$$

which, via (3.8), implies that $\lambda_\kappa^i = -\kappa m_i(\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i),\ i = 1, 2, \dots, N$. Substituting these values of the Lagrange multipliers into equations (3.7), the equations of motion and their constraints become

$$m_i\ddot{\mathbf{q}}_i = \widetilde{\nabla}_{\mathbf{q}_i}U_\kappa(\mathbf{q}) - m_i\kappa(\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i)\mathbf{q}_i, \quad \mathbf{q}_i \odot \mathbf{q}_i = \kappa^{-1}, \quad \kappa \neq 0, \quad i = 1, 2, \dots, N. \tag{3.9}$$

The $\mathbf{q}_i$-gradient of the curved force function, obtained from (3.1), has the form

$$\widetilde{\nabla}_{\mathbf{q}_i}U_\kappa(\mathbf{q}) = \sum_{j=1, j\neq i}^{N} \frac{\frac{m_i m_j(\sigma\kappa)^{1/2}\left(\sigma\kappa\mathbf{q}_j - \sigma\frac{\kappa^2\mathbf{q}_i\odot\mathbf{q}_j}{\kappa\mathbf{q}_i\odot\mathbf{q}_i}\mathbf{q}_i\right)}{\sqrt{\kappa\mathbf{q}_i\odot\mathbf{q}_i}\sqrt{\kappa\mathbf{q}_j\odot\mathbf{q}_j}}}{\left[\sigma - \sigma\left(\frac{\kappa\mathbf{q}_i\odot\mathbf{q}_j}{\sqrt{\kappa\mathbf{q}_i\odot\mathbf{q}_i}\sqrt{\kappa\mathbf{q}_j\odot\mathbf{q}_j}}\right)^2\right]^{3/2}},\ \kappa \neq 0,\ i = 1, 2, \dots, N, \tag{3.10}$$

which is equivalent to

$$\widetilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q}) = \sum_{\substack{j=1\\ j\neq i}}^{N} \frac{m_i m_j |\kappa|^{3/2} (\kappa\mathbf{q}_j \odot \mathbf{q}_j)[(\kappa\mathbf{q}_i \odot \mathbf{q}_i)\mathbf{q}_j - (\kappa\mathbf{q}_i \odot \mathbf{q}_j)\mathbf{q}_i]}{[\sigma(\kappa\mathbf{q}_i \odot \mathbf{q}_i)(\kappa\mathbf{q}_j \odot \mathbf{q}_j) - \sigma(\kappa\mathbf{q}_i \odot \mathbf{q}_j)^2]^{3/2}},\ i = 1, 2, \dots, N. \tag{3.11}$$

Using the fact that $\kappa\mathbf{q}_i \odot \mathbf{q}_i = 1,\ i = 1, 2, \dots, N$, we can write this gradient as

$$\widetilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q}) = \sum_{j=1, j\neq i}^{N} \frac{m_i m_j |\kappa|^{3/2} \left[\mathbf{q}_j - (\kappa\mathbf{q}_i \odot \mathbf{q}_j)\mathbf{q}_i\right]}{\left[\sigma - \sigma\left(\kappa\mathbf{q}_i \odot \mathbf{q}_j\right)^2\right]^{3/2}},\ \kappa \neq 0,\ i = 1, 2, \dots, N. \tag{3.12}$$

We can often use the simpler form (3.12) of the gradient of the force function, but whenever we need to exploit the homogeneity of the gradient, or have to differentiate it, we must revert to its original form (3.11). Thus equations (3.9) and (3.11) describe the N-body problem on surfaces of constant curvature for $\kappa \neq 0$. Though more complicated than the equations of motion Newton derived for the Euclidean space, system (3.9) is simple enough to allow an analytic approach.

3.5 Independence of curvature

Recently, Carles Simó pointed to us an important property, which will greatly simplify the study of the equations of motion. For $\mathbb{S}^2$, he found a change of coordinates and a rescaling of time that allow the elimination of the parameter κ, up to its sign, from the equations of motion. His idea can be easily generalized to spheres and hyperbolic manifolds of any dimension. So consider the coordinate and time-rescaling transformations given by

$$\mathbf{q}_i = |\kappa|^{-1/2}\mathbf{r}_i,\ i = 1, 2, \dots, N, \quad \text{and} \quad \tau = |\kappa|^{3/4} t. \tag{3.13}$$

The rescaling of the time variable is equivalent to writing $\frac{d}{dt} = |\kappa|^{3/4}\frac{d}{d\tau}$, which is a relationship between differentiation with respect to t and differentiation with respect to τ. Let $\mathbf{r}_i'$ and $\mathbf{r}_i''$ denote the first and second derivative of $\mathbf{r}_i$ with respect to the rescaled time variable τ. Then the equations of motion (3.9) take the form

$$\mathbf{r}_i'' = \sum_{j=1, j\neq i}^{N} \frac{m_j[\mathbf{r}_j - \sigma(\mathbf{r}_i \odot \mathbf{r}_j)\mathbf{r}_i]}{[\sigma - \sigma(\mathbf{r}_i \odot \mathbf{r}_j)^2]^{3/2}} - \sigma(\mathbf{r}_i' \odot \mathbf{r}_i')\mathbf{r}_i, \quad i = 1, 2, \dots, N, \tag{3.14}$$

where κ does not appear explicitly anymore. They only depend on the sign of κ, given that we must take $\sigma = 1$ for $\kappa > 0$ and $\sigma = -1$ for $\kappa < 0$. Moreover, the change of coordinates (3.13) shows that

$$\mathbf{r}_i \odot \mathbf{r}_i = |\kappa| \mathbf{q}_i \odot \mathbf{q}_i = |\kappa| \kappa^{-1} = \sigma.$$

Consequently, for positive curvature we have $\mathbf{r}_i \in \mathbb{S}^3,\ i = 1, 2, \ldots, N$, and for negative curvature we have $\mathbf{r}_i \in \mathbb{H}^3,\ i = 1, 2, \ldots, N$. This means that, from the qualitative point of view, the behavior of the orbits is independent of the curvature's value, and that, without loss of generality, we can restrict our study to the unit sphere, for positive curvature, and the unit hyperbolic sphere, for negative curvature. Of course, for any practical or quantitative purposes, which will not be addressed in this monograph, we would have to use system (3.9).

Instead of employing the new equations of motion (3.14), in which the $\mathbf{r}_i$s represent the coordinates, we will use the old notations in the particular cases $\kappa = 1$, which stands for positive curvature, and $\kappa = -1$, which stands for negative curvature. This is as if we would redenote the variables $\mathbf{r}_i$ by $\mathbf{q}_i$, the rescaled time τ by t, and use the upper dots instead of the primes for the derivatives. In other words, for positive curvature we will use the system

$$\ddot{\mathbf{q}}_i = \sum_{j=1, j\neq i}^{N} \frac{m_j[\mathbf{q}_j - (\mathbf{q}_i \cdot \mathbf{q}_j)\mathbf{q}_i]}{[1 - (\mathbf{q}_i \cdot \mathbf{q}_j)^2]^{3/2}} - (\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i)\mathbf{q}_i,\ \mathbf{q}_i \cdot \mathbf{q}_\mathbf{i} = 1, \quad i = 1, 2, \ldots, N, \quad (3.15)$$

where $\cdot$ is the standard inner product. The constraints show that the motion takes place on the unit sphere $\mathbb{S}^3$. For negative curvature we will use the system

$$\ddot{\mathbf{q}}_i = \sum_{j=1, j\neq i}^{N} \frac{m_j[\mathbf{q}_j + (\mathbf{q}_i \boxdot \mathbf{q}_j)\mathbf{q}_i]}{[(\mathbf{q}_i \boxdot \mathbf{q}_j)^2 - 1]^{3/2}} + (\dot{\mathbf{q}}_i \boxdot \dot{\mathbf{q}}_i)\mathbf{q}_i,\ \mathbf{q}_i \boxdot \mathbf{q}_\mathbf{i} = -1, \quad i = 1, 2, \ldots, N, \quad (3.16)$$

where $\boxdot$ is the Lorentz inner product. The constraints show that the motion takes place on the unit hyperbolic manifold $\mathbb{H}^3$. When referring to both equations simultaneously, we will consider the form

$$\ddot{\mathbf{q}}_i = \sum_{j=1, j\neq i}^{N} \frac{m_j[\mathbf{q}_j - \sigma(\mathbf{q}_i \odot \mathbf{q}_j)\mathbf{q}_i]}{[\sigma - \sigma(\mathbf{q}_i \odot \mathbf{q}_j)^2]^{3/2}} - \sigma(\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i)\mathbf{q}_i,\ \mathbf{q}_i \odot \mathbf{q}_\mathbf{i} = \sigma, \quad i = 1, 2, \ldots, N. \quad (3.17)$$

The last term in each equation, involving the Lagrange multipliers, occurs due to the constraints that keep the bodies moving on the manifold. In Euclidean space those terms vanish.

The force function and its gradient are then expressed as

$$U(\mathbf{q}) = \sum_{1 \le i < j \le N} \frac{\sigma m_i m_j \mathbf{q}_i \odot \mathbf{q}_j}{[\sigma(\mathbf{q}_i \odot \mathbf{q}_i)(\mathbf{q}_j \odot \mathbf{q}_j) - \sigma(\mathbf{q}_i \odot \mathbf{q}_j)^2]^{1/2}}, \quad (3.18)$$

$$\widetilde{\nabla}_{\mathbf{q}_i} U(\mathbf{q}) = \sum_{j=1, j\neq i}^{N} \frac{m_i m_j \left[\mathbf{q}_j - \sigma(\mathbf{q}_i \odot \mathbf{q}_j)\mathbf{q}_i\right]}{\left[\sigma - \sigma\left(\mathbf{q}_i \odot \mathbf{q}_j\right)^2\right]^{3/2}}, \tag{3.19}$$

respectively, and he kinetic energy is given by

$$T(\mathbf{q}, \dot{\mathbf{q}}) := \frac{1}{2}\sum_{i=1}^{N} m_i (\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i)(\sigma \mathbf{q}_i \odot \mathbf{q}_i). \tag{3.20}$$

3.6 Hamiltonian formulation

It is always desirable to place any new problem in a more general framework. The theory of Hamiltonian systems turns out to be the suitable structure in this case, the same as for the Euclidean N-body problem. In classical mechanics, a Hamiltonian system is a physical system that is momentum invariant. In mathematics, a Hamiltonian system is usually formulated in terms of Hamiltonian vector fields on a symplectic manifold or, more generally, on a Poisson manifold. Hamiltonian systems cover a wide range of applications, and their mathematical properties have been intensely investigated in recent times. Consequently the Hamiltonian character of the curved N-body problem adds another argument in favor of the statement that these equations naturally extend Newtonian gravitation to spaces of constant curvature.

The Hamiltonian function describing the motion of the curved N-body problem is provided by

$$H(\mathbf{q}, \mathbf{p}) = T(\mathbf{q}, \mathbf{p}) - U(\mathbf{q}),$$

where T is defined in (3.20) and U in (3.18). The Hamiltonian form of the equations of motion (3.17) is given by the system with constraints

$$\begin{cases} \dot{\mathbf{q}}_i = \widetilde{\nabla}_{\mathbf{p}_i} H(\mathbf{q}, \mathbf{p}) = m_i^{-1}\mathbf{p}_i, \\ \dot{\mathbf{p}}_i = -\widetilde{\nabla}_{\mathbf{q}_i} H(\mathbf{q}, \mathbf{p}) = \widetilde{\nabla}_{\mathbf{q}_i} U(\mathbf{q}) - m_i^{-1}(\mathbf{p}_i \odot \mathbf{p}_i)\mathbf{q}_i, \\ \mathbf{q}_i \odot \mathbf{q}_i = \sigma, \quad \mathbf{q}_i \odot \mathbf{p}_i = 0, \quad i = 1, 2, \dots, N. \end{cases} \tag{3.21}$$

The configuration space is the manifold $(\mathbb{M}^3)^N$, where, recall, $\mathbb{M}^3$ denotes $\mathbb{S}^3$ or $\mathbb{H}^3$. Then $(\mathbf{q}, \mathbf{p}) \in \mathbf{T}^*(\mathbb{M}^3)^N$, where $\mathbf{T}^*(\mathbb{M}^3)^N$ is the cotangent bundle, which represents the phase space. The constraints $\mathbf{q}_i \odot \mathbf{q}_i = \sigma$, $\mathbf{q}_i \odot \mathbf{p}_i = 0$, $i = 1, 2, \dots, N$, keep the bodies on the manifold and show that the position vectors and the momenta of each body are orthogonal to each other. They reduce the $8N$-dimensional system (3.21) by $2N$ dimensions. So, before taking into consideration the first integrals of motion, which we will compute in Section 3.8, the phase space has dimension

$6N$, as it should, given the fact that we are studying the motion of N bodies on 3-dimensional manifolds.

3.7 Invariance

In the Euclidean case, planes are invariant sets for the equations of motion. In other words, if the initial positions and momenta are contained in a plane, the motion takes place in that plane for all time for which the solution is defined. We can now prove the natural analogue of this result for the curved N-body problem. More precisely, we will show that, in $\mathbb{S}^3$ and $\mathbb{H}^3$, the motion can take place on 2-dimensional great spheres and 2-dimensional great hyperbolic spheres, respectively, if we assign suitable initial positions and momenta. This result is also true on 1-dimensional manifolds, letting aside that such motions may end up in singularities.

Proposition 1. *Let $N \geq 2$ and consider the bodies of masses $m_1, m_2, \dots, m_N > 0$ in $\mathbb{M}^3$. Assume that $\mathbb{M}^2$ is any 2-dimensional submanifold of $\mathbb{M}^3$ having the same curvature, i.e. a great sphere $\mathbb{S}^2$ for $\mathbb{S}^3$ or a great hyperbolic sphere $\mathbb{H}^2$ for $\mathbb{H}^3$. Then, for any nonsingular initial conditions $(\mathbf{q}(0), \mathbf{p}(0)) \in (\mathbb{M}^2)^N \times (T(\mathbb{M}^2))^N$, where $\times$ denotes the cartesian product of two sets and $T(\mathbb{M}^2)$ is the tangent space of $\mathbb{M}^2$, the motion takes place in $\mathbb{M}^2$.*

Proof. Without loss of generality, it is enough to prove the result for $\mathbf{M}^2_w$, where

$$\mathbf{M}^2_w := \{(w, x, y, z) \,|\, x^2 + y^2 + \sigma z^2 = \sigma,\ w = 0\}$$

is the great 2-dimensional sphere $\mathbf{S}^2_w$, for $\kappa = 1$, and the great 2-dimensional hyperbolic sphere $\mathbf{H}^2_w$, for $\kappa = -1$, which we defined in (2.5) and (2.9), respectively. Indeed, we can obviously restrict to this case since any great sphere or great hyperbolic sphere can be reduced to it by a suitable change of coordinates.

Let us denote the coordinates and the momenta of the body $m_i,\ i = 1, 2, \dots, N$, by

$$\mathbf{q}_i = (w_i, x_i, y_i, z_i) \ \text{ and } \ \mathbf{p}_i = (r_i, s_i, u_i, v_i),\ i = 1, 2, \dots, N,$$

which, when restricted to $\mathbf{M}^2_w$ and the tangent space $T(\mathbf{M}^2_w)$, respectively, become

$$\mathbf{q}_i = (0, x_i, y_i, z_i) \ \text{ and } \ \mathbf{p}_i = (0, s_i, u_i, v_i),\ i = 1, 2, \dots, N.$$

Relative to the first component, w, the equations of motion (3.21) have the form

$$\begin{cases} \dot w_i = m_i^{-1} r_i, \\ \dot r_i = \sum_{j=1, j\neq i}^N \frac{m_i m_j [w_j - \sigma(\mathbf{q}_i \odot \mathbf{q}_j) w_i]}{\left[\sigma - \sigma(\mathbf{q}_i \odot \mathbf{q}_j)^2\right]^{3/2}} - \sigma m_i^{-1} (\mathbf{p}_i \odot \mathbf{p}_i) w_i, \\ \mathbf{q}_i \odot \mathbf{q}_i = \sigma, \quad \mathbf{q}_i \odot \mathbf{p}_i = 0, \quad i = 1, 2, \dots, N. \end{cases}$$

For our purposes, we can view this first-order system of differential equations as linear in the variables $w_i, r_i,\ i = 1, 2, \ldots, N$. But on $\mathbf{M}^2_w$, the initial conditions are $w_i(0) = r_i(0) = 0,\ i = 1, 2, \ldots, N$, therefore $w_i(t) = r_i(t) = 0,\ i = 1, 2, \ldots, N$, for all t for which the corresponding solutions are defined. Consequently, given initial conditions $(\mathbf{q}(0), \mathbf{p}(0)) \in (\mathbf{M}^2_w)^N \times (T(\mathbf{M}^2_w))^N$, the motion is confined to $\mathbf{M}^2_w$, a remark that completes the proof. □

3.8 First integrals

In this section we will determine the first integrals of the equations of motion. These integrals lie at the foundation of the reduction method, which played an important role in the theory of differential equations ever since mathematicians discovered the existence of functions that remain constant along solutions. The classical N-body problem in $\mathbb{R}^3$ has 10 first integrals that are algebraic with respect to $\mathbf{q}$ and $\mathbf{p}$, known already to Lagrange in the mid 18th century, [100]. In 1887, Heinrich Bruns proved that there are no other first integrals, algebraic with respect to $\mathbf{q}$ and $\mathbf{p}$, [10].

In our case, the existing integrals follow from Noether's theorem, according to which differentiable symmetries generated by local actions correspond to a conserved flow. But we can also prove the existence of the first integrals of the curved N-body problem through elementary computations.

The Hamiltonian function provides the integral of energy,

$$H(\mathbf{q}, \mathbf{p}) = h, \tag{3.22}$$

where h is the energy constant. Indeed, $\odot$-multiplying system (3.9) by $\dot{\mathbf{q}}_i$, we obtain

$$\sum_{i=1}^N m_i \ddot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i = \sum_{i=1}^N [\widetilde{\nabla}_{\mathbf{q}_i} U(\mathbf{q})] \odot \dot{\mathbf{q}}_i - \sum_{i=1}^N \sigma m_i (\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i) \mathbf{q}_i \odot \dot{\mathbf{q}}_i = \frac{d}{dt} U(\mathbf{q}(t)).$$

Then equation (3.22) follows by integrating the first and last term in the above equation.

Unlike in the classical Euclidean case, there are no integrals of the center of mass and the linear momentum (for more details see [30]). This fact is far from surprising, given that N-body problems obtained by discretizing Einstein's field equations lack these integrals as well, [42], [46], [62], [63]. Their absence, however, complicates the study of the problem since many of the standard methods used in the classical case don't apply anymore.

We could, of course, define some artificial center of mass for the particle system, but this move would be to no avail. In general, forces do not cancel each other at such a point, neither do they make it move uniformly along a geodesic, as it happens in the Euclidean case, so no advantage can be gained from introducing this concept. Nevertheless, for particular orbits, most of which have many symmetries, there exist points on the manifold that behave like a center of mass, i.e. where the forces that act on them cancel each other or make the point move uniformly along a geodesic. Unlike in the Euclidean case, this point may not be unique, as it happens for instance in the case of Lagrangian solutions (equilateral triangles) of the curved 3-body problem in $\mathbb{S}^2$ (see Chapter 13), when 3 bodies of equal masses rotate along the equator. Then both the north pole, $(0, 0, 1)$, and the south pole, $(0, 0, -1)$, act like centers of mass in the sense that, at those points, forces cancel each other on the surface of the sphere.

As we will further show, equations (3.21) have 6 angular momentum integrals. To prove their existence, we need to introduce the notion of bivector, which generalizes the idea of vector. A scalar has dimension 0, a vector has dimension 1, and a bivector has dimension 2. Bivectors are constructed with the help of the wedge product, $\mathbf{a}\wedge\mathbf{b}$, defined below, of two vectors $\mathbf{a}$ and $\mathbf{b}$. Its magnitude can be intuitively understood as the oriented area of the parallelogram with edges $\mathbf{a}$ and $\mathbf{b}$. A bivector lies in a vector space different from that of the vectors it is generated from, therefore the wedge product is an exterior operation. The space of bivectors together with the wedge product is called a Grassmann algebra.

To make these concepts precise, let

$$\mathbf{e}_w = (1, 0, 0, 0),\ \mathbf{e}_x = (0, 1, 0, 0),\ \mathbf{e}_y = (0, 0, 1, 0),\ \mathbf{e}_z = (0, 0, 0, 1)$$

denote the elements of the canonical basis of $\mathbb{R}^4$, and let us consider the vectors $\mathbf{u} = (u_w, u_x, u_y, u_z)$ and $\mathbf{v} = (v_w, v_x, v_y, v_z)$. We define the wedge product (also called outer product or exterior product) of $\mathbf{u}$ and $\mathbf{v}$ of $\mathbb{R}^4$ as

$$\begin{aligned}\mathbf{u}\wedge\mathbf{v} := {} & (u_w v_x - u_x v_w) e_w \wedge e_x + (u_w v_y - u_y v_w) e_w \wedge e_y + \\ & (u_w v_z - u_z v_w) e_w \wedge e_z + (u_x v_y - u_y v_x) e_x \wedge e_y + \\ & (u_x v_z - u_z v_x) e_x \wedge e_z + (u_y v_z - u_z v_y) e_y \wedge e_z,\end{aligned} \tag{3.23}$$

where $\mathbf{e}_w \wedge \mathbf{e}_x, \mathbf{e}_w \wedge \mathbf{e}_y, \mathbf{e}_w \wedge \mathbf{e}_z, \mathbf{e}_x \wedge \mathbf{e}_y, \mathbf{e}_x \wedge \mathbf{e}_z, \mathbf{e}_y \wedge \mathbf{e}_z$ represent the bivectors that form a canonical basis of the exterior Grassmann algebra over $\mathbb{R}^4$ (for more details, see, e.g., [39]). In $\mathbb{R}^3$, the exterior product is equivalent with the cross product.

Let us define $\sum_{i=1}^N m_i \mathbf{q}_i \wedge \dot{\mathbf{q}}_i$ to be the total angular momentum of the particles of masses $m_1, m_2, \ldots, m_N > 0$ in $\mathbb{R}^4$. We will further show that the total angular momentum is conserved for the equations of motion, i.e.

$$\sum_{i=1}^N m_i \mathbf{q}_i \wedge \dot{\mathbf{q}}_i = \mathbf{c}, \tag{3.24}$$

where $\mathbf{c} = c_{wx}\mathbf{e}_w \wedge \mathbf{e}_x + c_{wy}\mathbf{e}_w \wedge \mathbf{e}_y + c_{wz}\mathbf{e}_w \wedge \mathbf{e}_z + c_{xy}\mathbf{e}_x \wedge \mathbf{e}_y + c_{xz}\mathbf{e}_x \wedge \mathbf{e}_z + c_{yz}\mathbf{e}_y \wedge \mathbf{e}_z$, with the coefficients $c_{wx}, c_{wy}, c_{wz}, c_{xy}, c_{xz}, c_{yz} \in \mathbb{R}$. Indeed, relations (3.24) follow by integrating the identity formed by the first and last term of the equations

$$\begin{aligned}\sum_{i=1}^{N} m_i\ddot{\mathbf{q}}_i \wedge \mathbf{q}_i = \sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} \frac{m_i m_j \mathbf{q}_i \wedge \mathbf{q}_j}{[\sigma - \sigma(\mathbf{q}_i \odot \mathbf{q}_j)^2]^{3/2}} \\ - \sum_{i=1}^{N}\left[\sum_{j=1,j\neq i}^{N} \frac{\sigma m_i m_j \mathbf{q}_i \odot \mathbf{q}_j}{[\sigma - \sigma(\mathbf{q}_i \odot \mathbf{q}_j)^2]^{3/2}} - \sigma m_i(\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i)\right]\mathbf{q}_i \wedge \mathbf{q}_i = \mathbf{0}, \quad (3.25)\end{aligned}$$

obtained after $\wedge$-multiplying the equations of motion (3.17) by $\mathbf{q}_i$ and summing up from $i = 1$ to $i = N$. The last of the above identities follows from the skew-symmetry of the $\wedge$ operation and the fact that $\mathbf{q}_i \wedge \mathbf{q}_i = \mathbf{0}$, $i = 1, 2, \ldots, N$, as can be easily seen from the definition (3.23) of the wedge product.

On components, the 6 integrals in (3.24) can be written as

$$\sum_{i=1}^{N} m_i(w_i\dot{x}_i - \dot{w}_i x_i) = c_{wx}, \qquad \sum_{i=1}^{N} m_i(w_i\dot{y}_i - \dot{w}_i y_i) = c_{wy}, \quad (3.26)$$

$$\sum_{i=1}^{N} m_i(w_i\dot{z}_i - \dot{w}_i z_i) = c_{wz}, \qquad \sum_{i=1}^{N} m_i(x_i\dot{y}_i - \dot{x}_i y_i) = c_{xy}, \quad (3.27)$$

$$\sum_{i=1}^{N} m_i(x_i\dot{z}_i - \dot{x}_i z_i) = c_{xz}, \qquad \sum_{i=1}^{N} m_i(y_i\dot{z}_i - \dot{y}_i z_i) = c_{yz}. \quad (3.28)$$

The physical interpretation of these six integrals is related to the geometry of $\mathbb{R}^4$. The coordinate axes Ow, Ox, Oy, and Oz determine six orthogonal planes, wx, wy, wz, xy, xz, and yz. We call them basis planes, since they correspond to the bivectors $\mathbf{e}_w \wedge \mathbf{e}_x$, $\mathbf{e}_w \wedge \mathbf{e}_y$, $\mathbf{e}_w \wedge \mathbf{e}_z$, $\mathbf{e}_x \wedge \mathbf{e}_y$, $\mathbf{e}_x \wedge \mathbf{e}_z$, and $\mathbf{e}_y \wedge \mathbf{e}_z$, respectively, that form a basis of the Grassmann algebra generated from the basis vectors $\mathbf{e}_w, \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ of $\mathbb{R}^4$. Then the constants $c_{wx}, c_{wy}, c_{wz}, c_{xy}, c_{xz}, c_{yz}$ measure the rotation of an orbit relative to a point in the plane their indices define. This reference point is the same for all 6 basis planes, namely the origin of the coordinate system.

To clarify this interpretation of rotations in $\mathbb{R}^4$, let us point out that, in $\mathbb{R}^3$, rotation is understood as a motion around a pointwise invariant axis orthogonal to a basis plane, which the rotation leaves globally (but not pointwise) invariant. In $\mathbb{R}^4$, there are infinitely many axes orthogonal to this plane, and the angular momentum is the same for them all, since each of the 6 equations of the total angular momentum depends only on the 2 coordinates of the plane and the corresponding velocities. It is, therefore, more convenient to think of these rotations in $\mathbb{R}^4$ as taking place around a point in a plane, in spite of the fact that the rotation moves points outside the plane too.

Whatever sense of rotation a scalar constant of the angular momentum determines, the opposite sign indicates the opposite sense. A zero scalar constant means that there is no rotation relative to the origin in that particular plane.

Notice that, after taking into account the integrals of motion described above, the dimension of the phase space can be reduced to $6n - 7$.

3.9 Singularities

Before we begin the study of relative equilibria, it is important to know whether there are configurations the bodies cannot achieve. It turns out that impossible configurations exist, and they occur when system (3.17) encounters singularities, i.e. if at least one denominator in the sum on the right hand sides of system (3.17) vanishes. So a configuration is singular when

$$(\mathbf{q}_i \odot \mathbf{q}_j)^2 = 1 \quad \text{for some} \quad i, j \in \{1, 2, \ldots, N\}, \; i \neq j,$$

which is the same as saying that

$$\mathbf{q}_i \odot \mathbf{q}_j = 1 \quad \text{or} \quad \mathbf{q}_i \odot \mathbf{q}_j = -1, \quad \text{for some} \quad i, j \in \{1, 2, \ldots, N\}, \; i \neq j.$$

The following result shows that the former case corresponds to collisions, i.e. to configurations for which at least two bodies have identical coordinates, whereas the latter case occurs in $\mathbb{S}^3$, but not in $\mathbb{H}^3$, and corresponds to antipodal configurations, i.e. when at least two bodies have coordinates of opposite signs. These are impossible initial configurations, which we must avoid in the following endeavors.

Proposition 2. (Collision and antipodal configurations) *Consider the 3-dimensional curved N-body problem, $N \geq 2$, with masses $m_1, m_2, \ldots, m_N > 0$. Then, in $\mathbb{S}^3$, if there are $i, j \in \{1, 2, \ldots, N\}$, $i \neq j$, such that $\mathbf{q}_i \cdot \mathbf{q}_j = 1$, the bodies m_i and m_j form a collision configuration, and if $\mathbf{q}_i \odot \mathbf{q}_j = -1$, they form an antipodal configuration. In $\mathbb{H}^3$, if there are $i, j \in \{1, 2, \ldots, N\}$, $i \neq j$, such that $\mathbf{q}_i \boxdot \mathbf{q}_j = -1$, the bodies m_i and m_j form a collision configuration, whereas configurations with $\mathbf{q}_i \boxdot \mathbf{q}_j = 1$ don't exist.*

Proof. Let us first prove the implication related to collision configurations for positive curvature. Assume that there exist $i, j \in \{1, 2, \ldots, N\}$, $i \neq j$, such that $\mathbf{q}_i \cdot \mathbf{q}_j = 1$, a relationship that can be written as

$$w_i w_j + x_i x_j + y_i y_j + z_i z_j = 1. \tag{3.29}$$

But since the bodies are in $\mathbb{S}^3$, the coordinates satisfy the conditions

$$w_i^2 + x_i^2 + y_i^2 + z_i^2 = w_j^2 + x_j^2 + y_j^2 + z_j^2 = 1.$$

Consequently we can write that

$$(w_iw_j + x_ix_j + y_iy_j + z_iz_j)^2 = (w_i^2 + x_i^2 + y_i^2 + \sigma z_i^2)(w_j^2 + x_j^2 + y_j^2 + z_j^2),$$

which is the equality case of the Cauchy-Schwarz inequality. Therefore there is a constant $\tau \neq 0$ such that $w_j = \tau w_i,\ x_j = \tau x_i,\ y_j = \tau y_i$, and $z_j = \tau z_i$. Substituting these values in equation (3.29), we obtain that

$$\tau(w_iw_j + x_ix_j + y_iy_j + z_iz_j) = 1. \tag{3.30}$$

Comparing (3.29) and (3.30), it follows that $\tau = 1$, so $w_i = w_j$, $x_i = x_j$, $y_i = y_j$, and $z_i = z_j$, therefore the bodies m_i and m_j form a collision configuration.

The proof of the implication related to antipodal configurations for positive curvature is very similar. Instead of relation (3.29), we have

$$w_iw_j + x_ix_j + y_iy_j + z_iz_j = -1.$$

Then, following the above steps, we obtain that $\tau = -1$, so $w_i = -w_j$, $x_i = -x_j$, $y_i = -y_j$, and $z_i = -z_j$, therefore the bodies m_i and m_j form an antipodal configuration.

Let us now prove the implication related to collision configurations in the case of negative curvature. Assume that there exist $i, j \in \{1, 2, \dots, N\},\ i \neq j$, such that $\mathbf{q}_i \boxdot \mathbf{q}_j = -1$, a relationship that can be written as

$$w_iw_j + x_ix_j + y_iy_j - z_iz_j = -1,$$

which is equivalent to

$$w_iw_j + x_ix_j + y_iy_j + 1 = z_iz_j. \tag{3.31}$$

But since the bodies are in $\mathbb{H}^3$, the coordinates satisfy the conditions

$$w_i^2 + x_i^2 + y_i^2 - z_i^2 = w_j^2 + x_j^2 + y_j^2 - z_j^2 = -1,$$

which are equivalent to

$$w_i^2 + x_i^2 + y_i^2 + 1 = z_i^2 \ \text{ and } \ w_j^2 + x_j^2 + y_j^2 + 1 = z_j^2. \tag{3.32}$$

From (3.31) and (3.32), we can conclude that

$$(w_iw_j + x_ix_j + y_iy_j + 1)^2 = (w_i^2 + x_i^2 + y_i^2 + 1)(w_j^2 + x_j^2 + y_j^2 + 1),$$

which is the same as

$$(w_i x_j - w_j x_i)^2 + (w_i y_j - w_j y_i)^2 + (x_i y_j - x_j y_i)^2 +$$
$$[(w_i - w_j)^2 + (x_i - x_j)^2 + (y_i - y_j)^2] = 0.$$

It follows from the above relation that $w_i = w_j,\ x_i = x_j$, and $y_i = y_j$. Then relation (3.32) implies that $z_i^2 = z_j^2$. But since for negative curvature all z coordinates are positive, we can conclude that $z_i = z_j$, so the bodies m_i and m_j form a collision configuration.

For negative curvature, we can now prove the non-existence of configurations with $\mathbf{q}_i \boxdot \mathbf{q}_j = 1$ with $i, j \in \{1, 2, \dots, N\}, i \neq j$. Let us assume that they exist. Then

$$w_i w_j + x_i x_j + y_i y_j = z_i z_j + 1. \tag{3.33}$$

But we also have that

$$w_i^2 + x_i^2 + y_i^2 = z_i^2 - 1 \quad \text{and} \quad w_j^2 + x_j^2 + y_j^2 = z_j^2 - 1. \tag{3.34}$$

It then follows from the Cauchy-Schwarz inequality that

$$(w_i w_j + x_i x_j + y_i y_j)^2 \leq (w_i^2 + x_i^2 + y_i^2)(w_j^2 + x_j^2 + y_j^2),$$

so, by (3.33) and (3.34), we can conclude that

$$(z_i z_j + 1)^2 \leq (z_i^2 - 1)(z_j^2 - 1),$$

which is equivalent to

$$(z_i + z_j)^2 \leq 0.$$

This relationship is satisfied only if $z_i = -z_j$, which is impossible because $z_i, z_j > 0$, a contradiction that completes the proof. $\square$

It is easy to construct solutions ending in collisions. In $\mathbb{S}^3$, we can place, for instance, 3 bodies of equal masses at the vertices of an Euclidean equilateral triangle, not lying on the same great circle, and release them with zero initial velocities. The bodies will end up in a triple collision. (If the bodies lie initially on a geodesic and have zero initial velocities, they won't move in time, a situation that corresponds to a fixed-point solution of the equations of motion, as we will show in Chapter 6.) The question of whether there exist solutions ending in antipodal or hybrid (collision-antipodal) singularities is harder, and it was treated in [36]. But since we are not touching on this subject when dealing with relative equilibria, we will not discuss it further. All we need to worry about in this monograph is to avoid placing the bodies at singular initial configurations, i.e. at collisions, for any curvature, or at antipodal positions for positive curvature.

3.10 Some physical remarks

The antipodal and the collision-antipodal singularities seem to obstruct the natural translation of the dynamical properties of the N-body problem from 0 to positive curvature. To better understand this issue, let us first compare how the force function and its gradient differ in the Euclidean and in the curved case. For simplicity, we will restrict to $\mathbb{R}^2$, $\mathbb{S}^2$ and $\mathbb{H}^2$, but all the remarks below can be extended to the 3-dimensional case.

Let's start with $N = 2$. The Euclidean force function, $U_0(\mathbf{q}) = m_1 m_2/|\mathbf{q}_1 - \mathbf{q}_2|$, is infinite at collision and tends to zero when the distance between bodies tends to infinity. The norm of the gradient, $|\nabla U_0(\mathbf{q})|$, has a similar behavior, which agrees with our perception that the gravitational force decreases when the distance between bodies increases. But the curved force function,

$$U(\mathbf{q}) = \frac{\sigma m_1 m_2 (\mathbf{q}_1 \odot \mathbf{q}_2)}{[\sigma - \sigma(\mathbf{q}_1 \odot \mathbf{q}_2)^2]^{1/2}},$$

and the norm of its gradient obtained from (3.11),

$$|\widetilde{\nabla} U(\mathbf{q})| = \left[\frac{m_1^2 m_2^2 (\mathbf{q}_2 \odot \mathbf{q}_2) |(\mathbf{q}_1 \odot \mathbf{q}_1)\mathbf{q}_2 - \sigma(\mathbf{q}_1 \odot \mathbf{q}_2)\mathbf{q}_1|^2}{[\sigma - \sigma(\mathbf{q}_1 \odot \mathbf{q}_2)^2]^3} \right.$$
$$\left. + \frac{m_1^2 m_2^2 (\mathbf{q}_1 \odot \mathbf{q}_1) |(\mathbf{q}_1 \odot \mathbf{q}_2)\mathbf{q}_1 - \sigma(\mathbf{q}_1 \odot \mathbf{q}_2)\mathbf{q}_2|^2}{[\sigma - \sigma(\mathbf{q}_1 \odot \mathbf{q}_2)^2]^3} \right]^{1/2},$$

which stems from a homogeneous function of degree -1, depend on the sign of the curvature. In $\mathbb{H}^2$ the motion behaves qualitatively as in $\mathbb{R}^2$. In $\mathbb{S}^2$, let's assume that one body is fixed at the north pole. Then U ranges from $+\infty$ at collision to $-\infty$ at the antipodal configuration, taking the value 0 when the second body is on the equator. The norm of the gradient is $+\infty$ at collision, and becomes smaller when the second body lies farther away from collision in the northern hemisphere; it takes a positive minimum value on the equator; and becomes larger the farther the second body is from the north pole while lying in the southern hemisphere; finally, the norm of the gradient becomes $+\infty$ when the two bodies lie at antipodes.

This behavior of the gradient seems to agree with our understanding of gravitation only when the second body doesn't leave the northern hemisphere, but not after it passes the equator, i.e. only when the arc distance between the two bodies does not exceed $\pi/2$. There is cosmological evidence that, in a hypothetical spherical universe with billions of objects ejected from a Big-Bang that took place at the north pole, all the bodies are still close to the origin of the explosion, i.e. far away from the equator, [6]. But when the expanding system approaches the equator, many bodies come close to antipodal singularities, so the potential energy becomes positive, like the kinetic energy. By the energy integral, the potential energy cannot grow beyond the value of the energy constant, which, when reached, makes the kinetic energy zero and stops any motion. (Of course, the larger the initial velocities are, the greater the energy constant will be, but finite nevertheless, so the moment when the system stops moving arrives sooner or later.) The motion then reverses from expansion to contraction, in agreement with the cosmological scenario of general relativity. So in a highly populated spherical universe, the motion would be contained in the northern hemisphere, away from the equator, never able to cross into the southern hemisphere. Obviously, all this happens only if all bodies are initially in the northern hemisphere, a restriction we don't have to take into consideration for a general dynamical study in $\mathbb{S}^2$. When the motion takes place in Euclidean or hyperbolic space, the Big-Bang could lead to a finite, eventually collapsing, or infinite, eternally expanding universe, depending on the initial velocities taken close to a singularity. These remarks explain the table presented at the end of the Preface.

3.11 The curved Kepler problem

As mentioned earlier, the Kepler problem describes the motion of a single body around a fixed point. An important property of the Kepler potential in the Euclidean case is that of being a harmonic function in $\mathbb{R}^3$, i.e. it satisfies Laplace's equation,

$$\frac{\partial^2 U(x,y,z)}{\partial x^2} + \frac{\partial^2 U(x,y,z)}{\partial y^2} + \frac{\partial^2 U(x,y,z)}{\partial z^2} = 0, \tag{3.35}$$

where

$$U\colon \mathbb{R}^3 \to \mathbb{R}, \quad U(x,y,z) = \frac{m}{(x^2+y^2+z^2)^{1/2}}$$

is the force function (recall that $-U$ is the potential) and (x,y,z) are the coordinates of the body of mass m, which moves around the origin of the coordinate system. Indeed,

$$\frac{\partial U(x,y,z)}{\partial x} = -\frac{mx}{(x^2+y^2+z^2)^{3/2}} \quad \text{and} \quad \frac{\partial^2 U(x,y,z)}{\partial x^2} = -\frac{m(2x^2-y^2-z^2)}{(x^2+y^2+z^2)^{5/2}},$$

with the corresponding expressions obtained by circular permutations for $\frac{\partial^2 U(x,y,z)}{\partial y^2}$ and $\frac{\partial^2 U(x,y,z)}{\partial z^2}$, which show that U satisfies equation (3.35). It is easy to see that this property is not satisfied if we restrict the Kepler potential to $\mathbb{R}^2$.

Similar things happen in curved space. The Laplace-Beltrami equation for some function $g\colon \mathbb{M}^3_\kappa \to \mathbb{R}$ can be written as

$$\frac{\partial^2 f(\mathbf{q})}{\partial w^2} + \frac{\partial^2 f(\mathbf{q})}{\partial x^2} + \frac{\partial^2 f(\mathbf{q})}{\partial y^2} + \sigma \frac{\partial^2 f(\mathbf{q})}{\partial z^2} = 0, \tag{3.36}$$

where $\mathbf{q} = (w, x, y, z)$ and $f(\mathbf{q}) = g(\mathbf{q}/\sqrt{\sigma \mathbf{q} \odot \mathbf{q}})$, see, e.g., [57]. In our case, g is the force function (3.18) when a single body rotates around a fixed point. To derive the expression of the curved Kepler force function, U, in this particular case, let us choose $\mathcal{N} = (0, 0, 0, 1)$ to play the role of the fixed point, assumed to act as if having mass 1. This choice of the fixed point does not restrict generality, but will make our computations easier. Let then $\mathbf{q} = (w, x, y, z)$ denote the position vector of the body of mass m. A straightforward computation shows that U, given by (3.18), becomes

$$U(\mathbf{q}) = U(w, x, y, z) = \frac{mz}{(w^2 + x^2 + y^2)^{1/2}}.$$

By computation, or simply invoking the fact that U is a homogeneous function of degree 0, we can see that

$$U(\mathbf{q}/\sqrt{\sigma \mathbf{q} \odot \mathbf{q}}) = U(\mathbf{q}),$$

which means that U can be taken as the function f in equation (3.36). Notice that

$$\frac{\partial U(w, x, y, z)}{\partial w} = -\frac{mwz}{(x^2 + y^2 + z^2)^{3/2}},$$

therefore

$$\frac{\partial^2 U(w, x, y, z)}{\partial w^2} = \frac{m(2w^2 - x^2 - y^2)z}{(x^2 + y^2 + z^2)^{5/2}},$$

with similar formulas, obtained by circular permutations, for the partial derivatives $\frac{\partial^2 U(w,x,y,z)}{\partial x^2}$ and $\frac{\partial^2 U(w,x,y,z)}{\partial y^2}$. Using the fact that

$$\frac{\partial U(w, x, y, z)}{\partial z} = \frac{m}{(w^2 + x^2 + y^2)^{1/2}} \quad \text{and} \quad \frac{\partial^2 U(w, x, y, z)}{\partial z^2} = 0,$$

we see that the curved Kepler force function, U, satisfies the Laplace-Beltrami equation (3.36), therefore U is a harmonic function in $\mathbb{M}^3$. It is an easy exercise to check that, as in the Euclidean case, this property is not satisfied if we restrict U to $\mathbb{M}^2$.

So both the classical Kepler potential in Euclidean space and its extension to spaces of constant curvature are harmonic functions in the 3-dimensional, but not in the 2-dimensional, case. This analogy explains why the extension introduced here for the Newtonian gravitational law to spaces of constant curvature, originally suggested by Bolyai and Lobachevsky for hyperbolic space, provides a natural gravitational model within the framework of classical mechanics.

Part II

Isometries and Relative Equilibria

Preamble

We will begin now to study the equations of gravitational motion for N point masses in $\mathbb{S}^3$ and $\mathbb{H}^3$. The goal of Part II is to define the concept of relative equilibrium for the curved N-body problem. Since for this kind of orbits the mutual distances remain constant during the motion, we need first to understand the isometric rotations in $\mathbb{S}^3$ and $\mathbb{H}^3$. Then we will define a natural class of relative equilibria for each possible isometry. At the end we will study a particular type of relative equilibrium, the fixed point, i.e. explore the possibility that the particles don't move at all. Such orbits don't occur in the classical case, and neither do they show up in $\mathbb{H}^3$, but we will prove that they exist in $\mathbb{S}^3$.

Chapter 4

Isometric rotations

In this chapter we will first introduce the isometries of $\mathbb{R}^4$ and $\mathbb{R}^{3,1}$ and connect them with the corresponding principal axis theorem. Then we aim to understand how these isometries act in $\mathbb{S}^3$ and $\mathbb{H}^3$. In fact we will be interested only in the transformations represented by the matrices A and B, defined in (4.1) and (4.2), respectively. As we will see in Section 7.6, the negative parabolic rotations represented by the matrix C, defined in (4.3), generate no relative equilibria in $\mathbb{H}^3$, so we don't need to worry about their geometric properties for the purposes of the research presented in this monograph. Our main reason for investigating these geometrical aspects is that of understanding how some previous research we did on the curved N-body problem in $\mathbb{S}^2$ and $\mathbb{H}^2$ can be geometrically connected to $\mathbb{S}^3$ and $\mathbb{H}^3$, respectively. In other words, we would like to see whether the above rotations preserve 2-dimensional spheres in $\mathbb{S}^3$ and $\mathbb{H}^3$ and 2-dimensional hyperbolic spheres in $\mathbb{H}^3$.

4.1 The principal axis theorems

This section introduces the isometric rotations in $\mathbb{S}^3$ and $\mathbb{H}^3$, since they play an essential role in defining the relative equilibria of the curved N-body problem, and connects them to the principal axis theorems of $\mathbb{R}^4$ and $\mathbb{R}^{3,1}$. There are many ways to express these rotations, but their matrix representations will suit our goals best, as they did in Section 2.1 for the 2-dimensional hyperbolic sphere $\mathbb{H}^2$.

For positive curvature, the isometric transformations of $\mathbb{S}^3$ are given by the elements of the Lie group $SO(4)$ of $\mathbb{R}^4$ that keep $\mathbb{S}^3$ invariant. They consist of all orthogonal transformations of the Lie group $O(4)$ represented by matrices of determinant 1, and have the form PAP^{-1}, with $P \in SO(4)$ and

$$A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix}, \tag{4.1}$$

where $\theta, \phi \in \mathbb{R}$. We will call these rotations positive elliptic-elliptic if $\theta \neq 0$ and $\phi \neq 0$, and positive elliptic if $\theta \neq 0$ and $\phi = 0$ (or $\theta = 0$ and $\phi \neq 0$, a possibility we ignore since it perfectly resembles the previous case). When $\theta = \phi = 0$, A is the identity matrix, so no rotation takes place. The above description is a generalization to $\mathbb{S}^3$ of Euler's principle axis theorem for $\mathbb{S}^2$. As we will next explain, the reference to a fixed axis is, from the geometric point of view, far from suggestive in $\mathbb{R}^4$.

The form of the matrix A given by (4.1) shows that the positive elliptic-elliptic transformations have two circular rotations, one relative to the origin of the coordinate system in the plane wx and the other relative to the same point in the plane yz. In this case, the only fixed point in $\mathbb{R}^4$ is the origin of the coordinate system. The positive elliptic transformations have a single rotation around the origin of the coordinate system that leaves infinitely many axes (in fact, an entire plan) of $\mathbb{R}^4$ pointwise fixed.

For negative curvature, the isometric transformations of $\mathbb{H}^3$ are given by the elements of the Lorentz group $\mathrm{Lor}(3,1)$, a Lie group in the Minkowski space $\mathbb{R}^{3,1}$. $\mathrm{Lor}(3,1)$ is formed by all orthogonal transformations of determinant 1 that keep $\mathbb{H}^3$ invariant. The elements of this group are negative elliptic, negative hyperbolic, and negative elliptic-hyperbolic transformations, on one hand, and negative parabolic transformations, on the other hand; they can be represented as matrices of the form PBP^{-1} and PCP^{-1}, respectively, with $P \in \mathrm{Lor}(3,1)$,

$$B = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cosh s & \sinh s \\ 0 & 0 & \sinh s & \cosh s \end{pmatrix}, \tag{4.2}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\xi & \xi \\ 0 & \xi & 1-\xi^2/2 & \xi^2/2 \\ 0 & \xi & -\xi^2/2 & 1+\xi^2/2 \end{pmatrix}, \tag{4.3}$$

where θ, s, ξ are some fixed values in $\mathbb{R}$. The negative elliptic, negative hyperbolic, and negative elliptic-hyperbolic transformations correspond to $\theta \neq 0$ and $s = 0$, to

$\theta = 0$ and $s \neq 0$, and to $\theta \neq 0$ and $s \neq 0$, respectively. The above description is a generalization to $\mathbb{H}^3$ of the principle axis theorem for $\mathbb{H}^2$, which we presented in Section 2.1. Again, as in the case of the group $SO(4)$, the reference to a fixed axis has no real geometric meaning in $\mathbb{R}^{3,1}$.

Indeed, from the geometric point of view, the negative elliptic transformations of $\mathbb{R}^{3,1}$ are similar to their counterpart, positive elliptic transformations, in $\mathbb{R}^4$, namely they have a single circular rotation around the origin of the coordinate system that leaves infinitely many axes of $\mathbb{R}^{3,1}$ pointwise invariant. The negative hyperbolic transformations correspond to a single hyperbolic rotation around the origin of the coordinate system that also leaves infinitely many axes of $\mathbb{R}^{3,1}$ pointwise invariant. The negative elliptic-hyperbolic transformations have two rotations, a circular one about the origin of the coordinate system, relative to the wx plane, and a hyperbolic one about the origin of the coordinate system, relative to the yz plane. The only point they leave fixed is the origin of the coordinate system. Finally, parabolic transformations correspond to parabolic rotations about the origin of the coordinate system that leave only the w axis pointwise fixed.

4.2 Invariance of 2-spheres

Let us start with the positive elliptic-elliptic rotations in $\mathbb{S}^3$ and consider first great spheres, which can be obtained, for instance, by the intersection of $\mathbb{S}^3$ with the hyperplane $z = 0$. We thus obtain the 2-dimensional great sphere

$$\mathbf{S}_z^2 = \{(w, x, y, 0) |\, w^2 + x^2 + y^2 = 1\}, \tag{4.4}$$

already defined in (2.5). Let $(w, x, y, 0)$ be a point on $\mathbf{S}_z^2$. Then the positive elliptic-elliptic transformation (4.1) takes $(w, x, y, 0)$ to the point (w_1, x_1, y_1, z_1) given by

$$\begin{pmatrix} w_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ 0 \end{pmatrix} = \tag{4.5}$$

$$\begin{pmatrix} w\cos\theta - x\sin\theta \\ w\sin\theta + x\cos\theta \\ y\cos\phi \\ y\sin\phi \end{pmatrix}.$$

Since, in general, y is not zero, it follows that $z_1 = y \sin\phi = 0$ only if $\phi = 0$, a case that corresponds to positive elliptic transformations. For a positive elliptic-elliptic transformation, (w_1, x_1, y_1, z_1) does not lie on $\mathbf{S}_z^2$ because this point is not in the hyperplane $z = 0$. Without loss of generality, we can always find a coordinate system in which the considered sphere is $\mathbf{S}_z^2$. We can therefore draw the following conclusion.

Remark 2. For every great sphere of $\mathbb{S}^3$, there is a suitable system of coordinates such that positive elliptic rotations leave the great sphere invariant. However, there is no system of coordinates for which we can find a positive elliptic-elliptic rotation that leaves a great sphere invariant.

Let us now see what happens with non-great spheres of $\mathbb{S}^3$. Such spheres can be obtained, for instance, by intersecting $\mathbb{S}^3$ with a hyperplane $z = z_0$, where $|z_0| < 1$ and $z_0 \neq 0$. These 2-dimensional spheres are of the form

$$\mathbf{S}^2_{\kappa_0, z_0} = \{(w, x, y, z) |\, w^2 + x^2 + y^2 = 1 - z_0^2,\, z = z_0\}, \tag{4.6}$$

where $\kappa_0 = (1 - z_0^2)^{-1/2}$ is the curvature.

Let (w, x, y, z_0) be a point on a non-great sphere $\mathbf{S}^2_{\kappa_0, z_0}$, given by some z_0 as above. Then the positive elliptic-elliptic transformation (4.1) takes the point (w, x, y, z_0) to the point (w_2, x_2, y_2, z_2) given by

$$\begin{pmatrix} w_2 \\ x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z_0 \end{pmatrix} = \tag{4.7}$$

$$\begin{pmatrix} w\cos\theta - x\sin\theta \\ w\sin\theta + x\cos\theta \\ y\cos\phi - z_0\sin\phi \\ y\sin\phi + z_0\cos\phi \end{pmatrix}.$$

Since, in general, y is not zero, it follows that $z_2 = y\sin\phi + z_0\cos\phi = z_0$ only if $\phi = 0$, a case that corresponds to positive elliptic transformations. In the case of a positive elliptic-elliptic transformation, the point (w_2, x_2, y_2, z_2) does not lie on $\mathbf{S}^2_{\kappa_0, z_0}$ because this point is not in the hyperplane $z = z_0$. Without loss of generality, we can always reduce the question we posed above to the sphere $\mathbf{S}^2_{\kappa_0, z_0}$. We can therefore draw the following conclusion, which resembles Remark 2.

Remark 3. For every non-great sphere of $\mathbb{S}^3$, there is a suitable system of coordinates such that positive elliptic rotations leave that non-great sphere invariant. However, there is no system of coordinates for which there exists a positive elliptic-elliptic rotation that leaves a non-great sphere invariant.

Since in $\mathbb{H}^3$ we have $z > 0$, 2-dimensional spheres cannot be centered around the origin of the coordinate system. We therefore look for 2-dimensional spheres centered on the z axis, with $z > 1$, because $z = 1$ is the smallest possible z coordinate in $\mathbb{H}^3$, attained only by the point $(0, 0, 0, 1)$. Such spheres can be obtained by intersecting $\mathbb{H}^3$ with a plane $z = z_0$, where $z_0 > 1$, and they are given by

$$\mathbf{S}^{2,h}_{\kappa_0,z_0} = \{(w, x, y, z) |\, w^2 + x^2 + y^2 = z_0^2 - 1,\, z = z_0\}, \tag{4.8}$$

where h indicates that the spheres lie in a 3-dimensional hyperbolic space, and $\kappa_0 = (z_0^2 - 1)^{-1/2} > 0$ is the curvature of the sphere.

Let (w, x, y, z_0) be a point on the sphere $\mathbf{S}^{2,h}_{\kappa_0,z_0}$. Then the negative elliptic transformation B, given by (4.2) with $s = 0$, takes the point (w, x, y, z_0) to the point (w_3, x_3, y_3, z_3) given by

$$\begin{pmatrix} w_3 \\ x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z_0 \end{pmatrix} = \begin{pmatrix} w\cos\theta - x\sin\theta \\ w\sin\theta + x\cos\theta \\ y \\ z_0 \end{pmatrix}, \tag{4.9}$$

which also lies on the sphere $\mathbf{S}^{2,h}_{\kappa_0,z_0}$. Indeed, since $z_3 = z_0$ and $w_3^2 + x_3^2 + y_3^2 = z_0^2 - 1$, it means that (w_3, x_3, y_3, z_3) also lies on the sphere $\mathbf{S}^{2,h}_{\kappa_0,z_0}$.

Since for any 2-dimensional sphere of $\mathbb{H}^3$ we can find a coordinate system and suitable values for κ_0 and z_0 such that the sphere has the form $\mathbf{S}^{2,h}_{\kappa_0,z_0}$, we can draw the following conclusion.

Remark 4. For every 2-dimensional sphere of $\mathbb{H}^3$, there is a system of coordinates such that negative elliptic rotations leave the sphere invariant.

Let us further see what happens with negative hyperbolic transformations in $\mathbb{H}^3$. Let (w, x, y, z_0) be a point on the sphere $\mathbf{S}^{2,h}_{\kappa_0,z_0}$. Then the negative hyperbolic transformation B, given by (4.2) with $\theta = 0$, takes the point (w, x, y, z_0) to (w_4, x_4, y_4, z_4) given by

$$\begin{pmatrix} w_4 \\ x_4 \\ y_4 \\ z_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh s & \sinh s \\ 0 & 0 & \sinh s & \cosh s \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z_0 \end{pmatrix} = \tag{4.10}$$

$$\begin{pmatrix} w \\ x \\ y\cosh s + z_0 \sinh s \\ y \sinh s + z_0 \cosh s \end{pmatrix},$$

which does not lie on $\mathbf{S}^{2,h}_{\kappa_0,z_0}$. Indeed, since $z_3 = y\sinh s + z_0 \cosh s = z_0$ only for $s = 0$, a case we exclude because the above transformation is the identity, the point (w_4, x_4, y_4, z_4) does not lie on a sphere of radius $\sqrt{z_0^2 - 1}$. Therefore we can draw the following conclusion.

Remark 5. Given a 2-dimensional sphere of curvature $\kappa_0 = (z_0 - 1)^{-1/2}$, with $z_0 > 1$, in $\mathbb{H}^3_\kappa$, there is no coordinate system for which some negative hyperbolic transformation would leave the sphere invariant. Consequently the same holds for negative elliptic-hyperbolic transformations.

We will further see how the problem of invariance relates to 2-dimensional hyperbolic spheres in $\mathbb{H}^3$.

4.3 Invariance of hyperbolic 2-spheres

Let us first check whether negative elliptic rotations preserve the great 2-dimensional hyperbolic spheres of $\mathbb{H}^3$. For this consider the 2-dimensional hyperbolic sphere

$$\mathbf{H}^2_y = \{(w, x, 0, z) \mid w^2 + x^2 - z^2 = -1\}, \tag{4.11}$$

already defined in (2.9), and obtained by intersecting $\mathbb{H}^3$ with the hyperplane $y = 0$. Let $(w, x, 0, z)$ be a point on $\mathbf{H}^2_y$. Then a negative elliptic rotation takes the point $(w, x, 0, z)$ to the point (w_5, x_5, y_5, z_5) given by

$$\begin{pmatrix} w_5 \\ x_5 \\ y_5 \\ z_5 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w \\ x \\ 0 \\ z \end{pmatrix} = \tag{4.12}$$

$$\begin{pmatrix} w\cos\theta - x\sin\theta \\ w\sin\theta + x\cos\theta \\ 0 \\ z \end{pmatrix},$$

which, obviously, also belongs to $\mathbf{H}^2_y$. Since for any 2-dimensional hyperbolic sphere of curvature κ we can find a coordinate system such that the hyperbolic sphere can be represented as $\mathbf{H}^2_y$, we can draw the following conclusion.

Remark 6. Given a great 2-dimensional hyperboloid in $\mathbb{H}^3$, there is a coordinate system for which the hyperbolic sphere is invariant to negative elliptic rotations.

Let us further check what happens in the case of negative hyperbolic rotations. Consider the great 2-dimensional hyperbolic sphere given by

$$\mathbf{H}_w^2 = \{(0, x, y, z) \mid x^2 + y^2 - z^2 = -1\}, \tag{4.13}$$

already defined in (2.9), and obtained by intersecting $\mathbb{H}^3$ with the hyperplane $w = 0$. Let $(0, x, y, z)$ be a point on $\mathbf{H}_w^2$. Then a negative hyperbolic rotation takes the point $(0, x, y, z)$ to the point (w_6, x_6, y_6, z_6) given by

$$\begin{pmatrix} w_6 \\ x_6 \\ y_6 \\ z_6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh s & \sinh s \\ 0 & 0 & \sinh s & \cosh s \end{pmatrix} \begin{pmatrix} 0 \\ x \\ y \\ z \end{pmatrix} = \tag{4.14}$$

$$\begin{pmatrix} 0 \\ x \\ y\cosh s + z \sinh s \\ y \sinh s + z \sinh s \end{pmatrix},$$

which, obviously, also belongs to $\mathbf{H}_w^2$. Since for any great 2-dimensional hyperbolic sphere we can find a coordinate system such that the hyperbolic sphere can be represented as $\mathbf{H}_w^2$, we can draw the following conclusions.

Remark 7. Given a great 2-dimensional hyperbolic sphere in $\mathbb{H}^3$, there is a coordinate system for which the hyperbolic sphere is invariant to negative hyperbolic rotations.

Remark 8. The coordinate system in Remark 6 is different from the coordinate system in Remark 7, so negative elliptic-hyperbolic transformation don't leave great 2-dimensional hyperbolic spheres invariant in $\mathbb{H}^3$.

The next step is to see whether negative elliptic rotations preserve the 2-dimensional hyperbolic spheres of curvature $\kappa_0 = -(y_0^2 + 1)^{-1/2}$ of $\mathbb{H}^3$. For this consider the 2-dimensional hyperbolic sphere

$$\mathbf{H}_{\kappa_0, y_0}^2 = \{(w, x, y, z) \mid w^2 + x^2 - z^2 = -1 - y_0^2,\ y = y_0\}, \tag{4.15}$$

obtained by intersecting $\mathbb{H}^3$ with the hyperplane $y = y_0$, with $y_0 \neq 0$. Let (w, x, y_0, z) be a point on $\mathbf{H}^2_y$. Then a negative elliptic rotation takes the point (w, x, y_0, z) to the point (w_7, x_7, y_7, z_7) given by

$$\begin{pmatrix} w_7 \\ x_7 \\ y_7 \\ z_7 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w \\ x \\ y_0 \\ z \end{pmatrix} = \begin{pmatrix} w\cos\theta - x\sin\theta \\ w\sin\theta + x\cos\theta \\ y_0 \\ z \end{pmatrix}, \tag{4.16}$$

which, obviously, also belongs to $\mathbf{H}^2_{\kappa_0,y_0}$. Since for any 2-dimensional hyperbolic sphere of curvature κ_0 we can find a coordinate system such that the hyperbolic sphere can be represented as $\mathbf{H}^2_{\kappa_0,y_0}$, we can draw the following conclusion.

Remark 9. Given a 2-dimensional hyperbolic sphere of curvature $\kappa_0 = -(y_0^2+1)^{-1/2}$ in $\mathbb{H}^3$, there is a coordinate system for which the hyperbolic sphere is invariant to negative elliptic rotations.

Consider finally the 2-dimensional hyperbolic sphere of curvature $\kappa_0 = -(w_0^2 + 1)^{-1/2}$ given by

$$\mathbf{H}^2_{\kappa_0,w_0} = \{(w, x, y, z) \mid x^2 + y^2 - z^2 = -1 - w_0^2,\ w = w_0\}, \tag{4.17}$$

obtained by intersecting $\mathbb{H}^3$ with the hyperplane $w = w_0 \neq 0$. Let the point (w_0, x, y, z) belong to $\mathbf{H}^2_{\kappa_0,w_0}$. Then a negative hyperbolic rotation takes the point (w_0, x, y, z) to the point (w_8, x_8, y_8, z_8) given by

$$\begin{pmatrix} w_8 \\ x_8 \\ y_8 \\ z_8 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh s & \sinh s \\ 0 & 0 & \sinh s & \cosh s \end{pmatrix} \begin{pmatrix} w_0 \\ x \\ y \\ z \end{pmatrix} = \tag{4.18}$$

$$\begin{pmatrix} w_0 \\ x \\ y\cosh s + z\sinh s \\ y\sinh s + z\sinh s \end{pmatrix},$$

which, obviously, also belongs to $\mathbf{H}^2_{\kappa_0,w_0}$. Since for any 2-dimensional hyperbolic sphere of curvature $\kappa_0 = -(w_0^2 + 1)^{-1/2}$ we can find a coordinate system such that the hyperbolic sphere can be represented as $\mathbf{H}^2_{\kappa_0,w_0}$, we can draw the following conclusions.

Remark 10. Given a 2-dimensional hyperbolic sphere of curvature $\kappa_0 = -(w_0^2 + 1)^{-1/2}$ in $\mathbb{H}^3$, there is a coordinate system for which the hyperbolic sphere is invariant to negative hyperbolic rotations.

Remark 11. Since the coordinate system in Remark 9 is different from the coordinate system in Remark 10, negative elliptic-hyperbolic transformations don't leave 2-dimensional hyperbolic spheres of curvature $\kappa_0 \neq -1$ invariant in $\mathbb{H}^3$.

Chapter 5

Relative equilibria (RE)

The goal of this chapter is to introduce the concepts we will explore in the rest of this monograph, namely the relative equilibrium solutions, also called relative equilibrium orbits or, simply, relative equilibria (from now on denoted by RE, whether in the singular or the plural form of the noun) of the curved N-body problem. For RE, the particle system behaves like a rigid body, i.e. all the mutual distances between the point masses remain constant during the motion. In other words, the bodies move under the action of an element belonging to a rotation group, so, in the light of Chapter 4, we can define six types of RE in $\mathbb{M}^3$: two in $\mathbb{S}^3$ and four in $\mathbb{H}^3$. In each case, we will bring the expressions involved in these natural definitions to simpler forms. In Section 7.6 we will see that one of the four types of RE we define in $\mathbb{H}^3$ does not translate into solutions of the equations of motion.

5.1 Positive elliptic RE

The first kind of RE we will introduce are inspired by the positive elliptic rotations of $\mathbb{S}^3$.

Definition 6. (Positive elliptic RE) *Let* $\mathbf{q}^0 = (\mathbf{q}_1^0, \mathbf{q}_2^0, \ldots, \mathbf{q}_N^0)$ *be a nonsingular initial position of the point particles of masses* $m_1, m_2, \ldots, m_N > 0, N \geq 2$, *on the manifold* $\mathbb{S}^3$, *where* $\mathbf{q}_i^0 = (w_i^0, x_i^0, y_i^0, z_i^0),\ i = 1, 2, \ldots, N$. *Then a solution of the form* $\mathbf{q} = (\mathcal{A}[\mathbf{q}_1^0]^T, \mathcal{A}[\mathbf{q}_2^0]^T, \ldots, \mathcal{A}[\mathbf{q}_N^0]^T)$ *of system* (3.17), *with*

$$\mathcal{A}(t) = \begin{pmatrix} \cos\alpha t & -\sin\alpha t & 0 & 0 \\ \sin\alpha t & \cos\alpha t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5.1}$$

where the upper ^{T} denotes the transpose of a vector and $\alpha \neq 0$ represents the frequency, is called a (simply rotating) positive elliptic RE.

Remark 12. In $\mathcal{A}$, the elements involving trigonometric functions could be in the lower right corner instead of the upper left corner of the matrix, but the behavior of the bodies would be similar, so we will always use the above form of the matrix.

If $r_i =: \sqrt{(w_i^0)^2 + (x_i^0)^2}$, we can find constants $a_i \in \mathbb{R},\ i = 1, 2, \dots, N$, such that $w_i^0 = r_i \cos a_i,\ x_i^0 = r_i \sin a_i,\ i = 1, 2, \dots, N$. Then

$$\mathcal{A}(t)[\mathbf{q}_i^0]^T = \begin{pmatrix} w_i^0 \cos \alpha t - x_i^0 \sin \alpha t \\ w_i^0 \sin \alpha t + x_i^0 \cos \alpha t \\ y_i^0 \\ z_i^0 \end{pmatrix} =$$

$$\begin{pmatrix} r_i \cos a_i \cos \alpha t - r_i \sin a_i \sin \alpha t \\ r_i \cos a_i \sin \alpha t + r_i \sin a_i \cos \alpha t \\ y_i^0 \\ z_i^0 \end{pmatrix} = \begin{pmatrix} r_i \cos(\alpha t + a_i) \\ r_i \sin(\alpha t + a_i) \\ y_i^0 \\ z_i^0 \end{pmatrix},$$

$i = 1, 2, \dots, N$.

5.2 Positive elliptic-elliptic RE

The second kind of RE we will introduce here are inspired by the positive elliptic-elliptic rotations of $\mathbb{S}^3$.

Definition 7. (Positive elliptic-elliptic RE) *Let $\mathbf{q}^0 = (\mathbf{q}_1^0, \mathbf{q}_2^0, \dots, \mathbf{q}_N^0)$ be a nonsingular initial position of the bodies of masses $m_1, m_2, \dots, m_N > 0, N \geq 2$, on the manifold $\mathbb{S}^3$, where $\mathbf{q}_i^0 = (w_i^0, x_i^0, y_i^0, z_i^0),\ i = 1, 2, \dots, N$. Then a solution of the form $\mathbf{q} = (\mathcal{B}[\mathbf{q}_1^0]^T, \mathcal{B}[\mathbf{q}_2^0]^T, \dots, \mathcal{B}[\mathbf{q}_N^0]^T)$ of system* (3.17)*, with*

$$\mathcal{B}(t) = \begin{pmatrix} \cos \alpha t & -\sin \alpha t & 0 & 0 \\ \sin \alpha t & \cos \alpha t & 0 & 0 \\ 0 & 0 & \cos \beta t & -\sin \beta t \\ 0 & 0 & \sin \beta t & \cos \beta t \end{pmatrix}, \tag{5.2}$$

where $\alpha, \beta \neq 0$ are the frequencies, is called a (doubly rotating) positive elliptic-elliptic RE.

If $r_i =: \sqrt{(w_i^0)^2 + (x_i^0)^2}$, $\rho_i := \sqrt{(y_i^0)^2 + (z_i^0)^2}$, we can find constants $a_i, b_i \in \mathbb{R}$, $i = 1, 2, \ldots, N$, such that $w_i^0 = r_i \cos a_i$, $x_i^0 = r_i \sin a_i$, $y_i^0 = \rho_i \cos b_i$, and $z_i^0 = \rho_i \sin b_i$, $i = 1, 2, \ldots, N$. Then

$$\mathcal{B}(t)[\mathbf{q}_i^0]^T = \begin{pmatrix} w_i^0 \cos \alpha t - x_i^0 \sin \alpha t \\ w_i^0 \sin \alpha t + x_i^0 \cos \alpha t \\ y_i^0 \cos \beta t - z_i^0 \sin \beta t \\ y_i^0 \sin \beta t + z_i^0 \cos \beta t \end{pmatrix} =$$

$$\begin{pmatrix} r_i \cos a_i \cos \alpha t - r_i \sin a_i \sin \alpha t \\ r_i \cos a_i \sin \alpha t + r_i \sin a_i \cos \alpha t \\ \rho_i \cos b_i \cos \beta t - \rho_i \sin b_i \sin \beta t \\ \rho_i \cos b_i \sin \beta t + \rho_i \sin b_i \cos \beta t \end{pmatrix} = \begin{pmatrix} r_i \cos(\alpha t + a_i) \\ r_i \sin(\alpha t + a_i) \\ \rho_i \cos(\beta t + b_i) \\ \rho_i \sin(\beta t + b_i) \end{pmatrix},$$

$i = 1, 2, \ldots, N$.

5.3 Negative elliptic RE

The third kind of RE we will introduce here are inspired by the negative elliptic rotations of $\mathbb{H}^3$.

Definition 8. (Negative elliptic RE) *Let $\mathbf{q}^0 = (\mathbf{q}_1^0, \mathbf{q}_2^0, \ldots, \mathbf{q}_N^0)$ be a nonsingular initial position of the point particles of masses $m_1, m_2, \ldots, m_N > 0, N \geq 2$, in $\mathbb{H}^3$, where $\mathbf{q}_i^0 = (w_i^0, x_i^0, y_i^0, z_i^0)$, $i = 1, 2, \ldots, N$. Then a solution of system* (3.17) *of the form $\mathbf{q} = (\mathcal{C}[\mathbf{q}_1^0]^T, \mathcal{C}[\mathbf{q}_2^0]^T, \ldots, \mathcal{C}[\mathbf{q}_N^0]^T)$, with*

$$\mathcal{C}(t) = \begin{pmatrix} \cos \alpha t & -\sin \alpha t & 0 & 0 \\ \sin \alpha t & \cos \alpha t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5.3}$$

where $\alpha \neq 0$ is the frequency, is called a (simply rotating) negative elliptic RE.

If $r_i =: \sqrt{(w_i^0)^2 + (x_i^0)^2}$, we can find $a_i \in \mathbb{R}$, $i = 1, 2, \ldots, N$, such that $w_i^0 = r_i \cos a_i$, $x_i^0 = r_i \sin a_i$, $i = 1, 2, \ldots, N$, so

$$\mathcal{C}(t)[\mathbf{q}_i^0]^T = \begin{pmatrix} w_i^0 \cos \alpha t - x_i^0 \sin \alpha t \\ w_i^0 \sin \alpha t + x_i^0 \cos \alpha t \\ y_i^0 \\ z_i^0 \end{pmatrix} = \tag{5.4}$$

$$\begin{pmatrix} r_i \cos a_i \cos \alpha t - r_i \sin a_i \sin \alpha t \\ r_i \cos a_i \sin \alpha t + r_i \sin a_i \cos \alpha t \\ y_i^0 \\ z_i^0 \end{pmatrix} = \begin{pmatrix} r_i \cos(\alpha t + a_i) \\ r_i \sin(\alpha t + a_i) \\ y_i^0 \\ z_i^0 \end{pmatrix},$$

$i = 1, 2, \dots, N$.

5.4 Negative hyperbolic RE

The fourth kind of RE we will introduce here are inspired by the negative hyperbolic rotations of $\mathbb{H}^3$.

Definition 9. (Negative hyperbolic RE) *Let $\mathbf{q}^0 = (\mathbf{q}_1^0, \mathbf{q}_2^0, \dots, \mathbf{q}_N^0)$ be a nonsingular initial position of the bodies of masses $m_1, m_2, \dots, m_N > 0, N \geq 2$, in $\mathbb{H}^3$, where the initial positions are $\mathbf{q}_i^0 = (w_i^0, x_i^0, y_i^0, z_i^0)$, $i = 1, 2, \dots, N$. Then a solution of system* (3.17) *of the form $\mathbf{q} = (\mathcal{D}[\mathbf{q}_1^0]^T, \mathcal{D}[\mathbf{q}_2^0]^T, \dots, \mathcal{D}[\mathbf{q}_N^0]^T)$, with*

$$\mathcal{D}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \cosh \beta t & \sinh \beta t \\ 0 & 0 & \sinh \beta t & \cosh \beta t \end{pmatrix}, \tag{5.5}$$

where $\beta \neq 0$ denotes the frequency, is called a (simply rotating) negative hyperbolic RE.

If $\eta_i := \sqrt{(z_i^0)^2 - (y_i^0)^2}$, we can find constants $b_i \in \mathbb{R}$, $i = 1, 2, \dots, N$, such that $y_i^0 = \eta_i \sinh b_i$ and $z_i^0 = \eta_i \cosh b_i$, $i = 1, 2, \dots, N$. Then

$$\mathcal{D}(t)[\mathbf{q}_i^0]^T = \begin{pmatrix} w_i^0 \\ x_i^0 \\ y_i^0 \cosh bt + z_i^0 \sinh bt \\ y_i^0 \sinh bt + z_i^0 \cosh bt \end{pmatrix} =$$

$$\begin{pmatrix} w_i^0 \\ x_i^0 \\ \eta_i \sinh b_i \cosh bt + \eta_i \cosh b_i \sinh bt \\ \eta_i \sinh b_i \sinh bt + \eta_i \cosh b_i \cosh bt \end{pmatrix} = \begin{pmatrix} w_i^0 \\ x_i^0 \\ \eta_i \sinh(bt + b_i) \\ \eta_i \cosh(bt + b_i) \end{pmatrix},$$

$i = 1, 2, \dots, N$.

5.5 Negative elliptic-hyperbolic RE

The fifth kind of RE we will introduce here are inspired by the negative elliptic-hyperbolic rotations of $\mathbb{H}^3$.

Definition 10. (Negative elliptic-hyperbolic RE) *Let $\mathbf{q}^0 = (\mathbf{q}_1^0, \mathbf{q}_2^0, \dots, \mathbf{q}_N^0)$ be a nonsingular initial position of the point particles of masses $m_1, m_2, \dots, m_N > 0, N \geq 2$, in $\mathbb{H}^3$, where $\mathbf{q}_i^0 = (w_i^0, x_i^0, y_i^0, z_i^0)$, $i = 1, 2, \dots, N$. Then a solution of system* (3.17) *of the form $\mathbf{q} = (\mathcal{E}[\mathbf{q}_1^0]^T, \mathcal{E}[\mathbf{q}_2^0]^T, \dots, \mathcal{E}[\mathbf{q}_N^0]^T)$, with*

$$\mathcal{E}(t) = \begin{pmatrix} \cos\alpha t & -\sin\alpha t & 0 & 0 \\ \sin\alpha t & \cos\alpha t & 0 & 0 \\ 0 & 0 & \cosh\beta t & \sinh\beta t \\ 0 & 0 & \sinh\beta t & \cosh\beta t \end{pmatrix}, \tag{5.6}$$

where $\alpha, \beta \neq 0$ denote the frequencies, is called a (doubly rotating) negative elliptic-hyperbolic RE.

If $r_i := \sqrt{(w_i^0)^2 + (x_i^0)^2}$, $\eta_i := \sqrt{(z_i^0)^2 - (y_i^0)^2}$, we can find constants $a_i, b_i \in \mathbb{R}$, $i = 1, 2, \dots, N$, such that $w_i^0 = r_i \cos a_i$, $x_i^0 = r_i \sin a_i$, $y_i^0 = \eta_i \sinh b_i$, and $z_i^0 = \eta_i \cosh b_i$, $i = 1, 2, \dots, N$. Then

$$\mathcal{E}(t)[\mathbf{q}_i^0]^T = \begin{pmatrix} w_i^0 \cos\alpha t - x_i^0 \sin\alpha t \\ w_i^0 \sin\alpha t + x_i^0 \cos\alpha t \\ y_i^0 \cosh\beta t + z_i^0 \sinh\beta t \\ y_i^0 \sinh\beta t + z_i^0 \cosh\beta t \end{pmatrix} =$$

$$\begin{pmatrix} r_i \cos a_i \cos\alpha t - r_i \sin a_i \sin\alpha t \\ r_i \cos a_i \sin\alpha t + r_i \sin a_i \cos\alpha t \\ \eta_i \sinh\beta_i \cosh\beta t + \eta_i \cosh\beta_i \sinh\beta t \\ \eta_i \sinh\beta_i \sinh\beta t + \eta_i \cosh\beta_i \cosh\beta t \end{pmatrix} = \begin{pmatrix} r_i \cos(\alpha t + a_i) \\ r_i \sin(\alpha t + a_i) \\ \eta_i \sinh(\beta t + b_i) \\ \eta_i \cosh(\beta t + b_i) \end{pmatrix},$$

$i = 1, 2, \dots, N$.

5.6 Negative parabolic RE

The sixth class of RE we will introduce here are inspired by the negative parabolic rotations of $\mathbb{H}^3$.

Definition 11. (Negative parabolic RE) *Consider a nonsingular initial position* $\mathbf{q}^0 = (\mathbf{q}_1^0, \mathbf{q}_2^0, \dots, \mathbf{q}_N^0)$ *of the point particles of masses* $m_1, m_2, \dots, m_N > 0, N \geq 2$, *on the manifold* $\mathbb{H}^3$, *where* $\mathbf{q}_i^0 = (w_i^0, x_i^0, y_i^0, z_i^0),\ i = 1, 2, \dots, N$. *Then a solution of system* (3.17) *of the form* $\mathbf{q} = (\mathcal{F}[\mathbf{q}_1^0]^T, \mathcal{F}[\mathbf{q}_2^0]^T, \dots, \mathcal{F}[\mathbf{q}_N^0]^T)$, *with*

$$\mathcal{F}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -t & t \\ 0 & t & 1 - t^2/2 & t^2/2 \\ 0 & t & -t^2/2 & 1 + t^2/2 \end{pmatrix}, \tag{5.7}$$

is called a (simply rotating) negative parabolic RE.

For simplicity, we denote $\alpha_i := w_i^0, \beta_i := x_i^0, \gamma_i := y_i^0, \delta_i := z_i^0,\ i = 1, 2, \dots, N$. Then parabolic RE take the form

$$\mathcal{F}(t)[\mathbf{q}_i^0]^T = \begin{pmatrix} w_i^0 \\ x_i^0 - y_i^0 t + z_i^0 t \\ x_i^0 t + y_i^0 (1 - t^2/2) + z_i^0 t^2/2 \\ x_i^0 t - y_i^0 t^2/2 + z_i^0 (1 + t^2/2) \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_i \\ \beta_i + (\delta_i - \gamma_i) t \\ \gamma_i + \beta_i t + (\delta_i - \gamma_i) t^2/2 \\ \delta_i + \beta_i t + (\delta_i - \gamma_i) t^2/2, \end{pmatrix},\ i = 1, 2, \dots, N.$$

5.7 Formal expressions of the RE

To summarize the previous findings, we can represent the above 6 types of RE of the 3-dimensional curved N-body problem in the form

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n), \quad \mathbf{q}_i = (w_i, x_i, y_i, z_i),\ i = 1, 2, \dots, N,$$

$$[\text{positive elliptic}]: \quad \begin{cases} w_i(t) = r_i \cos(\alpha t + a_i) \\ x_i(t) = r_i \sin(\alpha t + a_i) \\ y_i(t) = y_i \text{ (constant)} \\ z_i(t) = z_i \text{ (constant)}, \end{cases} \tag{5.8}$$

with $w_i^2 + x_i^2 = r_i^2,\ r_i^2 + y_i^2 + z_i^2 = 1,\ i = 1, 2, \ldots, N$;

$$\text{[positive elliptic}-\text{elliptic]} : \quad \begin{cases} w_i(t) = r_i \cos(\alpha t + a_i) \\ x_i(t) = r_i \sin(\alpha t + a_i) \\ y_i(t) = \rho_i \cos(\beta t + b_i) \\ z_i(t) = \rho_i \sin(\beta t + b_i), \end{cases} \tag{5.9}$$

with $w_i^2 + x_i^2 = r_i^2,\ y_i^2 + z_i^2 = \rho_i^2$, so $r_i^2 + \rho_i^2 = 1,\ i = 1, 2, \ldots, N$;

$$\text{[negative elliptic]} : \quad \begin{cases} w_i(t) = r_i \cos(\alpha t + a_i) \\ x_i(t) = r_i \sin(\alpha t + a_i) \\ y_i(t) = y_i \ \text{(constant)} \\ z_i(t) = z_i \ \text{(constant)}, \end{cases} \tag{5.10}$$

with $w_i^2 + x_i^2 = r_i^2,\ r_i^2 + y_i^2 - z_i^2 = -1,\ i = 1, 2, \ldots, N$;

$$\text{[negative hyperbolic]} : \quad \begin{cases} w_i(t) = w_i \ \text{(constant)} \\ x_i(t) = x_i \ \text{(constant)} \\ y_i(t) = \eta_i \sinh(\beta t + b_i) \\ z_i(t) = \eta_i \cosh(\beta t + b_i), \end{cases} \tag{5.11}$$

with $y_i^2 - z_i^2 = -\eta_i^2,\ w_i^2 + x_i^2 - \eta_i^2 = -1,\ i = 1, 2, \ldots, N$;

$$\text{[negative elliptic}-\text{hyperbolic]} : \quad \begin{cases} w_i(t) = r_i \cos(\alpha t + a_i) \\ x_i(t) = r_i \sin(\alpha t + a_i) \\ y_i(t) = \eta_i \sinh(\beta t + b_i) \\ z_i(t) = \eta_i \cosh(\beta t + b_i), \end{cases} \tag{5.12}$$

with $w_i^2 + x_i^2 = r_i^2,\ y_i^2 - z_i^2 = -\eta_i^2$, so $r_i^2 - \eta_i^2 = -1,\ i = 1, 2, \ldots, N$;

$$\text{[negative parabolic]} : \quad \begin{cases} w_i(t) = \alpha_i \ \text{(constant)} \\ x_i(t) = \beta_i + (\delta_i - \gamma_i)t \\ y_i(t) = \gamma_i + \beta_i t + (\delta_i - \gamma_i)t^2/2 \\ z_i(t) = \delta_i + \beta_i t + (\delta_i - \gamma_i)t^2/2, \end{cases} \tag{5.13}$$

with $\alpha_i^2 + \beta_i^2 + \gamma_i^2 - \delta_i^2 = -1,\ i = 1, 2, \ldots, N$.

Chapter 6

Fixed Points (FP)

In this chapter we will introduce the concept of fixed-point solution, also simply called fixed point (from now on denoted by FP, whether in the singular or the plural form of the noun) of the equations of motion, show that FP exist in $\mathbb{S}^3$, provide a couple of examples, and finally prove that they don't show up in $\mathbb{H}^3$ and in hemispheres of $\mathbb{S}^3$, provided that at least one body is not on the boundary of the hemisphere.

6.1 FP in $\mathbb{S}^3$

Although the goal of this monograph is to study RE of the curved N-body problem, some of these orbits can be generated from FP configurations by imposing on the initial positions of the bodies suitable nonzero initial velocities. It is therefore necessary to discuss FP as well. Let us start with their definition.

Definition 12. *A solution of system* (3.21) *is called a fixed point if it is a zero of the vector field, i.e.* $\mathbf{p}_i(t) = \widetilde{\nabla}_{\mathbf{q}_i} U(\mathbf{q}(t)) = \mathbf{0}$ *for all* $t \in \mathbb{R}$, $i = 1, 2, \dots, N$.

In [33] and [35], we showed that FP exist in $\mathbb{S}^2$, but don't exist in $\mathbb{H}^2$. Examples of FP are the equilateral triangle, and in general any regular N-gon of equal masses, N odd, lying on any great circle of $\mathbb{S}^2$, and the regular tetrahedron of equal masses inscribed in $\mathbb{S}^2$ in the 4-body case. There are also examples of FP of unequal masses. We showed in [28] that, for any acute triangle inscribed in a great circle of the sphere $\mathbb{S}^2$, there exist masses $m_1, m_2, m_3 > 0$ that can be placed at the vertices of the triangle such that they form a FP, and therefore can generate RE in the curved 3-body problem. The problem of the existence of N-gons with unequal masses is open, though very likely to be true, for $N > 3$.

6.2 Two examples

We can construct FP of $\mathbb{S}^3$ for which none of its great spheres contains them. A first simple example occurs in the 6-body problem if we take 6 bodies of equal positive masses, place 3 of them at the vertices of an equilateral triangle inscribed in a great circle of a great sphere, and place the other 3 bodies at the vertices of an equilateral triangle inscribed in a complementary great circle (see Definition 3) of another great sphere. Some straightforward computations show that 6 bodies of masses $m_1 = m_2 = m_3 = m_4 = m_5 = m_6 =: m > 0$, with zero initial velocities and initial conditions given, for instance, by

$$\begin{array}{llll}
w_1 = 1, & x_1 = 0, & y_1 = 0, & z_1 = 0, \\
w_2 = -\dfrac{1}{2}, & x_2 = \dfrac{\sqrt{3}}{2}, & y_2 = 0, & z_2 = 0, \\
w_3 = -\dfrac{1}{2}, & x_3 = -\dfrac{\sqrt{3}}{2}, & y_3 = 0, & z_3 = 0, \\
w_4 = 0, & x_4 = 0, & y_4 = 1, & z_4 = 0, \\
w_5 = 0, & x_5 = 0, & y_5 = -\dfrac{1}{2}, & z_5 = \dfrac{\sqrt{3}}{2}, \\
w_6 = 0, & x_6 = 0, & y_6 = -\dfrac{1}{2}, & z_6 = -\dfrac{\sqrt{3}}{2},
\end{array}$$

form a FP.

The second example is inspired from the theory of regular polytopes, [18], [19]. The simplest regular polytope in $\mathbb{R}^4$ is the pentatope (also called 5-cell, 4-simplex, pentachrone, pentahedroid, or hyperpyramid). The pentatope has Schläfli symbol $\{3, 3, 3\}$, which translates into: 3 regular polyhedra that have 3 regular polygons of 3 edges at every vertex (i.e. 3 regular tetrahedra) are attached to each of the pentatope's edges. (From the left to the right, the numbers in the Schläfli symbol are in the order we described them.)

A different way to understand the pentatope is to think of it as the generalization to $\mathbb{R}^4$ of the equilateral triangle of $\mathbb{R}^2$ or of the regular tetrahedron of $\mathbb{R}^3$. Then the pentatope can be constructed by adding to the regular tetrahedron a fifth vertex in $\mathbb{R}^4$ that connects the other four vertices with edges of the same length as those of the tetrahedron. Consequently the pentatope can be inscribed in the sphere $\mathbb{S}^3$, in which it has no antipodal vertices, so there is no danger of encountering singular configurations for the FP we want to construct. Specifically, the coordinates of the

5 vertices of a pentatope inscribed in the sphere $\mathbb{S}^3$ can be taken, for example, as

$$\begin{array}{llll}
w_1 = 1, & x_1 = 0, & y_1 = 0, & z_1 = 0, \\
w_2 = -\dfrac{1}{4}, & x_2 = \dfrac{\sqrt{15}}{4}, & y_2 = 0, & z_2 = 0, \\
w_3 = -\dfrac{1}{4}, & x_3 = -\dfrac{\sqrt{5}}{4\sqrt{3}}, & y_3 = \dfrac{\sqrt{5}}{\sqrt{6}}, & z_3 = 0, \\
w_4 = -\dfrac{1}{4}, & x_4 = -\dfrac{\sqrt{5}}{4\sqrt{3}}, & y_4 = -\dfrac{\sqrt{5}}{2\sqrt{6}}, & z_4 = \dfrac{\sqrt{5}}{2\sqrt{2}}, \\
w_5 = -\dfrac{1}{4}, & x_5 = -\dfrac{\sqrt{5}}{4\sqrt{3}}, & y_5 = -\dfrac{\sqrt{5}}{2\sqrt{6}}, & z_5 = -\dfrac{\sqrt{5}}{2\sqrt{2}}.
\end{array}$$

Straightforward computations show that the distance from the origin of the coordinate system to each of the 5 vertices is 1 and that, for equal masses, $m_1 = m_2 = m_3 = m_4 = m_5 =: m > 0$, this configuration produces a FP of system (3.15). Like in the previous example, the FP lying at the vertices of the pentatope is specific to $\mathbb{S}^3$ in the sense that there is no 2-dimensional sphere that contains it.

It is natural to ask whether other convex regular polytopes of $\mathbb{R}^4$ can form FPs in $\mathbb{S}^3$ if we place equal masses at their vertices. Apart from the pentatope, there are five other such geometrical objects: the tesseract (also called 8-cell, hypercube, or 4-cube, with 16 vertices), the orthoplex (also called 16-cell or hyperoctahedron, with 8 vertices), the octaplex (also called 24-cell or polyoctahedron, with 24 vertices), the dodecaplex (also called 120-cell, hyperdodecahedron, or polydodecahedron, with 600 vertices), and the tetraplex (also called 600-cell, hypericosahedron, or polytetrahedron, with 120 vertices). All these polytopes, however, are centrally symmetric, so they have antipodal vertices. Therefore, if we place bodies of equal masses at their vertices, we encounter singularities. Consequently the only convex regular polytope of $\mathbb{R}^4$ that can form a FP if we place equal masses at its vertices is the pentatope.

6.3 Cases of nonexistence

We will show further that there are no FP in $\mathbb{H}^3$ or in any hemisphere of $\mathbb{S}^3$. In the latter case, for FP not to exist it is necessary that at least one body is not on the boundary of the hemisphere.

Proposition 3. (No FP in $\mathbb{H}^3$) *There are no masses $m_1, m_2, \ldots, m_N > 0$, $N \geq 2$, that can form FP in $\mathbb{H}^3$.*

Proof. Consider N bodies of masses $m_1, m_2, \ldots, m_N > 0$, $N \geq 2$, lying in $\mathbb{H}^3$, in a nonsingular configuration (i.e. without collisions) and with zero initial velocities. Then one or more bodies, say, $m_1, m_2, \ldots, m_k$ with $k \leq N$, have the largest z coordinate. Consequently each of the bodies $m_{k+1}, \ldots, m_N$ will attract each of the bodies $m_1, m_2, \ldots, m_k$ along a geodesic hyperbola towards lowering the z coordinate of the latter. For any 2 bodies with the same largest z coordinate, the segment of hyperbola connecting them has points with lower z coordinates. Therefore these 2 bodies attract each other towards lowering their z coordinates as well. So each of the bodies $m_1, m_2, \ldots, m_k$ will move towards lowering their z coordinate, therefore the initial configuration of the bodies is not fixed. $\square$

Proposition 4. (No FP in hemispheres of $\mathbb{S}^3$) *There are no masses $m_1, m_2, \ldots, m_N > 0, N \geq 2$, that can form FP in any closed hemisphere of $\mathbb{S}^3$ (i.e. a hemisphere that contains its boundary), as long as at least one body doesn't lie on the boundary.*

Proof. The idea of the proof is similar to the idea of the proof we gave for Proposition 3. Let us assume, without loss of generality, that the bodies are in the hemisphere $z \leq 0$ and they form a nonsingular initial configuration (i.e. without collisions or antipodal positions), with at least one of the bodies not on the boundary $z = 0$, and with zero initial velocities. Then one or more bodies, say, $m_1, m_2, \ldots, m_k$, with $k < N$ (a strict inequality is essential to the proof), have the largest z coordinate, which can be at most 0. Consequently the bodies $m_{k+1}, \ldots, m_N$ have lower z coordinates. Each of the bodies $m_{k+1}, \ldots, m_N$ attract each of the bodies $m_1, m_2, \ldots, m_k$ along a geodesic arc of a great circle towards lowering the z coordinate of the latter.

The attraction between any 2 bodies among $m_1, m_2, \ldots, m_k$ is either towards lowering each other's z coordinate, when $z < 0$ or along the geodesic $z = 0$, when they are on that geodesic. In both cases, however, composing all the forces that act on each of the bodies $m_1, m_2, \ldots, m_k$ will make them move towards a lower z coordinate, which means that the initial configuration is not fixed. This remark completes the proof. $\square$

Part III

Criteria and Qualitative Behavior

Preamble

Now that we found natural definitions for the RE of the curved N-body problem, we must understand under what circumstances they may exist. So the goal of Part III is to provide existence criteria for each type of RE previously defined and to obtain some results that describe their qualitative behavior. For this purpose, we will also review some aspects of geometric topology related to $\mathbb{S}^3$ and $\mathbb{H}^3$, building on certain properties proved in Part I. Our results are exhaustive in the sense that no other type of dynamical behavior can occur for these orbits, but we don't deal here with proving their actual existence, an issue we will address in Part IV.

Chapter 7

Existence criteria

In this chapter we establish criteria for the existence of positive elliptic and elliptic-elliptic as well as negative elliptic, hyperbolic, and elliptic-hyperbolic RE. These criteria will be employed in later chapters to obtain concrete examples of such orbits. The proofs are similar in spirit, but for completeness and future reference we describe them all since the specifics differ in each case. We close this chapter by showing that negative parabolic RE do not exist in the curved N-body problem.

7.1 Criteria for RE

We will next provide a criterion for the existence of (simply rotating) positive elliptic RE and then prove a corollary that shows under what conditions such solutions can be generated from FP configurations.

Criterion 1. (Positive elliptic RE) *Let $m_1, m_2, \dots, m_N > 0$ represent the masses of $N \geq 2$ point particles moving in $\mathbb{S}^3$. Then system* (3.15) *admits a solution of the form*

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n),\ \ \mathbf{q}_i = (w_i, x_i, y_i, z_i),\ \ i = 1, 2, \dots, N,$$

$$w_i(t) = r_i \cos(\alpha t + a_i),\ \ x_i(t) = r_i \sin(\alpha t + a_i),\ \ y_i(t) = y_i,\ \ z_i(t) = z_i,$$

with $w_i^2 + x_i^2 = r_i^2$, $r_i^2 + y_i^2 + z_i^2 = 1$, and y_i, z_i constant, $i = 1, 2, \dots, N$, i.e. a (simply rotating) positive elliptic RE, if and only if there are constants r_i, a_i, y_i, z_i, $i = 1, 2, \dots, N$, and $\alpha \neq 0$, such that the following $4N$ conditions are satisfied:

$$\sum_{\substack{j=1\\ j\neq i}}^{N} \frac{m_j(r_j \cos a_j - \nu_{ij} r_i \cos a_i)}{(1-\nu_{ij}^2)^{3/2}} = (r_i^2 - 1)\alpha^2 r_i \cos a_i, \tag{7.1}$$

$$\sum_{\substack{j=1\\j\neq i}}^{N}\frac{m_j(r_j\sin a_j-\nu_{ij}r_i\sin a_i)}{(1-\nu_{ij}^2)^{3/2}}=(r_i^2-1)\alpha^2 r_i\sin a_i, \tag{7.2}$$

$$\sum_{\substack{j=1\\j\neq i}}^{N}\frac{m_j(y_j-\nu_{ij}y_i)}{(1-\nu_{ij}^2)^{3/2}}=\alpha^2 r_i^2 y_i, \tag{7.3}$$

$$\sum_{\substack{j=1\\j\neq i}}^{N}\frac{m_j(z_j-\nu_{ij}z_i)}{(1-\nu_{ij}^2)^{3/2}}=\alpha^2 r_i^2 z_i, \tag{7.4}$$

$i=1,2,\dots,N,$ *where* $\nu_{ij}=r_ir_j\cos(a_i-a_j)+y_iy_j+z_iz_j,\ i,j=1,2\dots,N,\ i\neq j.$

Proof. Consider a candidate, as described above, for a solution $\mathbf{q}$ of system (3.15). Some straightforward computations show that

$$\nu_{ij}:=\mathbf{q}_i\odot\mathbf{q}_j=r_ir_j\cos(a_i-a_j)+y_iy_j+z_iz_j,\ i,j=1,2\dots,n,\ i\neq j,$$

$$\dot{\mathbf{q}}_i\odot\dot{\mathbf{q}}_i=\alpha^2r_i^2,\ i=1,2,\dots,N,$$

$$\ddot{w}_i=-\alpha^2r_i\cos(\alpha t+a_i),\ \ddot{x}_i=-\alpha^2r_i\sin(\alpha t+a_i),$$

$$\ddot{y}_i=\ddot{z}_i=0,\ i=1,2,\dots,N.$$

Substituting the suggested solution and the above expressions into system (3.15), for the w coordinates we obtain conditions involving $\cos(\alpha t+a_i)$, whereas for the x coordinates we obtain conditions involving $\sin(\alpha t+a_i)$. In the former case, using the fact that $\cos(\alpha t+a_i)=\cos\alpha t\cos a_i-\sin\alpha t\sin a_i$, we can split each equation in two, one involving $\cos\alpha t$ and the other $\sin\alpha t$ as factors. The same thing happens in the latter case if we use the formula $\sin(\alpha t+a_i)=\sin\alpha t\cos a_i+\cos\alpha t\sin a_i$. Each of these equations are satisfied if and only if conditions (7.1) and (7.2) take place. Conditions (7.3) and (7.4) follow directly from the equations involving the coordinates y and z. This remark completes the proof. □

Criterion 2. (Positive elliptic RE generated from FP configurations) *Consider the point particles of masses* $m_1,m_2,\dots,m_N>0, N\geq 2$, *in* $\mathbb{S}^3$. *Then, for any* $\alpha\neq 0$, *system* (3.15) *admits a solution of the form* (5.8)*:*

$$\mathbf{q}=(\mathbf{q}_1,\mathbf{q}_2,\dots,\mathbf{q}_n),\mathbf{q}_i=(w_i,x_i,y_i,z_i),\ i=1,2,\dots,N,$$

$$w_i(t)=r_i\cos(\alpha t+a_i),\ x_i(t)=r_i\sin(\alpha t+a_i),\ y_i(t)=y_i,\ z_i(t)=z_i,$$

with $w_i^2+x_i^2 = r_i^2,\ r_i^2+y_i^2+z_i^2 = 1$, *and* y_i, z_i *constant,* $i = 1, 2, \dots, N$, *generated from a FP configuration, i.e. a (simply rotating) positive elliptic RE generated from the same initial positions that would form a FP for zero initial velocities, if and only if there are constants* $r_i, a_i, y_i, z_i,\ i = 1, 2, \dots, N$, *such that the following* $4N$ *conditions are satisfied:*

$$\sum_{\substack{j=1\\ j\neq i}}^{N} \frac{m_j(r_j\cos a_j - \nu_{ij}r_i\cos a_i)}{(1-\nu_{ij}^2)^{3/2}} = 0, \tag{7.5}$$

$$\sum_{\substack{j=1\\ j\neq i}}^{N} \frac{m_j(r_j\sin a_j - \nu_{ij}r_i\sin a_i)}{(1-\nu_{ij}^2)^{3/2}} = 0, \tag{7.6}$$

$$\sum_{\substack{j=1\\ j\neq i}}^{N} \frac{m_j(y_j - \nu_{ij}y_i)}{(1-\nu_{ij}^2)^{3/2}} = 0, \tag{7.7}$$

$$\sum_{\substack{j=1\\ j\neq i}}^{N} \frac{m_j(z_j - \nu_{ij}z_i)}{(1-\nu_{ij}^2)^{3/2}} = 0, \tag{7.8}$$

$i = 1, 2, \dots, N$, *where* $\nu_{ij} = r_ir_j\cos(a_i - a_j) + y_iy_j + z_iz_j,\ i, j = 1, 2, \dots, N,\ i \neq j$, *and one of the following two properties takes place:*

(i) $r_i = 1$ *for all* $i \in \{1, 2 \dots, N\}$,

(ii) there is a proper subset $\mathcal{I} \subset \{1, 2, \dots, N\}$ *such that* $r_i = 0$ *for all* $i \in \mathcal{I}$ *and* $r_j = 1$ *for all* $j \in \{1, 2, \dots, N\} \setminus \mathcal{I}$.

Proof. We are seeking a (simply rotating) elliptic RE, as in Criterion 1, that is valid for any $\alpha \neq 0$. But the solution is also generated from a FP configuration, so the left hand sides of equations (7.1), (7.2), (7.3), and (7.4) necessarily vanish, thus leading to conditions (7.5), (7.6), (7.7), and (7.8). However, the right hand sides of equations (7.1), (7.2), (7.3), and (7.4) must also vanish, so we have the $4N$ conditions:

$$(r_i^2 - 1)\alpha^2 r_i\cos a_i = 0,\ i = 1, 2, \dots, N,$$

$$(r_i^2 - 1)\alpha^2 r_i\sin a_i = 0,\ i = 1, 2, \dots, N,$$

$$\alpha^2 r_i^2 y_i = 0,\ i = 1, 2, \dots, N,$$

$$\alpha^2 r_i^2 z_i = 0,\ i = 1, 2, \dots, N.$$

Since $\alpha \neq 0$ and there is no $\gamma \in \mathbb{R}$ such that the quantities $\sin\gamma$ and $\cos\gamma$ vanish simultaneously, the above $4N$ conditions are satisfied in each of the following cases:

(a) $r_i = 0$ (consequently $w_i = x_i = 0$ and $y_i^2 + z_i^2 = 1$) for all $i \in \{1, 2, \dots, N\}$,

(b) $r_i = 1$ (consequently $w_i^2 + x_i^2 = 1$ and $y_i = z_i = 0$) for all $i \in \{1, 2 \dots, N\}$,

(c) there is a proper subset $\mathcal{I} \subset \{1, 2, \dots, N\}$ such that $r_i = 0$ (and consequently $y_i^2 + z_i^2 = 1$) for all $i \in \mathcal{I}$ and $r_j = 1$ (and consequently $y_j = z_j = 0$) for all $j \in \{1, 2, \dots, N\} \setminus \mathcal{I}$.

In case (a), we recover the FP, so there is no rotation of any kind, therefore this case does not lead to any simply rotating positive elliptic RE. As we will see in Theorem 2, case (b), which corresponds to *(i)* in the above statement, and case (c), which corresponds to *(ii)*, lead to RE of this kind. This remark completes the proof. □

7.2 Criteria for positive elliptic-elliptic RE

We can now provide a criterion for the existence of (doubly rotating) positive elliptic-elliptic RE and a criterion about how such orbits can be obtained from FP configurations.

Criterion 3. (Positive elliptic-elliptic RE) *Let $m_1, m_2, \dots, m_N > 0$ represent the masses of $N \geq 2$ point particles moving in $\mathbb{S}^3$. Then system* (3.15) *admits a solution of the form* (5.9)*:*

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n), \mathbf{q}_i = (w_i, x_i, y_i, z_i), \ i = 1, 2, \dots, N,$$

$$\begin{aligned} w_i(t) &= r_i \cos(\alpha t + a_i), & x_i(t) &= r_i \sin(\alpha t + a_i), \\ y_i(t) &= \rho_i \cos(\beta t + b_i), & z_i(t) &= \rho_i \sin(\beta t + b_i), \end{aligned}$$

with $w_i^2 + x_i^2 = r_i^2$, $y_i^2 + z_i^2 = \rho_i^2$, $r_i^2 + \rho_i^2 = 1$, $i = 1, 2, \dots, N$, i.e. a (doubly rotating) positive elliptic-elliptic RE, if and only if there are constants r_i, ρ_i, a_i, b_i, $i = 1, 2, \dots, N$, and $\alpha, \beta \neq 0$, such that the following $4N$ conditions are satisfied

$$\sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{m_j (r_j \cos a_j - \omega_{ij} r_i \cos a_i)}{(1 - \omega_{ij}^2)^{3/2}} = (\alpha^2 r_i^2 + \beta^2 \rho_i^2 - \alpha^2) r_i \cos a_i, \tag{7.9}$$

$$\sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{m_j (r_j \sin a_j - \omega_{ij} r_i \sin a_i)}{(1 - \omega_{ij}^2)^{3/2}} = (\alpha^2 r_i^2 + \beta^2 \rho_i^2 - \alpha^2) r_i \sin a_i, \tag{7.10}$$

$$\sum_{\substack{j=1\\j\neq i}}^{N}\frac{m_j(\rho_j\cos b_j-\omega_{ij}\rho_i\cos b_i)}{(1-\omega_{ij}^2)^{3/2}}=(\alpha^2r_i^2+\beta^2\rho_i^2-\beta^2)\rho_i\cos b_i, \tag{7.11}$$

$$\sum_{\substack{j=1\\j\neq i}}^{N}\frac{m_j(\rho_j\sin b_j-\omega_{ij}\rho_i\sin b_i)}{(1-\omega_{ij}^2)^{3/2}}=(\alpha^2r_i^2+\beta^2\rho_i^2-\beta^2)\rho_i\sin b_i, \tag{7.12}$$

$i=1,2,\dots,N,$ *where* $\omega_{ij}=r_ir_j\cos(a_i-a_j)+\rho_i\rho_j\cos(b_i-b_j), i,j=1,2,\dots,N,\ i\neq j.$

Proof. Consider a candidate $\mathbf{q}$ as above for a solution of system (3.15). Some straightforward computations show that

$$\omega_{ij}:=\mathbf{q}_i\odot\mathbf{q}_j=r_ir_j\cos(a_i-a_j)+\rho_i\rho_j\cos(b_i-b_j),\ i,j=1,2\dots,N,\ i\neq j,$$

$$\dot{\mathbf{q}}_i\odot\dot{\mathbf{q}}_i=\alpha^2r_i^2+\beta^2\rho_i^2,\ i=1,2,\dots,N,$$
$$\ddot{w}_i=-\alpha^2r_i\cos(\alpha t+a_i),\ \ddot{x}_i=-\alpha^2r_i\sin(\alpha t+a_i),$$
$$\ddot{y}_i=-\beta^2\rho_i\cos(\beta t+b_i),\ \ddot{z}_i=-\beta^2\rho_i\sin(\beta t+b_i),\ i=1,2,\dots,N.$$

Substituting $\mathbf{q}$ and the above expressions into system (3.15), for the w coordinates we obtain conditions involving $\cos(\alpha t+a_i)$, whereas for the x coordinates we obtain conditions involving $\sin(\alpha t+a_i)$. In the former case, using the fact that $\cos(\alpha t+a_i)=\cos\alpha t\cos a_i-\sin\alpha t\sin a_i$, we can split each equation in two, one involving $\cos\alpha t$ and the other $\sin\alpha t$ as factors. The same thing happens in the latter case if we use the formula $\sin(\alpha t+a_i)=\sin\alpha t\cos a_i+\cos\alpha t\sin a_i$. Each of these equations are satisfied if and only if conditions (7.9) and (7.10) take place.

For the y coordinate, the substitution of the above solution leads to conditions involving $\cos(\beta t+b_i)$, whereas for z coordinate it leads to conditions involving $\sin(\beta t+b_i)$. Then we proceed as we did for the w and x coordinates and obtain conditions (7.11) and (7.12). This remark completes the proof. □

Criterion 4. (Positive elliptic-elliptic RE generated from FP configurations) *Consider the point particles of masses* $m_1,m_2,\dots,\ m_N>0,N\geq 2,$ *moving in* $\mathbb{S}^3$*. Then, for any* $\alpha,\beta\neq 0,$ *system* (3.15) *admits a solution of the form* (5.9)*:*

$$\mathbf{q}=(\mathbf{q}_1,\mathbf{q}_2,\dots,\mathbf{q}_n),\mathbf{q}_i=(w_i,x_i,y_i,z_i),\ i=1,2,\dots,N,$$

$$w_i(t)=r_i\cos(\alpha t+a_i),\qquad x_i(t)=r_i\sin(\alpha t+a_i),$$
$$y_i(t)=\rho_i\cos(\beta t+b_i),\qquad z_i(t)=\rho_i\sin(\beta t+b_i),$$

with $w_i^2+x_i^2=r_i^2,\ y_i^2+z_i^2=\rho_i^2,\ r_i^2+\rho_i^2=1,\ i=1,2,\ldots,N$*, generated from a FP configuration, i.e. a (doubly rotating) positive elliptic-elliptic RE generated from the same initial positions that would form a FP for zero initial velocities, if and only if there are constants* $r_i,\rho_i,a_i,b_i,\ i=1,2,\ldots,N$*, such that the* $4N$ *relationships below are satisfied:*

$$\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{m_j(r_j\cos a_j-\omega_{ij}r_i\cos a_i)}{(1-\omega_{ij}^2)^{3/2}}=0, \tag{7.13}$$

$$\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{m_j(r_j\sin a_j-\omega_{ij}r_i\sin a_i)}{(1-\omega_{ij}^2)^{3/2}}=0, \tag{7.14}$$

$$\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{m_j(\rho_j\cos b_j-\omega_{ij}\rho_i\cos b_i)}{(1-\omega_{ij}^2)^{3/2}}=0, \tag{7.15}$$

$$\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{m_j(\rho_j\sin b_j-\omega_{ij}\rho_i\sin b_i)}{(1-\omega_{ij}^2)^{3/2}}=0, \tag{7.16}$$

$i=1,2,\ldots,N$*, where* $\omega_{ij}=r_ir_j\cos(a_i-a_j)+\rho_i\rho_j\cos(b_i-b_j),\ i,j=1,2,\ldots,N,\ i\neq j$*, and, additionally, one of the following properties takes place:*

(i) there is a proper subset $\mathcal{J}\subset\{1,2,\ldots,N\}$ *such that* $r_i=0$ *for all* $i\in\mathcal{J}$ *and* $\rho_j=0$ *for all* $j\in\{1,2,\ldots,N\}\setminus\mathcal{J}$*,*

(ii) the frequencies $\alpha,\beta\neq 0$ *satisfy the condition* $|\alpha|=|\beta|$*.*

Proof. A FP configuration requires that the left hand sides of equations (7.9), (7.10), (7.11), and (7.12) vanish, so we obtain the conditions (7.13), (7.14), (7.15), and (7.16). A RE can be generated from a FP configuration if and only if the right hand sides of (7.9), (7.10), (7.11), and (7.12) vanish as well, i.e.

$$(\alpha^2r_i^2+\beta^2\rho_i^2-\alpha^2)r_i\cos a_i=0,$$

$$(\alpha^2r_i^2+\beta^2\rho_i^2-\alpha^2)r_i\sin a_i=0,$$

$$(\alpha^2r_i^2+\beta^2\rho_i^2-\beta^2)\rho_i\cos b_i=0,$$

$$(\alpha^2r_i^2+\beta^2\rho_i^2-\beta^2)\rho_i\sin b_i=0,$$

where $r_i^2+\rho_i^2=1,\ i=1,2,\ldots,N$. Since there is no $\gamma\in\mathbb{R}$ such that $\sin\gamma$ and $\cos\gamma$ vanish simultaneously, the above expressions are zero in each of the following circumstances:

(a) $r_i = 0$, and consequently $\rho_i = 1$, for all $i \in \{1, 2, \ldots, N\}$;

(b) $\rho_i = 0$, and consequently $r_i = 1$, for all $i \in \{1, 2, \ldots, N\}$;

(c) there is a proper subset $\mathcal{J} \subset \{1, 2, \ldots, N\}$ such that $r_i = 0$ (consequently $\rho_i = 1$) for all $i \in \mathcal{J}$ and $\rho_j = 0$ (consequently $r_j = 1$) for all $j \in \{1, 2, \ldots, N\} \setminus \mathcal{J}$;

(d) $\alpha^2 r_i^2 + \beta^2 \rho_i^2 - \alpha^2 = \alpha^2 r_i^2 + \beta^2 \rho_i^2 - \beta^2 = 0,\ i \in \{1, 2, \ldots, N\}$.

Cases (a) and (b) correspond to (simply rotating) positive elliptic RE, thus recovering condition *(i)* in Criterion 2. Case (c) corresponds to *(i)* in the above statement. Since, from Definition 6, it follows that $\alpha, \beta \neq 0$, the identities in case (d) can obviously take place only if $\alpha^2 = \beta^2$, i.e. $|\alpha| = |\beta| \neq 0$, so (d) corresponds to condition *(ii)* in the above statement. This remark completes the proof. □

7.3 Criterion for negative elliptic RE

We further consider the motion of point masses in $\mathbb{H}^3$ and start with proving a criterion for the existence of simply rotating negative elliptic RE.

Criterion 5. (Negative elliptic RE) *Consider the point particles of masses* $m_1, m_2, \ldots, m_N > 0, N \geq 2$, *moving in* $\mathbb{H}^3$. *Then system* (3.16) *admits solutions of the form* (5.10)*:*

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n),\ \ \mathbf{q}_i = (w_i, x_i, y_i, z_i),\ \ i = 1, 2, \ldots, N,$$

$$w_i(t) = r_i \cos(\alpha t + a_i),\ \ x_i(t) = r_i \sin(\alpha t + a_i),\ \ y_i(t) = y_i,\ \ z_i(t) = z_i,$$

with $w_i^2 + x_i^2 = r_i^2,\ r_i^2 + y_i^2 - z_i^2 = -1$, *and* y_i, z_i *constant,* $i = 1, 2, \ldots, N$, *i.e. (simply rotating) negative elliptic RE, if and only if there are constants* $r_i, a_i, y_i, z_i,\ i = 1, 2, \ldots, N$, *and* $\alpha \neq 0$, *such that the following* $4N$ *conditions are satisfied:*

$$\sum_{\substack{j=1\\j\neq i}}^{N} \frac{m_j(r_j \cos a_j + \epsilon_{ij} r_i \cos a_i)}{(\epsilon_{ij}^2 - 1)^{3/2}} = -(r_i^2 + 1)\alpha^2 r_i \cos a_i, \tag{7.17}$$

$$\sum_{\substack{j=1\\j\neq i}}^{N} \frac{m_j(r_j \sin a_j + \epsilon_{ij} r_i \sin a_i)}{(\epsilon_{ij}^2 - 1)^{3/2}} = -(r_i^2 + 1)\alpha^2 r_i \sin a_i, \tag{7.18}$$

$$\sum_{\substack{j=1\\j\neq i}}^{N} \frac{m_j(y_j + \epsilon_{ij} y_i)}{(\epsilon_{ij}^2 - 1)^{3/2}} = -\alpha^2 r_i^2 y_i, \tag{7.19}$$

$$\sum_{\substack{j=1\\j\neq i}}^{N} \frac{m_j(z_j+\epsilon_{ij}z_i)}{(\epsilon_{ij}^2-1)^{3/2}} = -\alpha^2 r_i^2 z_i, \tag{7.20}$$

$i=1,2,\dots,N,$ *where* $\epsilon_{ij}=r_ir_j\cos(a_i-a_j)+y_iy_j-z_iz_j,\ i,j=1,2,\dots,N,\ i\neq j.$

Proof. Consider a candidate $\mathbf{q}$ as above for a solution of system (3.16). Some straightforward computations show that

$$\epsilon_{ij} := \mathbf{q}_i \odot \mathbf{q}_j = r_ir_j\cos(a_i-a_j)+y_iy_j-z_iz_j,\ i,j=1,2\dots,N,\ i\neq j,$$

$$\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i = \alpha^2 r_i^2,\ i=1,2,\dots,N,$$

$$\ddot{w}_i = -\alpha^2 r_i\cos(\alpha t+a_i),\ \ddot{x}_i=-\alpha^2 r_i\sin(\alpha t+a_i),$$

$$\ddot{y}_i=\ddot{z}_i=0,\ i=1,2,\dots,N.$$

Substituting $\mathbf{q}$ and the above expressions into the equations of motion (3.16), for the w coordinates we obtain conditions involving $\cos(\alpha t+a_i)$, whereas for the x coordinates we obtain conditions involving $\sin(\alpha t+a_i)$. In the former case, using the fact that $\cos(\alpha t+a_i)=\cos\alpha t\cos a_i-\sin\alpha t\sin a_i$, we can split each equation in two, one involving $\cos\alpha t$ and the other $\sin\alpha t$ as factors. The same thing happens in the latter case if we use the formula $\sin(\alpha t+a_i)=\sin\alpha t\cos a_i+\cos\alpha t\sin a_i$. Each of these equations are satisfied if and only if conditions (7.17) and (7.18) take place. Conditions (7.19) and (7.20) follow directly from the equations involving the coordinates y and z. This remark completes the proof. □

7.4 Criterion for negative hyperbolic RE

We continue our study of the hyperbolic space with proving a criterion that shows under what conditions (simply rotating) negative hyperbolic RE exist.

Criterion 6. (Negative hyperbolic RE) *Consider the point particles of masses* $m_1,m_2,\dots,m_N>0, N\geq 2$, *moving in* $\mathbb{H}^3$. *Then system* (3.16) *admits solutions of the form* (5.11)*:*

$$\mathbf{q}=(\mathbf{q}_1,\mathbf{q}_2,\dots,\mathbf{q}_n),\ \mathbf{q}_i=(w_i,x_i,y_i,z_i),\ i=1,2,\dots,N,$$

$$\begin{aligned} &w_i(t)=w_i \ \text{(constant)}, && x_i(t)=x_i\ \text{(constant)},\\ &y_i(t)=\eta_i\sinh(\beta t+b_i), && z_i(t)=\eta_i\cosh(\beta t+b_i), \end{aligned}$$

with $y_i^2 - z_i^2 = -\eta_i^2,\ w_i^2 + x_i^2 - \eta_i^2 = -1,\ i = 1, 2, \ldots, N,$ i.e. (simply rotating) negative hyperbolic RE, if and only if there are constants $\eta_i, w_i, x_i,\ i = 1, 2, \ldots, N,$ and $\beta \neq 0$, such that the following $4N$ conditions are satisfied:

$$\sum_{\substack{j=1\\ j\neq i}}^{N} \frac{m_j(w_j + \mu_{ij} w_i)}{(\mu_{ij}^2 - 1)^{3/2}} = -\beta^2 \eta_i^2 w_i, \tag{7.21}$$

$$\sum_{\substack{j=1\\ j\neq i}}^{N} \frac{m_j(x_j + \mu_{ij} x_i)}{(\mu_{ij}^2 - 1)^{3/2}} = -\beta^2 \eta_i^2 x_i, \tag{7.22}$$

$$\sum_{\substack{j=1\\ j\neq i}}^{N} \frac{m_j(\eta_j \sinh b_j + \mu_{ij}\eta_i \sinh b_i)}{(\mu_{ij}^2 - 1)^{3/2}} = (1 - \eta_i^2)\beta^2 \eta_i \sinh b_i, \tag{7.23}$$

$$\sum_{\substack{j=1\\ j\neq i}}^{N} \frac{m_j(\eta_j \cosh b_j + \mu_{ij}\eta_i \cosh b_i)}{(\mu_{ij}^2 - 1)^{3/2}} = (1 - \eta_i^2)\beta^2 \eta_i \cosh b_i, \tag{7.24}$$

$i = 1, 2, \ldots, N,$ where $\mu_{ij} = w_i w_j + x_i x_j - \eta_i \eta_j \cosh(b_i - b_j),\ i, j = 1, 2, \ldots, N,\ i \neq j.$

Proof. Consider a candidate $\mathbf{q}$ as above for a solution of system (3.16). Some straightforward computations show that

$$\mu_{ij} := \mathbf{q}_i \boxdot \mathbf{q}_j = w_i w_j + x_i x_j - \eta_i \eta_j \cosh(b_i - b_j),\ i, j = 1, 2, \ldots, N,\ i \neq j,$$

$$\dot{\mathbf{q}}_i \boxdot \dot{\mathbf{q}}_i = \beta^2 \eta_i^2,\ i = 1, 2, \ldots, N,$$

$$\ddot{w}_i = \ddot{x}_i = 0,$$

$$\ddot{y}_i = \beta^2 \eta_i \sinh(\beta t + b_i),\ \ddot{z}_i = \beta^2 \eta_i \cosh(\beta i + b_i),\ i = 1, 2, \ldots, N.$$

Substituting $\mathbf{q}$ and the above expressions into the equations of motion (3.16), we are led for the w and x coordinates to the equations (7.21) and (7.22), respectively. For the y and z coordinates we obtain conditions involving $\sinh(\beta t + b_i)$ and $\cosh(\beta t + b_i)$, respectively. In the former case, using the fact that $\sinh(\beta t + b_i) = \sinh \beta t \cosh b_i + \cosh \beta t \sinh b_i$, we can split each equation in two, one involving $\sinh \beta t$ and the other $\cosh \beta t$ as factors. The same thing happens in the later case if we use the formula $\cosh(\beta t + b_i) = \cosh \beta t \cosh b_i + \sinh \beta t \sinh b_i$. Each of these conditions are satisfied if and only of conditions (7.23) and (7.24) take place. This remark completes the proof. $\square$

7.5 Criterion for negative elliptic-hyperbolic RE

We end our study of existence criteria for RE in hyperbolic space with a result that shows under what conditions (doubly rotating) negative elliptic-hyperbolic RE exist.

Criterion 7. (Negative elliptic-hyperbolic RE) *Consider the point particles of masses* $m_1, m_2, \dots, m_N > 0, N \geq 2$*, moving in* $\mathbb{H}^3$*. Then system* (3.16) *admits solutions of the form* (5.12)*:*

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n), \mathbf{q}_i = (w_i, x_i, y_i, z_i),\ \ i = 1, 2, \dots, N,$$

$$\begin{aligned} w_i(t) &= r_i \cos(\alpha t + a_i), & x_i(t) &= r_i \sin(\alpha t + a_i), \\ y_i(t) &= \eta_i \sinh(\beta t + b_i), & z_i(t) &= \eta_i \cosh(\beta t + b_i), \end{aligned}$$

with $w_i^2 + x_i^2 = r_i^2,\ y_i^2 - z_i^2 = -\eta_i^2,\ r_i^2 - \eta_i^2 = -1$*, i.e. (doubly rotating) negative elliptic-hyperbolic RE, if and only if there are constants* $r_i, \eta_i, a_i, b_i,\ i = 1, 2 \dots, N$*, and* $\alpha, \beta \neq 0$*, such that the following* $4N$ *conditions are satisfied:*

$$\sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{m_j(r_j \cos a_j + \gamma_{ij} r_i \cos a_i)}{(\gamma_{ij}^2 - 1)^{3/2}} = -(\alpha^2 r_i^2 + \beta^2 \eta_i^2 + \alpha^2) r_i \cos a_i, \tag{7.25}$$

$$\sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{m_j(r_j \sin a_j + \gamma_{ij} r_i \sin a_i)}{(\gamma_{ij}^2 - 1)^{3/2}} = -(\alpha^2 r_i^2 + \beta^2 \eta_i^2 + \alpha^2) r_i \sin a_i, \tag{7.26}$$

$$\sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{m_j(\eta_j \sinh b_j + \gamma_{ij} \eta_i \sinh b_i)}{(\gamma_{ij}^2 - 1)^{3/2}} = (\beta^2 - \alpha^2 r_i^2 - \beta^2 \eta_i^2) \eta_i \sinh b_i, \tag{7.27}$$

$$\sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{m_j(\eta_j \cosh b_j + \gamma_{ij} \eta_i \cosh b_i)}{(\gamma_{ij}^2 - 1)^{3/2}} = (\beta^2 - \alpha^2 r_i^2 - \beta^2 \eta_i^2) \eta_i \cosh b_i, \tag{7.28}$$

$i = 1, 2, \dots, N$*, where* $\gamma_{ij} = r_i r_j \cos(a_i - a_j) - \eta_i \eta_j \cosh(b_i - b_j), i, j = 1, 2, \dots, N, i \neq j$.

Proof. Consider a candidate $\mathbf{q}$ as above for a solution of system (3.16). Some straightforward computations show that

$$\gamma_{ij} := \mathbf{q}_i \boxdot \mathbf{q}_j = r_i r_j \cos(a_i - a_j) - \eta_i \eta_j \cosh(b_i - b_j), i, j = 1, 2, \dots, N,\ \ i \neq j,$$

$$\dot{\mathbf{q}}_i \boxdot \dot{\mathbf{q}}_i = \alpha^2 r_i^2 + \beta^2 \eta_i^2,\ i = 1, 2, \dots, N,$$
$$\ddot{w}_i = -\alpha^2 r_i \cos(\alpha t + a_i),\ \ddot{x}_i = -\alpha^2 r_i \sin(\alpha t + a_i),$$
$$\ddot{y}_i = \beta^2 \eta_i \sinh(\beta t + b_i),\ \ddot{z}_i = \beta^2 \eta_i \cosh(\beta i + b_i),\ i = 1, 2, \dots, N.$$

Substituting these expression and those that define $\mathbf{q}$ into the equations of motion (3.16), we obtain for the w and x coordinates conditions involving $\cos(\alpha t + a_i)$ and $\sin(\alpha t + a_i)$, respectively. In the former case, using the fact that $\cos(\alpha t + a_i) = \cos \alpha t \cos a_i - \sin \alpha t \sin a_i$, we can split each equation in two, one involving $\cos \alpha t$ and the other $\sin \alpha t$ as factors. The same thing happens in the latter case if we use the formula $\sin(\alpha t + a_i) = \sin \alpha t \cos a_i + \cos \alpha t \sin a_i$. Each of these equations are satisfied if and only if conditions (7.25) and (7.26) take place.

For the y and z coordinates we obtain conditions involving $\sinh(\beta t + b_i)$ and $\cosh(\beta t + b_i)$, respectively. In the former case, using the fact that $\sinh(\beta t + b_i) = \sinh \beta t \cosh b_i + \cosh \beta t \sinh b_i$, we can split each equation in two, one involving $\sinh \beta t$ and the other $\cosh \beta t$ as factors. The same thing happens in the latter case if we use the formula $\cosh(\beta t + b_i) = \cosh \beta t \cosh b_i + \sinh \beta t \sinh b_i$. Each of these conditions are satisfied if and only of conditions (7.27) and (7.28) take place. These remarks complete the proof. □

7.6 Nonexistence of negative parabolic RE

As in the curved N-body problem restricted to $\mathbb{H}^2$, negative parabolic RE do not exist in $\mathbb{H}^3$. The idea of the proof exploits the fact that a RE of parabolic type would violate the conservation law of the angular momentum. Here are a formal statement and a proof of this result.

Proposition 5. (Nonexistence of negative parabolic RE) *Consider the point particles of masses $m_1, m_2, \dots, m_N > 0, N \geq 2$, in $\mathbb{H}^3$. Then system* (3.16) *does not admit solutions of the form* (5.13)*, which means that negative parabolic RE do not exist in the* 3*-dimensional curved N-body problem.*

Proof. Checking a solution of the form (5.13) into the last integral of (3.28), we obtain that

$$c_{yz} = \sum_{i=1}^N m_i(y_i \dot{z}_i - \dot{y}_i z_i) = \sum_{i=1}^N m_i \left[\gamma_i + \beta_i t + (\delta_i - \gamma_i)\frac{t^2}{2}\right] [\beta_i + (\delta_i - \gamma_i)t]$$
$$- \sum_{i=1}^N m_i \left[\delta_i + \beta_i t + (\delta_i - \gamma_i)\frac{t^2}{2}\right] [\beta_i + (\delta_i - \gamma_i)t]$$

$$= \sum_{i=1}^{N} m_i \beta_i (\gamma_i - \delta_i) - \sum_{i=1}^{N} m_i (\gamma_i - \delta_i)^2 t,\ \ i = 1, 2, \ldots, N.$$

Since c_{yz} is constant, it follows that $\gamma_i = \delta_i,\ i = 1, 2, \ldots, N$. But from (5.13) we obtain that $\alpha_i^2 + \beta_i^2 = -1$, a contradiction which proves that negative parabolic RE cannot exist. This remark completes the proof. □

Chapter 8

Qualitative behavior

In this chapter we will describe some qualitative dynamical properties for the positive elliptic, positive elliptic-elliptic, negative elliptic, negative hyperbolic, and negative elliptic-hyperbolic RE, under the assumption that they exist. (Examples of such solutions will be given in Part IV for various values of N and of the masses $m_1, m_2, \dots, m_N > 0$.) For this purpose we will also provide some geometric-topologic considerations about $\mathbb{S}^3$ and $\mathbb{H}^3$.

8.1 Some geometric topology in $\mathbb{S}^3$

Consider the circle of radius r in the wx plane of $\mathbb{R}^4$ and the circle of radius ρ in the yz plane of $\mathbb{R}^4$, with $r^2 + \rho^2 = 1$. Then $\mathbf{T}^2_{r\rho}$ is the cartesian product of these two circles, i.e. a 2-dimensional surface of genus 1, called a Clifford torus. Since these two circles are submanifolds embedded in $\mathbb{R}^2$, $\mathbf{T}^2_{r\rho}$ is embedded in $\mathbb{R}^4$. But $\mathbf{T}^2_{r\rho}$ also belongs to the sphere $\mathbb{S}^3$. Indeed, we can represent this torus as

$$\mathbf{T}^2_{r\rho} = \{(w, x, y, z) \mid r^2 + \rho^2 = 1,\ 0 \le \theta, \phi < 2\pi\}, \tag{8.1}$$

where $w = r\cos\theta,\ x = r\sin\theta,\ y = \rho\cos\phi$, and $z = \rho\sin\phi$, so the distance from the origin of the coordinate system to any point of the Clifford torus is

$$(r^2\cos^2\theta + r^2\sin^2\theta + \rho^2\cos^2\phi + \rho^2\sin^2\phi)^{1/2} = (r^2 + \rho^2)^{1/2} = 1.$$

When r (and, consequently, ρ) takes all the values between 0 and 1, the family of Clifford tori such defined foliates $\mathbb{S}^3$ (see Figure 8.1). Each Clifford torus splits $\mathbb{S}^3$ into two solid tori and forms the boundary between them. The two solid tori are congruent when $r = \rho = 1/\sqrt{2}$. For the sphere $\mathbb{S}^3$, this is the standard Heegaard splitting[1] of genus 1.

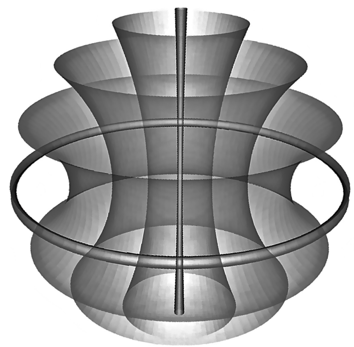

Figure 8.1: A 3-dimensional projection of a 4-dimensional foliation of the sphere $\mathbb{S}^3$ into Clifford tori.

Unlike regular tori embedded in $\mathbb{R}^3$, Clifford tori have zero Gaussian curvature at every point. Their flatness is due to the existence of an additional dimension in $\mathbb{R}^4$. Indeed, cylinders, obtained by pasting two opposite sides of a square, are flat surfaces both in $\mathbb{R}^3$ and $\mathbb{R}^4$. But to form a torus by pasting the other two sides of the square, cylinders must be stretched and squeezed in $\mathbb{R}^3$. In $\mathbb{R}^4$, the extra dimension allows pasting without stretching or squeezing.

8.2 RE in $\mathbb{S}^3$

The above considerations lead us to state and prove the following result, under the assumption that positive elliptic and positive elliptic-elliptic RE exist in $\mathbb{S}^3$.

Theorem 1. (Qualitative behavior of the RE in $\mathbb{S}^3$) *Assume that, in the curved N-body problem in $\mathbb{S}^3$, $N \geq 2$, with bodies of masses $m_1, m_2, \dots, m_N > 0$, positive elliptic and positive elliptic-elliptic RE exist. Then the corresponding solution $\mathbf{q}$ may have one of the following dynamical behaviors:*

(i) If $\mathbf{q}$ is given by (5.8), the orbit is a (simply rotating) positive elliptic RE, with the body of mass m_i moving on a (not necessarily geodesic) circle $\mathcal{C}_i, i = 1, 2, \dots, N$, of a 2-dimensional sphere in $\mathbb{S}^3$; in the hyperplanes wxy and wxz, the circles $\mathcal{C}_i$ are parallel with the plane wx. In particular, some bodies can rotate on a great circle of

[1] A Heegaard splitting, named after the Danish mathematician Poul Heegaard (1871-1943), is a decomposition of a compact, connected, oriented 3-dimensional manifold along its boundary into two manifolds having the same genus g, with $g = 0, 1, 2, \dots$

a great sphere, while the other bodies stay fixed on a complementary great circle of another great sphere.

(ii) If $\mathbf{q}$ *is given by* (5.9)*, the orbit is a (doubly rotating) positive elliptic-elliptic RE, with each body* m_i *moving on the Clifford torus* $\mathbf{T}^2_{r_i\rho_i}, i = 1, 2, \ldots, N$*. In particular, some bodies can rotate on a great circle of a great sphere, while the other bodies rotate on a complementary great circle of another great sphere.*

Proof. (i) The bodies move on circles $\mathcal{C}_i,\ i = 1, 2, \ldots, N$, because, by (5.8), the analytic expression of the orbit is given by

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n),\ \mathbf{q}_i = (w_i, x_i, y_i, z_i),\ i = 1, 2, \ldots, N,$$

$$w_i(t) = r_i \cos(\alpha t + a_i),\ x_i(t) = r_i \sin(\alpha t + a_i),\ y_i(t) = y_i,\ z_i(t) = z_i,$$

with $w_i^2 + x_i^2 = r_i^2,\ r_i^2 + y_i^2 + z_i^2 = 1$, and y_i, z_i constant, $i = 1, 2, \ldots, N$. This proves the first part of (i), except for the statements about parallelism.

In particular, if some bodies are on the circle

$$\mathbb{S}^1_{wx} = \{(0, 0, y, z) \mid y^2 + z^2 = 1\},$$

with $y_i(t) = y_i =$ constant and $z_i(t) = z_i =$ constant, then the elliptic rotation, which changes the coordinates w and x, does not act on the bodies, therefore they don't move. This remark proves the second part of statement (i).

To prove the parallelism statement from the first part of (i), let us first remark that, as the concept of two parallel lines makes sense only if the lines are contained in the same plane, the concept of two parallel planes has meaning only if the planes are contained in the same 3-dimensional space. This explains our formulation of the statement. Towards proving it, notice first that

$$c_{wx} = \sum_{i=1}^N m_i(w_i\dot{x}_i - \dot{w}_i x_i) = \alpha \sum_{i=1}^N m_i r_i^2$$

and

$$c_{yz} = \sum_{i=1}^N m_i(y_i\dot{z}_i - \dot{y}_i z_i) = 0.$$

These constants are independent of the bodies' position, a fact that confirms that they result from first integrals. To determine the values of the constants c_{wy}, c_{wz}, c_{xy}, and c_{xz}, we first compute that

$$c_{wy} = \sum_{i=1}^N m_i(w_i\dot{y}_i - \dot{w}_i y_i) = \alpha \sum_{i=1}^N m_i r_i y_i \sin(\alpha t + a_i),$$

$$c_{wz} = \sum_{i=1}^{N} m_i(w_i\dot{z}_i - \dot{w}_i z_i) = \alpha \sum_{i=1}^{N} m_i r_i z_i \sin(\alpha t + a_i),$$

$$c_{xy} = \sum_{i=1}^{N} m_i(x_i\dot{y}_i - \dot{x}_i y_i) = \alpha \sum_{i=1}^{N} m_i r_i y_i \cos(\alpha t + a_i),$$

$$c_{xz} = \sum_{i=1}^{N} m_i(x_i\dot{z}_i - \dot{x}_i z_i) = \alpha \sum_{i=1}^{N} m_i r_i z_i \cos(\alpha t + a_i).$$

Since they are constant, the first integrals must take the same value for the arguments $t = 0$ and $t = \pi/\alpha$. But at $t = 0$, we obtain

$$c_{wy} = \alpha \sum_{i=1}^{N} m_i r_i y_i \sin a_i, \quad c_{wz} = \alpha \sum_{i=1}^{N} m_i r_i z_i \sin a_i,$$

$$c_{xy} = \alpha \sum_{i=1}^{N} m_i r_i y_i \cos a_i, \quad c_{xz} = \alpha \sum_{i=1}^{N} m_i r_i z_i \cos a_i,$$

whereas at $t = \pi/\alpha$, we obtain

$$c_{wy} = -\alpha \sum_{i=1}^{N} m_i r_i y_i \sin a_i, \quad c_{wz} = -\alpha \sum_{i=1}^{N} m_i r_i z_i \sin a_i,$$

$$c_{xy} = -\alpha \sum_{i=1}^{N} m_i r_i y_i \cos a_i, \quad c_{xz} = -\alpha \sum_{i=1}^{N} m_i r_i z_i \cos a_i.$$

Consequently, $c_{wy} = c_{wz} = c_{xy} = c_{xz} = 0$. Since, as we already showed, $c_{yz} = 0$, it follows that the only nonzero constant of the total angular momentum is c_{wx}. This means that the particle system has nonzero total rotation with respect to the origin only in the wx plane.

To prove that the circles $\mathcal{C}_i, i = 1, 2, \dots, N$, are parallel with the plane wx in the hyperplanes wxy and wxz, assume that one circle, say $\mathcal{C}_1$, does not satisfy this property. Then some orthogonal projection of $\mathcal{C}_1$ (within either of the hyperplanes wxy and wxz) in at least one of the other base planes, say xy, is an ellipse, not a segment—as it would be otherwise. Then the angular momentum of the body of mass m_1 relative to the plane xy is nonzero. Should other circles have an elliptic projection in the plane xy, the angular momentum of the corresponding bodies would be nonzero as well. Moreover, all angular momenta would have the same sign because all bodies move in the same direction on the original circles. Consequently $c_{xy} \neq 0$, in contradiction with our previous findings. Therefore the circles $\mathcal{C}_i, i = 1, 2, \ldots, N$, must be parallel, as stated.

(ii) When a positive elliptic-elliptic (double) rotation acts on a system, if some bodies are on a great circle of a great sphere of $\mathbb{S}^3$, while other are on a complementary great circle of another great sphere, then the former bodies move only because of one rotation, while the latter bodies move only because of the other rotation. The special geometric properties of complementary circles leads to this kind of qualitative behavior.

To prove the general qualitative behavior, namely that the body of mass m_i of the (doubly rotating) positive elliptic-elliptic RE moves on the Clifford torus $\mathbf{T}^2_{r_i\rho_i}, i = 1, 2, \ldots, N$, of which the situation described above is a notable particular case, it is enough to compare the form of the orbit given in (5.9) with the characterization (8.1) of a Clifford torus. This remark completes the proof. $\square$

8.3 RE generated from FP configurations

We will further outline the dynamical consequences of Criterion 2 and Criterion 4, under the assumption that positive elliptic and positive elliptic-elliptic RE, both generated from FP configurations, exist in $\mathbb{S}^3$. This theorem deals with a subclass of the orbits whose qualitative behavior we have just described.

Theorem 2. (Qualitative behavior of the RE generated from FP configurations in $\mathbb{S}^3$) *Consider the bodies of masses $m_1, m_2, \ldots, m_N > 0, N \geq 2$, moving in $\mathbb{S}^3$. Then a RE $\mathbf{q}$ generated from a FP configuration may have one of the following characteristics:*

(i) $\mathbf{q}$ is a (simply rotating) positive elliptic RE for which all bodies rotate on the same great circle of a great sphere of $\mathbb{S}^3$;

(ii) $\mathbf{q}$ is a (simply rotating) positive elliptic RE for which some bodies rotate on a great circle of a great sphere, while the other bodies are fixed on a complementary great circle of a different great sphere;

(iii) $\mathbf{q}$ is a (doubly rotating) positive elliptic-elliptic RE for which some bodies rotate with frequency $\alpha \neq 0$ on a great circle of a great sphere, while the other bodies rotate with frequency $\beta \neq 0$ on a complementary great circle of a different sphere; the frequencies may be different in size, i.e. $|\alpha| \neq |\beta|$;

(iv) $\mathbf{q}$ is a (doubly rotating) positive elliptic-elliptic RE that does not rotate on complementary circles, a case in which the frequencies $\alpha, \beta \neq 0$ are equal in size, i.e. $|\alpha| = |\beta|$.

Proof. (i) From conclusion (i) of Criterion 2, a (simply rotating) positive elliptic RE of the form

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n), \mathbf{q}_i = (w_i, x_i, y_i, z_i),\ i = 1, 2, \dots, N,$$

$$w_i = r_i \cos(\alpha t + a_i),\ x_i(t) = r_i \sin(\alpha t + a_i),\ y_i(t) = y_i,\ z_i(t) = z_i,$$

with $w_i^2 + x_i^2 = r_i^2,\ r_i^2 + y_i^2 + z_i^2 = 1$ and y_i, z_i constant, $i = 1, 2, \dots, N$, generated from a FP configuration, must satisfy one of two additional conditions (besides the initial $4N$ equations), the first of which translates into

$$r_i = 1,\ i = 1, 2, \dots, N.$$

This property implies that $y_i = z_i = 0,\ i = 1, 2, \dots, N$, so all bodies rotate along the same great circle of radius 1, namely $\mathbf{S}^1_{yz}$, thus proving the statement in this case.

(ii) From conclusion (ii) of Criterion 2, there is a proper subset $\mathcal{I} \subset \{1, 2, \dots, N\}$ such that $r_i = 0$ for all $i \in \mathcal{I}$ and $r_j = 1$ for all $j \in \{1, 2, \dots, N\} \setminus \mathcal{I}$.

The bodies for which $r_i = 0$ must have $w_i = x_i = 0$ and $y_i^2 + z_i^2 = 1$, so they are fixed on the great circle $\mathbf{S}^1_{wx}$, since y_i and z_i are constant, $i \in \mathcal{I}$, and no rotation acts on the coordinates w and x.

As in the proof of (i) above, it follows that the bodies with $r_j = 1,\ j \in \{1, 2, \dots, N\} \setminus \mathcal{I}$ rotate on the circle $\mathbf{S}^1_{yz}$, which is complementary to $\mathbf{S}^1_{wx}$, so statement (ii) is also proved.

(iii) From conclusion (i) of Criterion 4, a (doubly rotating) positive elliptic-elliptic RE of the form

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n), \mathbf{q}_i = (w_i, x_i, y_i, z_i),\ i = 1, 2, \dots, N,$$

$$\begin{aligned} w_i(t) &= r_i \cos(\alpha t + a_i), & x_i(t) &= r_i \sin(\alpha t + a_i), \\ y_i(t) &= \rho_i \cos(\beta t + b_i), & z_i(t) &= \rho_i \sin(\beta t + b_i), \end{aligned}$$

with $w_i^2 + x_i^2 = r_i^2,\ y_i^2 + z_i^2 = \rho_i^2,\ r_i^2 + \rho_i^2 = 1,\ i = 1, 2, \dots, N$, generated from a FP configuration, must satisfy one of two additional conditions (besides the initial $4N$ equations), the first of which says that there is a proper subset $\mathcal{J} \subset \{1, 2, \dots, N\}$ such that $r_i = 0$ for all $i \in \mathcal{J}$ and $\rho_j = 0$ for all $j \in \{1, 2, \dots, N\} \setminus \mathcal{J}$. But this means that the bodies m_i with $i \in \mathcal{J}$ have $w_i = x_i = 0$ and $y_i^2 + z_i^2 = \rho_i^2$, so one rotation acts along the great circle $\mathbf{S}^1_{wx}$, while the bodies with $m_i, i \in \{1, 2, \dots, N\} \setminus \mathcal{J}$ satisfy the conditions $w_i^2 + x_i^2 = r_i^2$ and $y_i = z_i = 0$, so the other rotation acts on them along the great circle $\mathbf{S}^1_{yz}$, which is complementary to $\mathbf{S}^1_{wx}$. Moreover, since the bodies are distributed on two complementary circles, there are no constraints on the frequencies $\alpha, \beta \neq 0$, which can be independent of each other, a remark that proves the statement.

(iv) From statement (d) in the proof of Criterion 4, a (doubly rotating) positive elliptic-elliptic RE may exist also when the bodies are not necessarily on complementary circles but the frequencies satisfy the condition $|\alpha| = |\beta|$, a case that concludes the last statement of this result. □

8.4 Some geometric topology in $\mathbb{H}^3$

Usually, compact higher-dimensional manifolds have a richer geometry than noncompact manifolds of the same dimension. This fact is also true about $\mathbb{S}^3$ if compared to $\mathbb{H}^3$. Nevertheless, we will be able to characterize the relative equilibria of $\mathbb{H}^3$ in geometric-topologic terms.

The surface we are introducing in this section, which will play for our dynamical analysis in $\mathbb{H}^3$ the same role the Clifford torus played in $\mathbb{S}^3$, is homeomorphic to a cylinder. Consider a circle of radius r in the wx plane of $\mathbb{R}^4$ and the upper branch of the hyperbola $r^2 - \eta^2 = -1$ in the yz plane of $\mathbb{R}^4$. Then we will call the surface $\mathbf{C}^2_{r\eta}$ obtained by taking the cartesian product between the circle and the hyperbola a hyperbolic cylinder since it equidistantly surrounds a branch of a geodesic hyperbola in $\mathbb{H}^3$. Indeed, we can represent this cylinder as

$$\mathbf{C}^2_{r\eta} = \{(w, x, y, z) \mid r^2 - \eta^2 = -1,\ 0 \leq \theta < 2\pi,\ \xi \in \mathbb{R}\}, \tag{8.2}$$

where $w = r\cos\theta,\ x = r\sin\theta,\ y = \eta\sinh\xi,\ z = \eta\cosh\xi$. But the hyperbolic cylinder $\mathbf{C}^2_{r\eta}$ also lies in $\mathbb{H}^3$ because the coordinates w, x, y, z, endowed with the Lorentz inner product, satisfy the equations

$$w^2 + x^2 + y^2 - z^2 = r^2 - \eta^2 = -1.$$

As in the case of $\mathbb{S}^3$, which is foliated by a family of Clifford tori, $\mathbb{H}^3$ can be foliated by a family of hyperbolic cylinders. The foliation is, of course, not unique. But unlike the Clifford tori of $\mathbb{R}^4$, the hyperbolic cylinders of $\mathbb{R}^{3,1}$ are not flat surfaces. In general, they have constant positive Gaussian curvature, which varies with the size of the cylinder, becoming zero only when the cylinder degenerates into a geodesic hyperbola.

8.5 RE in $\mathbb{H}^3$

The above considerations allow us to state and prove the following result, under the assumption that negative elliptic, hyperbolic, and elliptic-hyperbolic RE exist. Notice that, on one hand, due to the absence of complementary circles, and, on the other hand, the absence of FP in $\mathbb{H}^3$, the dynamical behavior of RE is less complicated than in $\mathbb{S}^3$.

Theorem 3. (Qualitative behavior of the RE in $\mathbb{H}^3$) *In the curved N-body problem in $\mathbb{H}^3$, $N \geq 2$, with bodies of masses $m_1, m_2, \ldots, m_N > 0$, every RE $\mathbf{q}$ has one of the following potential behaviors:*

(i) if $\mathbf{q}$ is given by (5.10)*, the orbit is a (simply rotating) negative elliptic RE, with the body of mass m_i moving on a circle $\mathcal{C}^i$, $i = 1, 2, \ldots, N$, of a 2-dimensional hyperbolic sphere in $\mathbb{H}^3$; in the hyperplanes wxy and wxz, the planes of the circles $\mathcal{C}^i$ are parallel with the plane wx;*

(ii) if $\mathbf{q}$ is given by (5.11)*, the orbit is a (simply rotating) negative hyperbolic RE, with the body of mass m_i moving on some (not necessarily geodesic) hyperbola $\mathcal{H}_i$ of a 2-dimensional hyperbolic sphere in $\mathbb{H}^3$, $i = 1, 2, \ldots, N$; in the hyperplanes wyz and xyz, the planes of the hyperbolas $\mathcal{H}_i$ are parallel with the plane yz;*

(iii) if $\mathbf{q}$ is given by (5.12)*, the orbit is a (doubly rotating) negative elliptic-hyperbolic RE, with the body of mass m_i moving on the hyperbolic cylinder $\mathbf{C}^2_{r_i\eta_i}$, $i = 1, 2, \ldots, N$.*

Proof. (i) The bodies move on circles, $\mathcal{C}^i$, $i = 1, 2, \ldots, N$, because, by (5.10), the analytic expression of the orbit is given by

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n), \ \mathbf{q}_i = (w_i, x_i, y_i, z_i), i = 1, 2, \ldots, N,$$

$$w_i(t) = r_i \cos(\alpha t + a_i), \ x_i(t) = r_i \sin(\alpha t + a_i), \ y_i(t) = y_i, \ z_i(t) = z_i,$$

with $w_i^2 + x_i^2 = r_i^2$, $r_i^2 + y_i^2 - z_i^2 = -1$, and y_i, z_i constant, $i = 1, 2, \ldots, N$. The parallelism of the planes of the circles in the hyperplanes wxy and wxz follows exactly as in the proof of part (i) in Theorem 1, using the integrals of the total angular momentum.

(ii) The bodies move on hyperbolas, $\mathcal{H}_i$, $i = 1, 2, \ldots, N$, because, by (5.11), the analytic expression of the orbit is given by

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n), \mathbf{q}_i = (w_i, x_i, y_i, z_i), \ i = 1, 2, \ldots, N,$$

$$\begin{aligned} w_i(t) &= w_i \text{ (constant)}, & x_i(t) &= x_i \text{ (constant)}, \\ y_i(t) &= \eta_i \sinh(\beta t + b_i), & z_i(t) &= \eta_i \cosh(\beta t + b_i), \end{aligned}$$

with $y_i^2 - z_i^2 = -\eta_i^2,\ w_i^2 + x_i^2 - \eta_i^2 = -1,\ i = 1, 2, \ldots, N$.

Let us now prove the parallelism statement for the planes containing the hyperbolas $\mathcal{H}_i$. For this purpose, notice that

$$c_{wx} = \sum_{i=1}^{N} m_i(w_i\dot{x}_i - \dot{w}_i x_i) = 0$$

and

$$c_{yz} = \sum_{i=1}^{N} m_i(y_i\dot{z}_i - \dot{y}_i z_i) = -\beta \sum_{i=1}^{N} m_i\eta_i^2.$$

These constants are independent of the bodies' position, a fact which confirms that they result from first integrals. To determine the values of the constants c_{wy}, c_{wz}, c_{xy}, and c_{xz}, we first compute that

$$c_{wy} = \sum_{i=1}^{N} m_i(w_i\dot{y}_i - \dot{w}_i y_i) = \beta \sum_{i=1}^{N} m_i w_i \eta_i \cosh(\beta t + b_i),$$

$$c_{wz} = \sum_{i=1}^{N} m_i(w_i\dot{z}_i - \dot{w}_i z_i) = \beta \sum_{i=1}^{N} m_i w_i \eta_i \sinh(\beta t + b_i),$$

$$c_{xy} = \sum_{i=1}^{N} m_i(x_i\dot{y}_i - \dot{x}_i y_i) = \beta \sum_{i=1}^{N} m_i x_i \eta_i \cosh(\beta t + b_i),$$

$$c_{xz} = \sum_{i=1}^{N} m_i(x_i\dot{z}_i - \dot{x}_i z_i) = \beta \sum_{i=1}^{N} m_i x_i \eta_i \sinh(\beta t + b_i).$$

We next show that $c_{wy} = 0$. For this, notice first that, using the formula $\cosh(\beta t + b_i) = \cosh b_i \cosh \beta t + \sinh b_i \sinh \beta t$, we can write

$$c_{wy} = \beta[A(t) + B(t)], \tag{8.3}$$

where

$$A(t) = \sum_{i=1}^{N} m_i w_i \eta_i \cosh b_i \cosh \beta t \quad \text{and} \quad B(t) = \sum_{i=1}^{N} m_i w_i \eta_i \sinh b_i \sinh \beta t.$$

But the function cosh is even, whereas sinh is odd. Therefore A is even and B is odd. Since c_{wy} is constant, we also have

$$c_{wy} = \beta[A(-t) + B(-t)] = \beta[A(t) - B(t)]. \tag{8.4}$$

From (8.3) and (8.4) and the fact that $\beta \neq 0$, we can conclude that

$$c_{wy} = \beta A(t) \quad \text{and} \quad B(t) = 0,$$

so $\frac{d}{dt}B(t) = 0$. However, $\frac{d}{dt}B(t) = \beta A(t)$, which proves that $c_{wy} = 0$.

The fact that $c_{xy} = 0$ can be proved exactly the same way. The only difference when showing that $c_{wz} = 0$ and $c_{xz} = 0$ is the use of the corresponding hyperbolic formula, $\sinh(\beta t + b_i) = \sinh \beta t \cosh b_i + \cosh \beta t \sinh b_i$. In conclusion,

$$c_{wx} = c_{wy} = c_{wz} = c_{xy} = c_{xz} = 0 \quad \text{and} \quad c_{yz} \neq 0,$$

which means that the hyperbolic rotation takes place relative to the origin of the coordinate system solely with respect to the plane yz.

Using a similar reasoning as in the proof of (i) for Theorem 1, it can be shown that the above conclusion proves the parallelism of the planes that contain the hyperbolas $\mathcal{H}_i$ in the 3-dimensional hyperplanes wyz and xyz.

(iii) To prove that (doubly rotating) negative elliptic-hyperbolic RE move on hyperbolic cylinders, it is enough to compare the form of the orbit given in (5.11) with the characterization (8.2) of a hyperbolic cylinder. $\square$

Part IV

Examples

Preamble

The goal of Part IV is to provide examples of RE for each type of qualitative behavior described in Part III. The dynamics appear to be richer and more interesting in $\mathbb{S}^3$ than in $\mathbb{H}^3$, probably because, unlike the latter, the former is a compact manifold. Nevertheless, the curved N-body problem in $\mathbb{H}^3$ exhibits certain specific, as well as unexpected, RE as well, a fact we will use in Part V to make a first step towards proving that, for distances comparable to those of our solar system, space is Euclidean.

Chapter 9

Positive elliptic RE

Since Theorems 1, 2, and 3 provide us with the qualitative behavior of all the five classes of RE that we expect to find in $\mathbb{S}^3$ and $\mathbb{H}^3$, we know what kind of rigid-body-type orbits to look for in the curved N-body problem for various values of $N \geq 3$. Ideal, of course, would be to find them all, but this problem appears to be very difficult, and it might never be completely solved. As a first step towards this (perhaps unreachable) goal, we will show that each type of orbit described in the above criteria and theorems exists for some values of $N \geq 3$ and $m_1, m_2, \dots, m_N > 0$.

To appreciate the difficulty of the above mentioned question, we remark that its Euclidean analogue is that of finding all central configurations for the Newtonian potential. The notoriety of this problem has been recognized for at least seven decades, [100]. In fact, we don't even know whether, for some given masses $m_1, m_2, \dots, m_N > 0$, with $N \geq 5$, the number of classes of central configurations (after we factorize the central configurations by size and rotation) is finite or not[1] and, should it be infinite, whether the set of classes of central configurations is discrete or contains a continuum. The finiteness of the number of classes of central configurations is Problem 6 on Steven Smale's list of mathematics problems for the 21st century, [96]. Its analogue in our case would be that of deciding whether, for given masses, $m_1, m_2, \dots, m_N > 0$, the number of classes of RE of the 3-dimensional curved N-body problem is finite or not.

In this chapter, we will provide specific examples of positive elliptic RE, i.e. orbits on the sphere $\mathbb{S}^3$ that have a single rotation. The first example is a RE in a 3-body problem in which 3 equal masses are at the vertices of an equilateral triangle that

[1] Alain Albouy and Vadim Kaloshin have recently reported to have proved the finiteness of central configurations in the planar Euclidean 5-body problem for any given positive masses, except perhaps for a codimension 2 submanifold of the mass space, [3].

rotates along a not necessarily great circle of a great or non-great sphere. The second example is a RE of a 3-body problem in which 3 unequal masses move at the vertices of an acute scalene triangle along a great circle of a great sphere. The third example is a RE generated from a FP configuration in a 6-body problem of equal masses for which 3 bodies move along a great circle of a great sphere at the vertices of an equilateral triangle, while the other 3 masses are fixed on a complementary great circle of another great sphere at the vertices of an equilateral triangle. The fourth, and last example of this chapter, generalizes the third example to the case of acute scalene, not necessarily congruent, triangles and unequal masses.

9.1 Lagrangian RE

In the light of Remark 2, we expect to find solutions in $\mathbb{S}^3$ that move on 2-dimensional spheres. A simple example is that of the Lagrangian RE (i.e. equilateral triangles) of equal masses in the curved 3-body problem. Their existence in $\mathbb{S}^2$, and the fact that they occur only when the masses are equal, was first proved in [35]. This proof will also be presented in Part V. So, in this case, we have $N = 3$ and $m_1 = m_2 = m_3 =: m > 0$. The solution of the corresponding system (3.15) we are seeking is of the form

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3),\ \ \mathbf{q}_i = (w_i, x_i, y_i, z_i),\ \ i = 1, 2, 3, \tag{9.1}$$

$$\begin{aligned}
w_1(t) &= r\cos\omega t, & x_1(t) &= r\sin\omega t,\\
y_1(t) &= y \text{ (constant)}, & z_1(t) &= z \text{ (constant)},\\
w_2(t) &= r\cos(\omega t + 2\pi/3), & x_2(t) &= r\sin(\omega t + 2\pi/3),\\
y_2(t) &= y \text{ (constant)}, & z_2(t) &= z \text{ (constant)},\\
w_3(t) &= r\cos(\omega t + 4\pi/3), & x_3(t) &= r\sin(\omega t + 4\pi/3),\\
y_3(t) &= y \text{ (constant)}, & z_3(t) &= z \text{ (constant)},
\end{aligned}$$

with $r^2 + y^2 + z^2 = 1$. Consequently, for the equations occurring in Criterion 1, we have

$$r_1 = r_2 = r_3 =: r,\ \ \ a_1 = 0,\ \ a_2 = 2\pi/3,\ \ a_3 = 4\pi/3,$$

$$y_1 = y_2 = y_3 =: y \text{ (constant)},\ \ \ z_1 = z_2 = z_3 =: z \text{ (constant)}.$$

Substituting these values into the equations (7.1), (7.2), (7.3), (7.4), we obtain either identities or the equation

$$\omega^2 = \frac{8m}{\sqrt{3}r^3(4-3r^2)^{3/2}}.$$

Therefore, given $m > 0$, $r \in (0,1)$, and y, z with $r^2 + y^2 + z^2 = 1$, we can always find two frequencies,

$$\omega_1 = \frac{2}{r}\sqrt{\frac{2m}{\sqrt{3}r(4-3r^2)^{3/2}}} \quad \text{and} \quad \omega_2 = -\frac{2}{r}\sqrt{\frac{2m}{\sqrt{3}r(4-3r^2)^{3/2}}},$$

such that system (3.17) has a solution of the form (9.1). The positive frequency corresponds to one sense of rotation, whereas the negative frequency corresponds to the opposite sense.

Notice that if $r = 1$, i.e. when the bodies move along a great circle of a great sphere, equations (7.1), (7.2), (7.3), (7.4) are identities for any $\omega \in \mathbb{R}$, so any frequency leads to a solution. This phenomenon happens because, under those circumstances, the motion is generated from a FP configuration, a case in which we can apply Criterion 2, whose statement is independent of the frequency.

The bodies move on the great circle $\mathbf{S}^1_{yz}$ of a great sphere only if $y = z = 0$. Otherwise they move on non-great circles of great or non-great spheres. So we can also interpret this example as existing in the light of Remark 3, which says that there are positive elliptic rotations that leave non-great spheres invariant.

The constants of the angular momentum are

$$c_{wx} = 3m\omega \neq 0 \quad \text{and} \quad c_{wy} = c_{wz} = c_{xy} = c_{xz} = c_{yz} = 0,$$

which means that the rotation takes place around the origin of the coordinate system only relative to the plane wx.

9.2 Scalene triangles

It is natural to ask whether solutions such as the one in the previous example also exist for unequal masses. The answer is positive, and it was first answered in [28], where we proved that, given a 2-dimensional sphere and any triangle inscribed in a great circle of it (for instance inside the equator $z = 0$), there are masses, $m_1, m_2, m_3 > 0$, such that the bodies form a relative equilibrium that rotates around the z axis.

We will consider a similar solution here in $\mathbb{S}^3$, which moves on the great circle $\mathbf{S}^1_{yz}$ of the great sphere $\mathbf{S}^2_y$ or $\mathbf{S}^2_z$. The expected analytic expression of the solution depends on the shape of the triangle, i.e. it has the form

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3), \ \mathbf{q}_i = (w_i, x_i, y_i, z_i), \ i = 1, 2, 3, \tag{9.2}$$

$$\begin{aligned}
w_1(t) &= \cos(\omega t + a_1), & x_1(t) &= \sin(\omega t + a_1),\\
y_1(t) &= 0, & z_1(t) &= 0,\\
w_2(t) &= \cos(\omega t + a_2), & x_2(t) &= \sin(\omega t + a_2),\\
y_2(t) &= 0, & z_2(t) &= 0,\\
w_3(t) &= \cos(\omega t + a_3), & x_3(t) &= \sin(\omega t + a_3),\\
y_3(t) &= 0, & z_3(t) &= 0,
\end{aligned}$$

where the constants a_1, a_2, a_3, with $0 \le a_1 < a_2 < a_3 < 2\pi$, determine the triangle's shape. The other constants involved in the description of this orbit are

$$r_1 = r_2 = r_3 = 1. \tag{9.3}$$

We can use now Criterion 2 to prove that (9.2) is a positive elliptic RE for any frequency $\omega \neq 0$. Indeed, we know from [28] that, for any shape of the triangle, there exist masses that yield a FP on the great circle $\mathbf{S}^1_{yz}$, so the corresponding equations (7.5), (7.6), (7.7), (7.8) are satisfied. Since conditions (9.3) are also satisfied, the proof that (9.2) is a solution of (3.15) is complete.

The constants of the angular momentum integrals are

$$c_{wx} = (m_1 + m_2 + m_3)\omega \neq 0 \ \text{ and } \ c_{wy} = c_{wz} = c_{xy} = c_{xz} = c_{yz} = 0,$$

which means that the bodies rotate in $\mathbb{R}^4$ around the origin of the coordinate system only relative to the plane wx.

9.3 A regular RE with a fixed subsystem

The following example of a (simply rotating) positive elliptic RE in the curved 6-body problem corresponds to the second type of orbit described in part (i) of Theorem 1, and it is interesting from two points of view. First, it is an orbit that is specific to $\mathbb{S}^3$ in the sense that it cannot exist on any 2-dimensional sphere. Second, 3 bodies of equal masses move on a great circle of a great sphere at the vertices of an equilateral triangle, while the other 3 bodies of masses equal to the first stay fixed on a complementary great circle of another great sphere at the vertices of an equilateral triangle. So consider the equal masses

$$m_1 = m_2 = m_3 = m_4 = m_5 = m_6 =: m > 0.$$

Then a solution as described above has the form

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4, \mathbf{q}_5, \mathbf{q}_6), \ \mathbf{q}_i = (w_i, x_i, y_i, z_i), \ i = 1, 2, \dots, 6,$$

$$
\begin{array}{llll}
w_1 = \cos\alpha t, & x_1 = \sin\alpha t, & y_1 = 0, & z_1 = 0,\\
w_2 = \cos(\alpha t + 2\pi/3), & x_2 = \sin(\alpha t + 2\pi/3), & y_2 = 0, & z_2 = 0,\\
w_3 = \cos(\alpha t + 4\pi/3), & x_3 = \sin(\alpha t + 4\pi/3), & y_3 = 0, & z_3 = 0,\\
w_4 = 0, & x_4 = 0, & y_4 = 1, & z_4 = 0,\\
w_5 = 0, & x_5 = 0, & y_5 = -\dfrac{1}{2}, & z_5 = \dfrac{\sqrt{3}}{2},\\
w_6 = 0, & x_6 = 0, & y_6 = -\dfrac{1}{2}, & z_6 = -\dfrac{\sqrt{3}}{2}.
\end{array}
$$

A straightforward computation shows that this attempted orbit, which is generated from a FP configuration, satisfies Criterion 2, therefore it is indeed a solution of system (3.15) for $N = 6$ and for any frequency $\alpha \neq 0$.

The constants of the angular momentum are

$$c_{wx} = 3m\alpha \neq 0 \quad \text{and} \quad c_{wy} = c_{wz} = c_{xy} = c_{xz} = c_{yz} = 0,$$

which implies that the rotation takes place around the origin of the coordinate system only relative to the wx plane.

9.4 An irregular RE with a fixed subsystem

To prove the existence of positive elliptic RE with unequal masses not contained in any 2-dimensional sphere, we have only to combine the ideas of Sections 9.2 and 9.3. More precisely, we consider a 6-body problem in which 3 bodies of unequal masses, $m_1, m_2, m_3 > 0$, rotate on a great circle (lying, say, in the plane wx) of a great sphere at the vertices of an acute scalene triangle, while the other 3 bodies of unequal masses, $m_4, m_5, m_6 > 0$, are fixed on a complementary great circle of another great sphere (lying, as a consequence of our choice of the previous circle, in the plane yz) at the vertices of another acute scalene triangle, which is not necessarily congruent with the first.

Notice that, as shown in [28], we must first choose the shapes of the triangles and then determine the masses that correspond to them, not the other way around. The reason for proceeding in this order is that not any 3 positive masses can rotate along a great circle of a great sphere. Like the solution in Section 9.3, this orbit is specific to $\mathbb{S}^3$ in the sense that it cannot exist on any 2-dimensional sphere. Its analytic expression depends on the shapes of the two triangles, i.e.

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4, \mathbf{q}_5, \mathbf{q}_6), \ \mathbf{q}_i = (w_i, x_i, y_i, z_i), \ i = 1, 2, \ldots, 6, \tag{9.4}$$

$$
\begin{array}{llll}
w_1 = \cos(\alpha t + a_1), & x_1 = \sin(\alpha t + a_1), & y_1 = 0, & z_1 = 0, \\
w_2 = \cos(\alpha t + a_2), & x_2 = \sin(\alpha t + a_2), & y_2 = 0, & z_2 = 0, \\
w_3 = \cos(\alpha t + a_3), & x_3 = \sin(\alpha t + a_3), & y_3 = 0, & z_3 = 0, \\
w_4 = 0, & x_4 = 0, & y_4 = \cos b_4, & z_4 = \sin b_4, \\
w_5 = 0, & x_5 = 0, & y_5 = \cos b_5, & z_5 = \sin b_5, \\
w_6 = 0, & x_6 = 0, & y_6 = \cos b_6, & z_6 = \sin b_6.
\end{array}
$$

where the constants a_1, a_2, and a_3, with $0 \le a_1 < a_2 < a_3 < 2\pi$, and b_4, b_5, and b_6, with $0 \le b_4 < b_5 < b_6 < 2\pi$, determine the shape of the first and second triangle, respectively. For $t = 0$, we obtain the configuration given by the coordinates

$$
\begin{array}{llll}
w_1 = \cos a_1, & x_1 = \sin a_1, & y_1 = 0, & z_1 = 0, \\
w_2 = \cos a_2, & x_2 = \sin a_2, & y_2 = 0, & z_2 = 0, \\
w_3 = \cos a_3, & x_3 = \sin a_3, & y_3 = 0, & z_3 = 0, \\
w_4 = 0, & x_4 = 0, & y_4 = \cos b_4, & z_4 = \sin b_4, \\
w_5 = 0, & x_5 = 0, & y_5 = \cos b_5, & z_5 = \sin b_5, \\
w_6 = 0, & x_6 = 0, & y_6 = \cos b_6, & z_6 = \sin b_6.
\end{array}
$$

We will prove next that this is a FP configuration. For this purpose, let us first compute that

$$\nu_{12} = \nu_{21} = \cos(a_1 - a_2),$$

$$\nu_{13} = \nu_{31} = \cos(a_1 - a_3),$$

$$\nu_{23} = \nu_{32} = \cos(a_2 - a_3),$$

$$\nu_{14} = \nu_{41} = \nu_{15} = \nu_{51} = \nu_{16} = \nu_{61} = 0,$$

$$\nu_{24} = \nu_{42} = \nu_{25} = \nu_{52} = \nu_{26} = \nu_{62} = 0,$$

$$\nu_{34} = \nu_{43} = \nu_{35} = \nu_{53} = \nu_{36} = \nu_{63} = 0,$$

$$\nu_{45} = \nu_{54} = \cos(b_4 - a_5),$$

$$\nu_{46} = \nu_{64} = \cos(b_4 - b_6),$$

$$\nu_{56} = \nu_{65} = \cos(b_5 - b_6).$$

Since $y_1 = y_2 = y_3 = z_1 = z_2 = z_3 = 0$, it follows that, at $t = 0$, the equations involving the force for the coordinates $w_1, x_1, w_2, x_2, w_3, x_3$ have only the constants $m_1, m_2, m_3, a_1, a_2, a_3$. In other words, for these coordinates, the forces acting on the masses m_1, m_2, and m_3 do not involve the masses m_4, m_5, and m_6. But the bodies m_1, m_2, and m_3 are on the great circle $\mathbf{S}^1_{yz}$, which can be seen as lying on the great sphere $\mathbf{S}^2_z$. This means that, by applying the result of [28], the bodies m_1, m_2, and m_3 form an independent FP configuration. Similarly, we can show that the masses m_4, m_5, and m_6 form an independent FP configuration. Therefore all 6 bodies form a FP configuration.

Consequently, we can now use Criterion 2 to check whether $\mathbf{q}$ given by (9.4) is a positive elliptic RE generated from a FP configuration. We can approach this problem in two ways. One is computational, and it consists of using the fact that the positions at $t = 0$ form a FP configuration to determine the relationships between the constants m_1, m_2, m_3, a_1, a_2, and a_3, on one hand, and the constants m_4, m_5, m_6, b_4, b_5, and b_6, on the other hand. It turns out that these relationships reduce to conditions (7.5), (7.6), (7.7), and (7.8). Then we only need to remark that

$$r_1 = r_2 = r_3 = 1 \quad \text{and} \quad r_4 = r_5 = r_6 = 0,$$

which means that condition (ii) of Criterion 2 is satisfied. The other approach is to invoke again the result of [28] and a reasoning similar to the one we used to show that the position at $t = 0$ is a FP configuration. Each approach helps us conclude that $\mathbf{q}$ given by (9.4) is a solution of system (3.15) for any $\alpha \neq 0$.

The constants of the angular momentum are

$$c_{wx} = (m_1 + m_2 + m_3)\alpha \neq 0 \quad \text{and} \quad c_{wy} = c_{wz} = c_{xy} = c_{xz} = c_{yz} = 0,$$

which means, as expected, that the bodies rotate in $\mathbb{R}^4$ around the origin of the coordinate system only relative to the plane wx.

Chapter 10

Positive elliptic-elliptic RE

In this chapter we will construct examples of positive elliptic-elliptic RE, i.e. orbits with two elliptic rotations on the sphere $\mathbb{S}^3$. The first example is that of a 3-body problem in which 3 bodies of equal masses are at the vertices of an equilateral triangle, which has two rotations of the same frequency. The second example is that of a 4-body problem in which 4 equal masses are at the vertices of a regular tetrahedron, which has two rotations of the same frequency. The third example is that of a 5-body problem in which 5 equal masses lying at the vertices of a pentatope have two rotations of same-size frequencies. This is the only regular polytope that allows RE, because the five other existing regular polytopes of $\mathbb{R}^4$ have antipodal vertices, so they introduce singularities. As in the previous example, this motion cannot take place on any 2-dimensional sphere. The fourth example is that of a 6-body problem, with 3 bodies of equal masses rotating at the vertices of an equilateral triangle along a great circle of a great sphere, while the other 3 bodies, of the same mass as the others, rotate at the vertices of an equilateral triangle along a complementary great circle of another great sphere. In general, the frequencies of the two rotations are distinct. The fifth example generalizes the fourth example in the sense that the triangles are scalene, acute, not necessarily congruent, and the masses as well as the frequencies of the rotations are distinct, in general.

10.1 Equilateral triangle with equal frequencies

The example we will now construct is that of a (doubly rotating) positive elliptic-elliptic equilateral triangle of equal masses in the curved 3-body problem in $\mathbb{S}^3$ for which the rotations have the same frequency. Such solutions cannot be found on 2-dimensional spheres. So we consider the masses $m_1 = m_2 = m_3 =: m > 0$ in $\mathbb{S}^3$.

Then the solution we check for system (3.15) with $N = 3$ has the form:

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3),\ \ \mathbf{q}_i = (w_i, x_i, y_i, z_i),\ \ i = 1, 2, 3, \tag{10.1}$$

$$\begin{aligned}
w_1 &= r\cos\alpha t, & x_1 &= r\sin\alpha t,\\
y_1 &= \rho\cos\alpha t, & z_1 &= \rho\sin\alpha t,\\
w_2 &= r\cos\left(\alpha t + 2\pi/3\right), & x_2 &= r\sin\left(\alpha t + 2\pi/3\right),\\
y_2 &= \rho\cos(\alpha t + 2\pi/3), & z_2 &= \rho\sin(\alpha t + 2\pi/3),\\
w_3 &= r\cos\left(\alpha t + 4\pi/3\right), & x_3 &= r\sin\left(\alpha t + 4\pi/3\right),\\
y_3 &= \rho\cos(\alpha t + 4\pi/3), & z_3 &= \rho\sin(\alpha t + 4\pi/3),
\end{aligned}$$

with $r^2 + \rho^2 = 1$. For $t = 0$, the above attempted solution gives for the 3 bodies the coordinates

$$\begin{aligned}
w_1 &= r, & x_1 &= 0, & y_1 &= \rho, & z_1 &= 0,\\
w_2 &= -\frac{r}{2}, & x_2 &= \frac{r\sqrt{3}}{2}, & y_2 &= -\frac{\rho}{2}, & z_2 &= \frac{\rho\sqrt{3}}{2},\\
w_3 &= -\frac{r}{2}, & x_3 &= -\frac{r\sqrt{3}}{2}, & y_3 &= -\frac{\rho}{2}, & z_3 &= -\frac{\rho\sqrt{3}}{2},
\end{aligned}$$

which is a FP configuration, since the bodies have equal masses and are at the vertices of an equilateral triangle inscribed in a great circle of a great sphere. Consequently, we can use Criterion 4 to check whether a solution of the form (10.1) satisfies system (3.15) for any $\alpha \neq 0$. A straightforward computation shows that the first $4N$ conditions are satisfied. Moreover, since the two rotations have the same frequency, it follows that condition (ii) of Criterion 4 is verified, therefore (10.1) is indeed a solution of system (3.15) for any $\alpha \neq 0$.

The angular momentum constants are

$$c_{wx} = 3m\alpha r^2,\ \ c_{wy} = 0,\ \ c_{wz} = 3m\alpha r\rho,$$

$$c_{xy} = -3m\alpha r\rho,\ \ c_{xz} = 0,\ \ c_{yz} = 3m\alpha\rho^2,$$

which show that rotations around the origin of the coordinate system take place relative to 4 planes: wx, wz, xy, and yz. Consequently the bodies don't move on circles, but on the same Clifford torus, namely $\mathbf{T}^2_{r\rho}$, a case that agrees with the qualitative result described in part (ii) of Theorem 1.

Notice that for $r = 1$ and $\rho = 0$, the orbit becomes a (simply rotating) positive elliptic RE that rotates along a great circle of a great sphere in $\mathbb{S}^3$, i.e. an orbit such as the one we described in Section 9.1.

10.2 Regular tetrahedron

We will further construct a (doubly rotating) positive elliptic-elliptic RE of the 4-body problem in $\mathbb{S}^3$, in which 4 equal masses are at the vertices of a regular tetrahedron that has rotations of equal frequencies. So let us fix $m_1 = m_2 = m_3 = m_4 =: m > 0$ and consider the initial position of the 4 bodies to be given as in the first example of Section 6.2, i.e. by the coordinates

$$
\begin{array}{llll}
w_1^0 = 0, & x_1^0 = 0, & y_1^0 = 0, & z_1^0 = 1, \\
w_2^0 = 0, & x_2^0 = 0, & y_2^0 = \dfrac{2\sqrt{2}}{3}, & z_2^0 = -\dfrac{1}{3}, \\
w_3^0 = 0, & x_3^0 = -\dfrac{\sqrt{6}}{3}, & y_3^0 = -\dfrac{\sqrt{2}}{3}, & z_3^0 = -\dfrac{1}{3}, \\
w_4^0 = 0, & x_4^0 = \dfrac{\sqrt{6}}{3}, & y_4^0 = -\dfrac{\sqrt{2}}{3}, & z_4^0 = -\dfrac{1}{3},
\end{array}
$$

which is a FP configuration. Indeed, the masses are equal and the bodies are at the vertices of a regular tetrahedron inscribed in a great sphere of $\mathbb{S}^3$.

For this choice of initial positions, we can compute that

$$
r_1 = r_2 = 0, \qquad \rho_1 = \rho_2 = 1, \qquad r_3 = r_4 = \frac{\sqrt{6}}{3}, \qquad \rho_3 = \rho_4 = \frac{\sqrt{3}}{3},
$$

which means that m_1 and m_2 move on the Clifford torus with $r = 0$ and $\rho = 1$ (i.e. one of the two Clifford tori, within the class of a given foliation of $\mathbb{S}^3$, which is also a great circle of $\mathbb{S}^3$, see Figure 8.1, the other corresponding to $r = 1$ and $\rho = 0$), while we expect m_3 and m_4 to move on the Clifford torus with $r = \frac{\sqrt{6}}{3}$ and $\rho = \frac{\sqrt{3}}{3}$.

These considerations allow us to obtain the constants that determine the angles. Indeed, a_1 and a_2 can take any values,

$$
a_3 = 3\pi/2, \; a_4 = \pi/2, \; b_1 = \pi/2,
$$

and b_2, b_3, b_4 are such that

$$
\sin b_2 = -1/3, \quad \cos b_2 = 2\sqrt{2}/3,
$$

$$
\cos b_3 = -\sqrt{6}/3, \quad \sin b_3 = -\sqrt{3}/3
$$

$$
\cos b_4 = -\sqrt{6}/3, \quad \sin b_4 = -\sqrt{3}/3,
$$

which means that $b_3 = b_4$.

We can now compute the form of the candidate for a solution generated from the above FP configuration. Using the above values of r_i, ρ_i, a_i and b_i, $i = 1, 2, 3, 4$, we obtain from the equations

$$w_i^0 = r_i \cos a_i,\ x_i^0 = r_i \sin a_i,\ y_i^0 = \rho_i \cos b_i,\ z_i^0 = \rho_i \sin b_i,\ i = 1, 2, 3, 4,$$

that the candidate for a solution is given by

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4),\ \mathbf{q}_i = (w_i, x_i, y_i, z_i),\ i = 1, 2, 3, 4, \tag{10.2}$$

$$\begin{aligned}
w_1 &= 0, & x_1 &= 0,\\
y_1 &= \cos(\alpha t + \pi/2), & z_1 &= \sin(\alpha t + \pi/2),\\
w_2 &= 0, & x_2 &= 0,\\
y_2 &= \cos(\alpha t + b_2), & z_2 &= \sin(\alpha t + b_2),\\
w_3 &= \frac{\sqrt{6}}{3}\cos(\beta t + 3\pi/2), & x_3 &= \frac{\sqrt{6}}{3}\sin(\beta t + 3\pi/2),\\
y_3 &= \frac{\sqrt{3}}{3}\cos(\beta t + b_3), & z_3 &= \frac{\sqrt{3}}{3}\sin(\beta t + b_3),\\
w_4 &= \frac{\sqrt{6}}{3}\cos(\beta t + \pi/2), & x_4 &= \frac{\sqrt{6}}{3}\sin(\beta t + \pi/2),\\
y_4 &= \frac{\sqrt{3}}{3}\cos(\beta t + b_4), & z_4 &= \frac{\sqrt{3}}{3}\sin(\beta t + b_4).
\end{aligned}$$

If we invoke Criterion 4, do a straightforward computation, and use the fact that the frequencies of the two rotations have the same size, i.e. are equal in absolute value, we can conclude that $\mathbf{q}$, given by (10.2), satisfies system (3.15), so it is indeed a solution of the curved 4-body problem in $\mathbb{S}^3$.

Straightforward computations lead us to the following values of the angular momentum constants:

$$c_{wx} = \frac{4}{3}m\alpha,\ c_{wy} = c_{wz} = c_{xy} = c_{xz} = 0,\ c_{yz} = \frac{8}{3}m\alpha,$$

for $\beta = \alpha$, and

$$c_{wx} = \frac{4}{3}m\alpha,\ c_{wy} = c_{wz} = c_{xy} = c_{xz} = 0,\ c_{yz} = -\frac{8}{3}m\alpha,$$

for $\beta = -\alpha$, a fact which shows that rotations around the origin of the coordinate system takes place only relative to the planes wx and yz.

10.3 Regular pentatope

We will next construct a (doubly rotating) positive elliptic-elliptic RE of the 5-body problem in $\mathbb{S}^3$, in which 5 equal masses are at the vertices of a regular pentatope that has two rotations of equal-size frequencies. So let us fix $m_1 = m_2 = m_3 = m_4 = m_5 =: m > 0$ and consider the initial position of the 5 bodies to be given as in the second example of Section 6.2, i.e. by the coordinates

$$
\begin{aligned}
&w_1^0 = 1, && x_1^0 = 0, && y_1^0 = 0, && z_1^0 = 0,\\
&w_2^0 = -\frac{1}{4}, && x_2^0 = \frac{\sqrt{15}}{4}, && y_2^0 = 0, && z_2^0 = 0,\\
&w_3^0 = -\frac{1}{4}, && x_3^0 = -\frac{\sqrt{5}}{4\sqrt{3}}, && y_3^0 = \frac{\sqrt{5}}{\sqrt{6}}, && z_3^0 = 0,\\
&w_4^0 = -\frac{1}{4}, && x_4^0 = -\frac{\sqrt{5}}{4\sqrt{3}}, && y_4^0 = -\frac{\sqrt{5}}{2\sqrt{6}}, && z_4^0 = \frac{\sqrt{5}}{2\sqrt{2}},\\
&w_5^0 = -\frac{1}{4}, && x_5^0 = -\frac{\sqrt{5}}{4\sqrt{3}}, && y_5^0 = -\frac{\sqrt{5}}{2\sqrt{6}}, && z_5^0 = -\frac{\sqrt{5}}{2\sqrt{2}},
\end{aligned}
$$

which is a FP configuration because the masses are equal and the bodies are at the vertices of a regular pentatope inscribed in $\mathbb{S}^3$.

For this choice of initial positions, we can compute that

$$
\begin{aligned}
&r_1 = r_2 = 1, && \rho_1 = \rho_2 = 0,\\
&r_3 = r_4 = r_5 = 1/\sqrt{6}, && \rho_3 = \rho_4 = \rho_5 = \sqrt{5}/\sqrt{6},
\end{aligned}
$$

which means that m_1 and m_2 move on the Clifford torus with $r = 1$ and $\rho = 0$ (i.e. one of the two Clifford tori, in a class of a given foliation of $\mathbb{S}^3$, which is also a great circle of $\mathbb{S}^3$, the other corresponding to $r = 0$ and $\rho = 1$), while we expect m_3, m_4, and m_5 to move on the Clifford torus with $r = \frac{1}{\sqrt{6}}$ and $\rho = \frac{\sqrt{5}}{\sqrt{6}}$.

These considerations allow us to compute the constants that determine the angles. We obtain that

$$a_1 = 0,$$

a_2 is such that

$$\cos a_2 = -1/4, \quad \sin a_2 = -\sqrt{15}/4$$

and a_3, a_4, a_5 are such that

$$\cos a_3 = -\sqrt{6}/4, \quad \sin a_3 = -\sqrt{10}/4,$$

$$\cos a_4 = -\sqrt{6}/4, \quad \sin a_4 = -\sqrt{10}/4,$$
$$\cos a_5 = -\sqrt{6}/4, \quad \sin a_5 = -\sqrt{10}/4,$$

which means that $a_3 = a_4 = a_5$. We further obtain that, since $\rho_1 = \rho_2 = 0$, the constants b_1 and b_2 can be anything, in particular 0. Further computations lead us to the conclusion that

$$b_1 = b_2 = b_3 = 0, \quad b_4 = 2\pi/3, \quad b_5 = 4\pi/3.$$

We can now compute the form of the candidate for a solution generated from the above FP configuration. Using the above values of r_i, ρ_i, a_i and b_i, $i = 1, 2, 3, 4, 5$, we obtain from the equations

$$w_i^0 = r_i \cos a_i,\ x_i^0 = r_i \sin a_i,\ y_i^0 = \rho_i \cos b_i,\ z_i^0 = \rho_i \sin b_i,\ i = 1, 2, 3, 4, 5$$

that the candidate for a solution is given by

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4, \mathbf{q}_5),\ \mathbf{q}_i = (w_i, x_i, y_i, z_i),\ i = 1, 2, 3, 4, 5, \tag{10.3}$$

$$\begin{aligned}
w_1 &= \cos \alpha t, & x_1 &= \sin \alpha t,\\
y_1 &= 0, & z_1 &= 0,\\
w_2 &= \cos(\alpha t + a_2), & x_2 &= \sin(\alpha t + a_2),\\
y_2 &= 0, & z_2 &= 0,\\
w_3 &= \frac{1}{\sqrt{6}} \cos(\alpha t + a_3), & x_3 &= \frac{1}{\sqrt{6}} \sin(\alpha t + a_3),\\
y_3 &= \frac{\sqrt{5}}{\sqrt{6}} \cos \beta t, & z_3 &= \frac{\sqrt{5}}{\sqrt{6}} \sin \beta t,\\
w_4 &= \frac{1}{\sqrt{6}} \cos(\alpha t + a_4), & x_4 &= \frac{1}{\sqrt{6}} \sin(\alpha t + a_4),\\
y_4 &= \frac{\sqrt{5}}{\sqrt{6}} \cos(\beta t + 2\pi/3), & z_4 &= \frac{\sqrt{5}}{\sqrt{6}} \sin(\beta t + 2\pi/3),\\
w_5 &= \frac{1}{\sqrt{6}} \cos(\alpha t + a_5), & x_5 &= \frac{1}{\sqrt{6}} \sin(\alpha t + a_5),\\
y_5 &= \frac{\sqrt{5}}{\sqrt{6}} \cos(\beta t + 4\pi/3), & z_5 &= \frac{\sqrt{5}}{\sqrt{6}} \sin(\beta t + 4\pi/3),
\end{aligned}$$

If we invoke Criterion 4, do a straightforward computation, and use the fact that the frequencies of the two rotations have the same size, i.e. $|\alpha| = |\beta|$, we can conclude that $\mathbf{q}$, given by (10.3), satisfies system (7.13), (7.14), (7.15), (7.16) and condition (ii), so it is indeed a (doubly rotating) positive elliptic-elliptic RE generated from a FP configuration, i.e. a solution of system (3.15) with $N = 5$ for any value of α and β with $|\alpha| = |\beta| \neq 0$.

A straightforward computation shows that the constants of the angular momentum are

$$c_{wx} = \frac{5}{2}m\alpha,\ c_{wy} = c_{wz} = c_{xy} = c_{xz} = 0,\ c_{yz} = \frac{5}{2}m\alpha$$

for $\beta = \alpha$ and

$$c_{wx} = \frac{5}{2}m\alpha,\ c_{wy} = c_{wz} = c_{xy} = c_{xz} = 0,\ c_{yz} = -\frac{5}{2}m\alpha$$

for $\beta = -\alpha$, which means that the bodies rotate around the origin of the coordinate system only relative to the planes wx and yz.

10.4 Pair of equilateral triangles

We will next construct an example in the 6-body problem in $\mathbb{S}^3$ in which 3 bodies of equal masses move along a great circle at the vertices of an equilateral triangle, while the other 3 bodies of masses equal to those of the previous bodies move along a complementary circle of another great sphere, also at the vertices of an equilateral triangle. So consider the masses $m_1 = m_2 = m_3 = m_4 = m_5 = m_6 =: m > 0$ and the frequencies $\alpha, \beta \neq 0$, which, in general, we can take as distinct, $\alpha \neq \beta$. Then a candidate for a solution as described above has the form

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4, \mathbf{q}_5, \mathbf{q}_6),\ \mathbf{q}_i = (w_i, x_i, y_i, z_i),\ i = 1, 2, \dots, 6, \tag{10.4}$$

$$\begin{aligned}
w_1 &= \cos\alpha t, & x_1 &= \sin\alpha t,\\
y_1 &= 0, & z_1 &= 0,\\
w_2 &= \cos(\alpha t + 2\pi/3), & x_2 &= \sin(\alpha t + 2\pi/3),\\
y_2 &= 0, & z_2 &= 0,\\
w_3 &= \cos(\alpha t + 4\pi/3), & x_3 &= \sin(\alpha t + 4\pi/3),\\
y_3 &= 0, & z_3 &= 0,\\
w_4 &= 0, & x_4 &= 0,\\
y_4 &= \cos\beta t, & z_4 &= \sin\beta t,
\end{aligned}$$

$$
\begin{aligned}
w_5 &= 0, & x_5 &= 0,\\
y_5 &= \cos(\beta t + 2\pi/3), & z_5 &= \sin(\beta t + 2\pi/3),\\
w_6 &= 0, & x_6 &= 0,\\
y_6 &= \cos(\beta t + 4\pi/3), & z_6 &= \sin(\beta t + 4\pi/3).
\end{aligned}
$$

For $t = 0$, we obtain the FP configuration specific to $\mathbb{S}^3$ similar to the one constructed in Section 6.2, namely

$$
\begin{aligned}
w_1 &= 0, & x_1 &= 1, & y_1 &= 0, & z_1 &= 0,\\
w_2 &= -\frac{1}{2}, & x_2 &= \frac{\sqrt{3}}{2}, & y_2 &= 0, & z_2 &= 0,\\
w_3 &= -\frac{1}{2}, & x_3 &= -\frac{\sqrt{3}}{2}, & y_3 &= 0, & z_3 &= 0,\\
w_4 &= 0, & x_4 &= 0, & y_4 &= 1, & z_4 &= 0,\\
w_5 &= 0, & x_5 &= 0, & y_5 &= -\frac{1}{2}, & z_5 &= \frac{\sqrt{3}}{2},\\
w_6 &= 0, & x_6 &= 0, & y_6 &= -\frac{1}{2}, & z_6 &= -\frac{\sqrt{3}}{2}.
\end{aligned}
$$

To prove that $\mathbf{q}$ given by (10.4) is a solution of system (3.15), we can therefore apply Criterion 4. A straightforward computation shows that the $4N$ conditions (7.13), (7.14), (7.15), (7.16) are satisfied, and then we can observe that condition (i) is also verified because

$$
\begin{aligned}
r_1 &= r_2 = r_3 = 1, & \rho_1 &= \rho_2 = \rho_3 = 0,\\
r_4 &= r_5 = r_6 = 0, & \rho_4 &= \rho_5 = \rho_6 = 1.
\end{aligned}
$$

Consequently $\mathbf{q}$ given by (10.4) is a positive elliptic-elliptic RE of the 6-body problem given by system (3.15) with $N = 6$ for any $\alpha, \beta \neq 0$. If α/β is rational, a case in which the set of frequency pairs has measure zero in $\mathbb{R}^2$, the corresponding orbits are periodic. In general, however, α/β is irrational, so the orbits are quasiperiodic.

This property is quite interesting since, in $\mathbb{R}^3$, no quasiperiodic RE were ever found. Nevertheless, quasiperiodic RE were discovered for the Newtonian N-body problem in $\mathbb{R}^4$, [2], [17].

A straightforward computation shows that the constants of the angular momentum integrals are

$$
c_{wx} = 3m\alpha \neq 0, \quad c_{yz} = 3m\beta \neq 0, \quad c_{wy} = c_{wz} = c_{xy} = c_{xz} = 0,
$$

which means that the rotation takes place around the origin of the coordinate system only relative to the planes wx and yz.

Notice that, in the light of [28], the kind of example constructed here in the 6-body problem can be easily generalized to any $(N+M)$-body problem of equal masses, $N, M \geq 3$ and odd, in which N bodies rotate along a great circle of a great sphere at the vertices of a regular N-gon, while the other M bodies rotate along a complementary great circle of another great sphere at the vertices of a regular M-gon. The same as in the 6-body problem discussed here, the rotation takes place around the origin of the coordinate system only relative to 2 out of 6 reference planes.

10.5 Pair of scalene triangles

We will next extend the example constructed in Section 10.4 to unequal masses. The idea is the same as the one we used in Section 9.4, based on the results proved in [28], according to which, given an acute scalene triangle inscribed in a great circle of a great sphere, we can find 3 masses such that this configuration forms a FP. The difference is that we don't keep the configuration fixed here by assigning zero initial velocities, but make it rotate uniformly, thus leading to a RE. In fact, in this 6-body problem, 3 bodies of unequal masses rotate along a great circle of a great sphere at the vertices of an acute scalene triangle, while the other 3 bodies rotate along a complementary great circle of another great sphere at the vertices of another acute scalene triangle, not necessarily congruent with the previous one.

So consider the allowable masses $m_1, m_2, m_3, m_4, m_5, m_6 > 0$, which, in general, are not equal. Then a candidate for a solution as described above has the form

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4, \mathbf{q}_5, \mathbf{q}_6), \ \mathbf{q}_i = (w_i, x_i, y_i, z_i), \ i = 1, 2, \dots, 6, \tag{10.5}$$

$$\begin{aligned}
w_1 &= \cos(\alpha t + a_1), & x_1 &= \sin(\alpha t + a_1),\\
y_1 &= 0, & z_1 &= 0,\\
w_2 &= \cos(\alpha t + a_2), & x_2 &= \sin(\alpha t + a_2),\\
y_2 &= 0, & z_2 &= 0,\\
w_3 &= \cos(\alpha t + a_3), & x_3 &= \sin(\alpha t + a_3),\\
y_3 &= 0, & z_3 &= 0,\\
w_4 &= 0, & x_4 &= 0,\\
y_4 &= \cos(\beta t + b_4), & z_4 &= \sin(\beta t + b_4),
\end{aligned}$$

$$
\begin{aligned}
w_5 &= 0, & x_5 &= 0,\\
y_5 &= \cos(\beta t + b_5), & z_5 &= \sin(\beta t + b_5),\\
w_6 &= 0, & x_6 &= 0,\\
y_6 &= \cos(\beta t + b_6), & z_6 &= \sin(\beta t + b_6),
\end{aligned}
$$

where the constants a_1, a_2, and a_3, with $0 \leq a_1 < a_2 < a_3 < 2\pi$, and b_1, b_2, and b_3, with $0 \leq b_4 < b_5 < b_6 < 2\pi$, determine the shape of the first and second triangle, respectively, in agreement with the values of the masses.

Notice that for $t = 0$, the position of the bodies is the FP configuration described and proved to be as such in Section 9.4. Therefore we can apply Criterion 4 to check whether $\mathbf{q}$ given in (10.5) is a positive elliptic-elliptic RE. Again, as in Section 9.4, we can approach this problem in two ways. One is computational, and it consists of using the fact that the positions at $t = 0$ form a FP configuration to determine the relationships between the constants m_1, m_2, m_3, a_1, a_2, and a_3, on one hand, and the constants m_4, m_5, m_6, b_4, b_5, and b_6, on the other hand. It turns out that they reduce to conditions (7.13), (7.14), (7.15), and (7.16). Then we only need to remark that

$$r_1 = r_2 = r_3 = 1 \quad \text{and} \quad r_4 = r_5 = r_6 = 0,$$

which means that condition (i) of Criterion 4 is satisfied. The other approach is to invoke again the result of [28] and a reasoning similar to the one we used to show that the position at $t = 0$ is a FP configuration. Both help us conclude that $\mathbf{q}$ given by (10.5) is a solution of system (3.15) for any $\alpha, \beta \neq 0$. Again, when α/β is rational, a case that corresponds to a negligible set of frequency pairs, the solutions are periodic. In the generic case, when α/β is irrational, the solutions are quasiperiodic.

A straightforward computation shows that the constants of the total angular momentum integrals are

$$c_{wx} = (m_1 + m_2 + m_3)\alpha \neq 0, \quad c_{yz} = (m_1 + m_2 + m_3)\beta \neq 0,$$

$$c_{wy} = c_{wz} = c_{xy} = c_{xz} = 0,$$

which means that the rotation takes place around the origin of the coordinate system only relative to the planes wx and yz.

Chapter 11

Negative RE

In this chapter we will provide examples of negative RE, one for each type of orbit of this kind: elliptic, hyperbolic, and elliptic-hyperbolic. The first is the Lagrangian RE of equal masses, which is a negative elliptic RE of the 3-body problem in $\mathbb{H}^3$, the second is the Eulerian orbit of equal masses, which is a negative hyperbolic RE of the 3-body problem in $\mathbb{H}^3$, and the third is an elliptic-hyperbolic orbit that combines the previous two examples in the sense that it inherits their rotations.

11.1 Negative elliptic RE

The class of examples we construct here is the analogue of the one presented in Section 9.1 in the case of the sphere, namely Lagrangian solutions (i.e. equilateral triangles) of equal masses in $\mathbb{H}^3$. In the light of Remark 9, we expect that the bodies move on a 2-dimensional hyperbolic sphere, whose curvature is not necessarily the same as the one of $\mathbb{H}^3$. The existence of these orbits in $\mathbb{H}^2$, and the fact that they occur only when the masses are equal, was first proved in [35]. So, in this case, we have $N = 3$ and $m_1 = m_2 = m_3 =: m > 0$. The solution of the corresponding system (3.16) we are seeking is of the form

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3),\ \mathbf{q}_i = (w_i, x_i, y_i, z_i),\ i = 1, 2, 3, \tag{11.1}$$

$$\begin{aligned}
w_1(t) &= r\cos\omega t, & x_1(t) &= r\sin\omega t,\\
y_1(t) &= y \text{ (constant)}, & z_1(t) &= z \text{ (constant)},\\
w_2(t) &= r\cos(\omega t + 2\pi/3), & x_2(t) &= r\sin(\omega t + 2\pi/3),\\
y_2(t) &= y \text{ (constant)}, & z_2(t) &= z \text{ (constant)},
\end{aligned}$$

$$w_3(t) = r\cos(\omega t + 4\pi/3), \qquad x_3(t) = r\sin(\omega t + 4\pi/3),$$
$$y_3(t) = y \text{ (constant)}, \qquad z_3(t) = z \text{ (constant)},$$

with $r^2 + y^2 - z^2 = -1$. Consequently, for the equations occurring in Criterion 5, we have

$$r_1 = r_2 = r_3 =: r,\ a_1 = 0,\ a_2 = 2\pi/3,\ a_3 = 4\pi/3,$$

$$y_1 = y_2 = y_3 =: y \text{ (constant)}, \quad z_1 = z_2 = z_3 =: z \text{ (constant)}.$$

Substituting these values into the equations (7.17), (7.18), (7.19), (7.20), we obtain either identities or the same equation as in Section 9.1, namely

$$\alpha^2 = \frac{8m}{\sqrt{3}r^3(4-3r^2)^{3/2}}.$$

Consequently, given $m > 0$, $r > 0$, and y, z with $r^2 + y^2 - z^2 = -1$ and $z > 1$, we can always find two frequencies,

$$\alpha_1 = \frac{2}{r}\sqrt{\frac{2m}{\sqrt{3}r(4-3r^2)^{3/2}}} \quad \text{and} \quad \alpha_2 = -\frac{2}{r}\sqrt{\frac{2m}{\sqrt{3}r(4-3r^2)^{3/2}}},$$

such that system (3.16) has a solution of the form (11.1). The positive frequency corresponds to one sense of rotation, whereas the negative frequency corresponds to the opposite sense.

Notice that the bodies move on the 2-dimensional hyperbolic sphere

$$\mathbb{H}^2_{\kappa_0, y_0} = \{(w, x, y_0, z) \,|\, w^2 + x^2 - z^2 = -1 - y_0^2,\ y_0 = \text{constant},\ z > 0\},$$

which has curvature $\kappa_0 = -(1+y_0^2)^{-1/2}$. When $y_0 = 0$, we have a great 2-dimensional hyperbolic sphere, i.e. its curvature is 1, so the motion is in agreement with Remark 6.

A straightforward computation shows that the constants of the angular momentum are

$$c_{wx} = 3m\alpha \neq 0 \quad \text{and} \quad c_{wy} = c_{wz} = c_{xy} = c_{xz} = c_{yz} = 0,$$

which means that the rotation takes place around the origin of the coordinate system only relative to the plane wx.

11.2 Negative hyperbolic RE

In this section we will construct a class of negative hyperbolic RE for which, in agreement with Remark 7, the bodies rotate on a 2-dimensional hyperbolic sphere of the same curvature as $\mathbb{H}^3$. In the 2-dimensional case, the existence of a similar orbit was already pointed out in [33], where we have also proved it to be unstable. So let us check a solution of the form

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3),\ \ \mathbf{q}_i = (w_i, x_i, y_i, z_i),\ \ i = 1, 2, 3, \tag{11.2}$$

$$\begin{array}{llll} w_1 = 0, & x_1 = 0, & y_1 = \sinh \beta t, & z_1 = \cosh \beta t, \\ w_2 = 0, & x_2 = x \text{ (constant)}, & y_2 = \eta \sinh \beta t, & z_2 = \eta \cosh \beta t, \\ w_3 = 0, & x_3 = -x \text{ (constant)}, & y_3 = \eta \sinh \beta t, & z_3 = \eta \cosh \beta t, \end{array}$$

with $x^2 - \eta^2 = -1$. Consequently

$$\eta_1 = 1,\ \ \eta_2 = \eta_3 =: \eta \text{ (constant)},\ \ b_1 = b_2 = b_3 = 0.$$

We then compute that

$$\mu_{12} = \mu_{21} = \mu_{13} = \mu_{23} = -\eta,\ \ \mu_{23} = \mu_{32} = 1 - 2\eta^2.$$

We can now use Criterion 6 to determine whether a candidate $\mathbf{q}$ given by (11.2) is a (simply rotating) negative hyperbolic RE. Straightforward computations lead us from equations (7.21), (7.22), (7.23), and (7.24) either to identities or to the equation

$$\beta^2 = \frac{1 + 4\eta^2}{4\eta^3(\eta^2 - 1)^{3/2}}.$$

Therefore, given $m, x, \eta > 0$ with $x^2 - \eta^2 = -1$, there exist two nonzero frequencies,

$$\beta_1 = \frac{1}{2\eta}\sqrt{\frac{1 + 4\eta^2}{\eta(\eta^2 - 1)^{3/2}}} \ \ \text{and} \ \ \beta_2 = -\frac{1}{2\eta}\sqrt{\frac{1 + 4\eta^2}{\eta(\eta^2 - 1)^{3/2}}},$$

such that $\mathbf{q}$ given by (11.2) is a (simply rotating) positive hyperbolic relative equilibrium. Notice that the motion takes place on the 2-dimensional hyperbolic sphere

$$\mathbb{H}^2_w = \{(0, x, y, z)\,|\, x^2 + y^2 - z^2 = -1,\ \ z > 0\}.$$

These orbits are neither periodic nor quasiperiodic. A straightforward computation shows that the constants of the angular momentum are

$$c_{wx} = c_{wy} = c_{wz} = c_{xy} = c_{xz} = 0,\ \ c_{yz} = m\beta(1 - 2\eta^2),$$

which means that the rotation takes place about the origin of the coordinate system only relative to the yz plane.

11.3 Negative elliptic-hyperbolic RE

In this section we will construct a class of (doubly rotating) negative elliptic-hyperbolic RE. In the light of Remark 11, we expect that the motion cannot take place on any 2-dimensional hyperboloid of $\mathbb{H}^3$. In fact, as we know from Theorem 3, relative equilibria of this type may rotate on hyperbolic cylinders, which is also the case with the solution we introduce here.

Consider the masses $m_1 = m_2 = m_3 =: m > 0$. We will check a solution of the form

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3),\ \mathbf{q}_i = (w_i, x_i, y_i, z_i),\ i = 1, 2, 3, \tag{11.3}$$

$$\begin{array}{llll} w_1 = 0, & x_1 = 0, & y_1 = \sinh \beta t, & z_1 = \cosh \beta t, \\ w_2 = r \cos \alpha t, & x_2 = r \sin \alpha t, & y_2 = \eta \sinh \beta t, & z_2 = \eta \cosh \beta t, \\ w_3 = -r \cos \alpha t, & x_3 = -r \sin \alpha t, & y_3 = \eta \sinh \beta t, & z_3 = \eta \cosh \beta t. \end{array}$$

In terms of the form (5.12) of an elliptic-hyperbolic RE, (11.3) is realized when

$$r_1 = 0, r_2 = r_3 =: r,\ \eta_1 = 1,\ \eta_2 = \eta_3 =: \eta,$$

$$a_1 = a_2 = 0,\ a_3 = \pi,\ b_1 = b_2 = b_3 = 0.$$

Substituting these values into the equations (7.25), (7.26), (7.27), (7.28) of Criterion 7 and using the fact that $r^2 - \eta^2 = -1$, we obtain the equation

$$\alpha^2 + \beta^2 = \frac{m(4\eta^2 + 1)}{4\eta^3(\eta^2 - 1)^{3/2}},$$

which is satisfied for infinitely many values of α and β. Therefore, for any masses $m_1 = m_2 = m_3 =: m > 0$, and r, η with $r^2 - \eta^2 = -1$, there are infinitely many frequencies α and β that correspond to negative elliptic-hyperbolic RE of the form (11.3). These orbits are neither periodic nor quasiperiodic.

The bodies m_2 and m_3 move on the same hyperbolic cylinder, namely $\mathbf{C}^2_{r\eta}$, which has constant positive curvature, while m_1 moves on the degenerate hyperbolic cylinder $\mathbf{C}^2_{01}$, which is a geodesic hyperbola, therefore has zero curvature.

A straightforward computation shows that the constants of the angular momentum are

$$c_{wx} = 2m\alpha r^2, c_{yz} = -1 - 2\beta\eta^2, c_{wy} = c_{wz} = c_{xy} = c_{xz} = 0,$$

which means that the rotation takes place around the origin of the coordinate system only relative to the wx and yz planes.

Part V

The 2-dimensional case

Preamble

The goal of Part V is to study some properties of the curved N-body problem on the surfaces $\mathbb{S}^2$ and $\mathbb{H}^2$, which are invariant for the equations of motion. We will focus on polygonal RE for $N \geq 3$ as well as on Lagrangian and Eulerian RE for $N = 3$. The Lagrangian RE are behind a modest first step towards proving that space is Euclidean for distances comparable to those of our solar system. We will show that such orbits exist in the curved case only if the masses are equal. In Euclidean space they also occur for unequal masses, both in theory and in nature, such as the approximate equilateral triangles formed by the Sun, Jupiter, and the Trojan/Greek asteroids. We will end this monograph with extending the formulation of Saari's conjecture to the curved N-body problem in the 2-dimensional case and proving it when the motion of the bodies is restricted to a rotating geodesic.

Chapter 12

Polygonal RE

The goal of this chapter is to study polygonal RE in $\mathbb{S}^2$ and $\mathbb{H}^2$. Since these manifolds are embedded in $\mathbb{R}^3$, we will drop the w coordinate from now on and use an xyz frame. Given the fact that the dimension is reduced by one, we will not encounter positive elliptic-elliptic and negative elliptic-hyperbolic RE anymore. So the only orbits we will deal with from now on are the positive and negative elliptic as well as the negative hyperbolic RE.

We will first show that FP configurations lying on geodesics of $\mathbb{S}^2$ can generate RE for any nonzero value of the angular frequency. Then we will prove that if the bodies' initial configuration is on a great circle of $\mathbb{S}^2$, then a RE can be generated only along that great circle. We will further show that RE formed by regular N-gons having equal masses at their vertices can also move on non-great circles. Finally we will prove that this result is also true in $\mathbb{H}^2$.

Both in the classical and the curved N-body problem, given the size of a configuration, a RE exists for it, if at all, only for certain angular frequencies, i.e. for two angular velocities of equal size and opposite signs. The exception from this rule occurs for RE generated from some FP configurations. Let us prove this fact in $\mathbb{S}^2$.

Theorem 4. *Consider a nonsingular FP configuration given by the bodies of masses $m_1, m_2, \dots, m_N > 0, N \geq 2$, that initially lie on a great circle of $\mathbb{S}^2$. Then, for every nonzero angular velocity applied in the plane of the circle, this FP configuration generates a positive elliptic RE that rotates along that great circle.*

Proof. Without loss of generality, we assume that the great circle is the equator $z = 0$ and that for some given masses $m_1, m_2, \dots, m_N > 0$ there exist $\alpha_1, \alpha_2, \dots, \alpha_N$ such that

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N), \ \ \mathbf{q}_i = (x_i, y_i, 0), \tag{12.1}$$

$$x_i = \cos(\omega t + \alpha_i), y_i = \sin(\omega t + \alpha_i),\ i = 1, 2, \dots, N,$$

is a FP for $\omega = 0$. This configuration can also be interpreted as being $\mathbf{q}(0)$, i.e. the solution $\mathbf{q}$ at $t = 0$ for any $\omega \neq 0$. So we can conclude that

$$\nabla_{\mathbf{q}_i} U(\mathbf{q}(0)) = \mathbf{0},\ i = 1, 2, \dots, N.$$

But then, for $t = 0$, the equations of motion (3.15) reduce to

$$\begin{cases} \ddot{x}_i = -(\dot{x}_i^2 + \dot{y}_i^2)x_i \\ \ddot{y}_i = -(\dot{x}_i^2 + \dot{y}_i^2)y_i, \end{cases} \tag{12.2}$$

$i = 1, 2, \dots, N$. Notice, however, that

$$\dot{x}_i = -\omega \sin(\omega t + \alpha_i),\ \ddot{x}_i = -\omega^2 \cos(\omega t + \alpha_i),$$

$$\dot{y}_i = -\omega \cos(\omega t + \alpha_i),\ \ddot{y}_i = -\omega^2 \sin(\omega t + \alpha_i),$$

therefore $\dot{x}_i^2 + \dot{y}_i^2 = \omega^2$. Using these expressions, it is easy to see that $\mathbf{q}$ given by (12.1) is a solution of (12.2) for every t. Since $\nabla_{\mathbf{q}_i} U(\mathbf{q}(0)) = \mathbf{0},\ i = 1, 2, \dots, N$, it follows that the gravitational forces are in equilibrium at the initial moment, so no gravitational forces act on the bodies. Consequently, the rotation imposed by $\omega \neq 0$ makes the system move like a rigid body, so the gravitational forces further remain in equilibrium, consequently $\nabla_{\mathbf{q}_i} U(\mathbf{q}(t)) = \mathbf{0},\ i = 1, 2, \dots, N$, for all t. Therefore $\mathbf{q}$ given by (12.1) satisfies equations (3.15), so it is a positive elliptic RE. This remark completes the proof. □

12.1 Polygonal RE on geodesics of $\mathbb{S}^2$

The following result shows that positive elliptic RE generated from FP configurations given by regular N-gons of equal masses on a great circle of $\mathbb{S}^2$ can occur only if certain conditions are satisfied.

Theorem 5. *Consider an odd number of equal bodies, initially at the vertices of a regular N-gon inscribed in a great circle of $\mathbb{S}^2$. Then the only positive elliptic RE that can be generated from this configuration are the ones that rotate in the plane of the original great circle.*

Proof. Without loss of generality, we can prove this result for the equator $z = 0$. Consider therefore a potential positive elliptic RE of the form

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N),\ \mathbf{q}_i = (x_i, y_i, z_i), \tag{12.3}$$

$$x_i = r_i \cos(\omega t + \alpha_i),\ y_i = r_i \sin(\omega t + \alpha_i),\ z_i = \pm(1 - r_i^2)^{1/2},$$

$i = 1, 2, \dots, N$, with $+$ taken for $z_i > 0$ and $-$ for $z_i < 0$. The only condition we impose on this solution is that r_i and α_i, $i = 1, 2, \dots, N$, are chosen such that, at all times, the configuration is a regular N-gon inscribed in a moving great circle of $\mathbb{S}^2$. Therefore the plane of the N-gon can have any angle with, say, the z-axis. This solution has the derivatives

$$\dot{x}_i = -r_i\omega \sin(\omega t + \alpha_i),\ \dot{y}_i = r_i\omega \cos(\omega t + \alpha_i),\ \dot{z}_i = 0,\ i = 1, 2, \dots, N,$$

$$\ddot{x}_i = -r_i\omega^2 \cos(\omega t + \alpha_i),\ \ddot{y}_i = -r_i\omega^2 \sin(\omega t + \alpha_i),\ \ddot{z}_i = 0,\ i = 1, 2, \dots, N.$$

Then

$$\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2 = r_i^2\omega^2,\ i = 1, 2, \dots, N.$$

Since any N-gon solution with N odd satisfies the conditions

$$\nabla_{\mathbf{q}_i} U(\mathbf{q}) = \mathbf{0},\ i = 1, 2, \dots, N,$$

system (3.15) reduces to

$$\begin{cases} \ddot{x}_i = -(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)x_i, \\ \ddot{y}_i = -(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)y_i, \\ \ddot{z}_i = -(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)z_i,\ i = 1, 2, \dots, N. \end{cases}$$

Then the substitution of (12.3) into the above system leads to:

$$\begin{cases} r_i(1 - r_i^2)\omega^2 \cos(\omega t + \alpha_i) = 0, \\ r_i(1 - r_i^2)\omega^2 \sin(\omega t + \alpha_i) = 0,\ i = 1, 2, \dots, N. \end{cases}$$

But assuming $\omega \neq 0$, this system is nontrivially satisfied if and only if $r_i = 0$ for some $i \in \{1, 2, \dots, N\}$ and $r_i = 1$ for the rest of the indices. But given the fact that the configuration is an N-gon, the former possibility cannot take place, so we obtain that $z_i = 0,\ i = 1, 2, \dots, N$. Therefore the bodies must rotate along the equator $z = 0$. □

12.2 Polygonal RE on non-great circles of $\mathbb{S}^2$

Theorem 5 raises the question whether positive elliptic RE given by regular polygons can rotate on other curves than geodesics. The answer is positive and is given by the following result, which we prove in the case of equal masses and N odd, to avoid singular configurations. However, we expect that the result is true in general for any nonsingular shape of the polygon and the masses that correspond to such a configuration.

Theorem 6. *Consider the curved N-body problem with equal masses, $m_1 = m_2 = \cdots = m_N =: m > 0$ in $\mathbb{S}^2$. Then, for any N odd, $m > 0$, and $z \in (-1, 1)$, there are a positive and a negative ω that produce positive elliptic RE for which the bodies are at the vertices of a regular N-gon rotating in the plane $z =$ constant. If N is even, this property is still true if we exclude the case when the motion takes place along the equator $z = 0$.*

Proof. The proof could be presented in general, but for the transparency of the exposition we will discuss the two possible cases: (i) N odd and (ii) N even.

(i) To simplify the presentation, we further denote the bodies by $m_i, i = -s, -s+1, \dots, -1, 0, 1, \dots, s-1, s$, where s is a positive integer. Without loss of generality we can further check into system (3.15) a solution candidate of the form (12.3) with i as above, $\alpha_{-s} = -\frac{4s\pi}{2s+1}, \dots, \alpha_{-1} = -\frac{2\pi}{2s+1}, \alpha_0 = 0,\ \alpha_1 = \frac{2\pi}{2s+1}, \dots, \alpha_s = \frac{4s\pi}{2s+1}$, $r := r_i, z := z_i$, and consider only the equations for $i = 0$. The study of this case suffices due to the involved symmetry, which yields the same conclusions for any other value of i.

The equation corresponding to the z_0 coordinate takes the form

$$\sum_{j=-s, j\neq 0}^{s} \frac{m(z - k_{0j}z)}{(1 - k_{0j}^2)^{3/2}} - r^2\omega^2 z = 0,$$

where $k_{0j} = x_0x_j + y_0y_j + z_0z_j = \cos\alpha_j - z^2\cos\alpha_j + z^2$. Using the fact that $r^2 + z^2 = 1$, $\cos\alpha_j = \cos\alpha_{-j}$, and $k_{0j} = k_{0(-j)}$, this equation becomes

$$\sum_{j=1}^{s} \frac{2(1 - \cos\alpha_j)}{(1 - k_{0j}^2)^{3/2}} = \frac{\omega^2}{m}. \tag{12.4}$$

Now we need to check whether the equations corresponding to x_0 and y_0 lead to the same relationship. In fact, checking for x_0, and ignoring y_0, suffices due to the same symmetry reasons invoked earlier or the duality of the trigonometric functions sin and cos. The substitution of the the above functions into the first equation of (3.15) leads us to

$$(r^2 - 1)\omega^2\cos\omega t = \sum_{j=-s, j\neq 0}^{s} \frac{m[\cos(\omega t + \alpha_j) - k_{0j}\cos\omega t]}{(1 - k_{0j}^2)^{3/2}}.$$

A straightforward computation, which uses the fact that $r^2 + z^2 = 1$, $\sin\alpha_j = -\sin\alpha_{-j}$, $\cos\alpha_j = \cos\alpha_{-j}$, and $k_{0j} = k_{0(-j)}$, yields the same equation (12.4). Writing the denominator of equation (12.4) explicitly, we are led to

$$\sum_{j=1}^{s} \frac{2}{(1-\cos\alpha_j)^{1/2}(1-z^2)^{3/2}[2-(1-\cos\alpha_j)(1-z^2)]^{3/2}} = \frac{\omega^2}{m}. \tag{12.5}$$

The left hand side is always positive, so for any $m > 0$ and $z \in (-1, 1)$ fixed, there are a positive and a negative ω that satisfy the equation. Therefore the N-gon with an odd number of sides is a positive elliptic RE.

(ii) To simplify the presentation when n is even, we denote the bodies by m_i, $i = -s+1, \dots, -1, 0, 1, \dots, s-1, s$, where s is a positive integer. Without loss of generality, we can substitute into equations (3.15) a solution candidate of the form (12.3) with i as above, $\alpha_{-s+1} = \frac{(-s+1)\pi}{s}, \dots, \alpha_{-1} = -\frac{\pi}{s}$, $\alpha_0 = 0$, $\alpha_1 = \frac{\pi}{s}, \dots, \alpha_{s-1} = \frac{(s-1)\pi}{s}$, $\alpha_s = \pi$, $r := r_i$, $z := z_i$, and consider as in the previous case only the equations for $i = 0$. Then using the fact that $k_{0j} = k_{0(-j)}$, $\cos\alpha_j = \cos\alpha_{-j}$, and $\cos\pi = -1$, a straightforward computation brings the equation corresponding to z_0 to the form

$$\sum_{j=1}^{s-1} \frac{2(1-\cos\alpha_j)}{(1-k_{0j}^2)^{3/2}} + \frac{2}{(1-k_{0s}^2)^{3/2}} = \frac{\omega^2}{m}. \tag{12.6}$$

Using additionally the relations $\sin\alpha_j = -\sin\alpha_{-j}$ and $\sin\pi = 0$, we obtain for the equation corresponding to x_0 the same form (12.6), which—for k_{0j} and k_{0s} written explicitly—becomes

$$\sum_{j=1}^{s-1} \frac{2}{(1-\cos\alpha_j)^{1/2}(1-z^2)^{3/2}[2-(1-\cos\alpha_j)(1-z^2)]^{3/2}} + \frac{1}{4z^2|z|(1-z^2)^{3/2}} = \frac{\omega^2}{m}.$$

Since the left hand side of this equations is positive and finite, given any $m > 0$ and $z \in (-1, 0) \cup (0, 1)$, there are a positive and a negative ω that satisfy it. So except for the case $z = 0$, which introduces antipodal singularities, the rotating N-gon with an even number of sides is a positive elliptic RE. $\square$

12.3 Polygonal RE in $\mathbb{H}^2$

We will further show that negative elliptic RE, similar to the ones proved in Theorem 6, also exist in $\mathbb{H}^2$.

Theorem 7. *Consider the curved N-body problem with equal masses, $m_1 = m_2 = \cdots = m_N =: m > 0$ in $\mathbb{H}^2$. Then, for any $m > 0$ and $z > 1$, there are a positive and a negative ω that produce negative elliptic RE in which the bodies are at the vertices of a regular N-gon rotating in the plane $z =$ constant.*

Proof. The proof works in the same way as for Theorem 6, by considering the cases N odd and even separately. The only differences are that we replace r with ρ, the relation $r^2+z^2 = 1$ with $z^2 = \rho^2+1$, and the denominator $(1-k_{0j}^2)^{3/2}$ with $(c_{0j}^2-1)^{3/2}$, wherever it appears, where $c_{0j} = -k_{0j}$ replaces k_{0j}. Unlike in $\mathbb{S}^2$, the case N even is satisfied for all admissible values of z. □

Chapter 13

Lagrangian and Eulerian RE

The case $N = 3$ presents particular interest in Euclidean space because the equilateral triangle is a RE for any values of the masses, a property discovered by Joseph Louis Lagrange in 1772, [100]. We will further show that this is not the case in $\mathbb{S}^2$ and $\mathbb{H}^2$, where the positive and negative elliptic Lagrangian RE exist only if the masses are equal. This conclusion provides a first step towards understanding with the help of these equations whether space is Euclidean for distances of the order of 10 AU because Lagrangian orbits of unequal masses show up in our solar system, as for example the approximate equilateral triangle formed by the Sun, Jupiter, and the Trojan/Greek asteroids. Following this result, we will prove the existence of positive elliptic Eulerian RE in $\mathbb{S}^2$ and negative elliptic Eulerian RE in $\mathbb{H}^2$, i.e. orbits formed by 3 bodies lying on a rotating geodesic. In the end we will show that the curved 3-body problem in $\mathbb{H}^2$ also exhibits negative hyperbolic Eulerian RE, i.e. orbits that lie on a geodesic that rotates hyperbolically.

13.1 Positive Elliptic Lagrangian RE

We start with a result that refines Theorem 6 in the case $N = 3$ in $\mathbb{S}^2$ by clarifying exactly when the equilateral triangle of equal masses is a Lagrangian RE.

Proposition 6. *Consider the 3-body problem with equal masses, $m_1 = m_2 = m_3 =: m$, in $\mathbb{S}^2$. Then for any $m > 0$ and $z \in (-1, 1)$, there are a positive and a negative ω that produce positive elliptic RE in which the bodies are at the vertices of an equilateral triangle that rotates in the plane $z =$ constant. Moreover, for every ω^2/m there are two values of z that lead to relative equilibria if $\omega^2/m \in (8/\sqrt{3}, \infty) \cup \{3\}$, three values if $\omega^2/m = 8/\sqrt{3}$, and four values if $\omega^2/m \in (3, 8/\sqrt{3})$.*

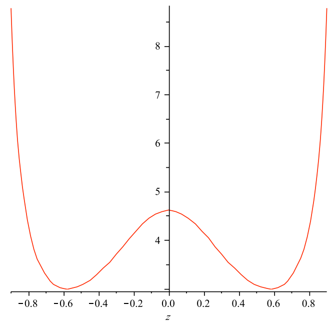

Figure 13.1: The graph of the function $f(z) = \frac{8}{\sqrt{3}(1+2z^2-3z^4)^{3/2}}$ for $z \in (-1, 1)$.

Proof. The first part of the statement is a consequence of Theorem 6 for $N = 3$. Alternatively, we can substitute into system (3.15) a solution of the form (12.3) with $i = 1, 2, 3$, $r := r_1 = r_2 = r_3$, $z = \pm(1 - r^2)^{1/2}$, $\alpha_1 = 0, \alpha_2 = 2\pi/3, \alpha_3 = 4\pi/3$, and obtain the equation

$$\frac{8}{\sqrt{3}(1 + 2z^2 - 3z^4)^{3/2}} = \frac{\omega^2}{m}. \tag{13.1}$$

The left hand side is positive for $z \in (-1, 1)$ and tends to infinity when $z \to \pm 1$ (see Figure 13.1). So for any z in this interval and $m > 0$, there are a positive and a negative ω for which the above equation is satisfied. A qualitative argument justifying Figure 13.1 and a straightforward computation also clarify the second part of the statement. $\square$

Remark 13. A result similar to Proposition 6 can be proved for 2 equal masses that rotate on a non-geodesic circle when the bodies are situated at opposite ends of a rotating diameter. Then, for $m > 0$ and $z \in (-1, 0) \cup (0, 1)$, the analogue of relationship (13.1) is the equation

$$\frac{1}{4z^2|z|(1 - z^2)^{3/2}} = \frac{\omega^2}{m}.$$

The case $z = 0$ yields no solution because it involves an antipodal singularity.

We have reached now the point when we can decide whether the equilateral triangle can be a positive elliptic RE in $\mathbb{S}^2$ if the masses are not equal. The following result shows that, unlike in the Euclidean case, the answer is negative when the bodies move on the sphere in the same Euclidean plane.

Proposition 7. *In the* 3*-body problem in* $\mathbb{S}^2$*, if the bodies* $m_1, m_2, m_3 > 0$ *are initially at the vertices of an equilateral triangle in the plane* $z =$ *constant for some* $z \in (-1, 1)$*, then there are initial velocities that lead to a positive elliptic RE in which the triangle rotates in its own plane if and only if* $m_1 = m_2 = m_3$*.*

Proof. The implication which shows that if $m_1 = m_2 = m_3$, the rotating equilateral triangle is a RE, follows from Proposition 6. To prove the other implication, we substitute into equations (3.15) a solution of the form (12.3) with $i = 1, 2, 3,\ r := r_1, r_2, r_3,\ z := z_1 = z_2 = z_3 = \pm(1 - r^2)^{1/2}$, and $\alpha_1 = 0, \alpha_2 = 2\pi/3, \alpha_3 = 4\pi/3$. The computations then lead to the system

$$\begin{cases} m_1 + m_2 = \gamma\omega^2 \\ m_2 + m_3 = \gamma\omega^2 \\ m_3 + m_1 = \gamma\omega^2, \end{cases} \tag{13.2}$$

where $\gamma = \sqrt{3}(1 + 2z^2 - 3z^4)^{3/2}/4$. But for any $z =$ constant in the interval $(-1, 1)$, the above system has a solution only for $m_1 = m_2 = m_3 = \gamma\omega^2/2$. Therefore the masses must be equal. □

The next result leads to the conclusion that Lagrangian solutions in $\mathbb{S}^2$ can take place only in Euclidean planes of $\mathbb{R}^3$. This property is known to be true in the Euclidean case for all RE, [100], but Wintner's proof doesn't work in our case because it uses the integral of the center of mass. Most importantly, our result also implies that Lagrangian orbits with unequal masses cannot exist in $\mathbb{S}^2$.

Theorem 8. *For all positive elliptic Lagrangian RE of the curved* 3*-body problem in* $\mathbb{S}^2$*, the masses* $m_1, m_2, m_3 > 0$ *have to rotate on the same circle, whose plane must be orthogonal to the rotation axis, and therefore* $m_1 = m_2 = m_3$*.*

Proof. Consider a positive elliptic Lagrangian RE in $\mathbb{S}^2$ with bodies $m_1, m_2, m_3 > 0$. This means that the solution must have the form

$$\begin{aligned}
x_1 &= r_1 \cos\omega t, & y_1 &= r_1 \sin\omega t, & z_1 &= (1 - r_1^2)^{1/2}, \\
x_2 &= r_2 \cos(\omega t + a), & y_2 &= r_2 \sin(\omega t + a), & z_2 &= (1 - r_2^2)^{1/2}, \\
x_3 &= r_3 \cos(\omega t + b), & y_3 &= r_3 \sin(\omega t + b), & z_3 &= (1 - r_3^2)^{1/2},
\end{aligned}$$

with $b > a > 0$. In other words, we assume that this equilateral triangle forms a constant angle with the rotation axis, z, such that each body describes its own circle on $\mathbb{S}^2$. But for such a solution to exist, the total angular momentum must be either zero or is given by a constant vector parallel with the z axis. Otherwise this vector rotates around the z axis, in violation of the total angular-momentum integrals. This means that at least the first two components of the vector $\sum_{i=1}^{3} m_i\mathbf{q}_i \times \dot{\mathbf{q}}_i$ must be zero, where $\times$ represents the cross product. A straightforward computation shows this constraint to lead to the condition

$$m_1 r_1 z_1 \sin\omega t + m_2 r_2 z_2 \sin(\omega t + a) + m_3 r_3 z_3 \sin(\omega t + b) = 0,$$

assuming that $\omega \neq 0$. For $t = 0$, this equation becomes

$$m_2 r_2 z_2 \sin a = -m_3 r_3 z_3 \sin b. \tag{13.3}$$

Using now the fact that

$$\alpha := x_1x_2 + y_1y_2 + z_1z_2 = x_1x_3 + y_1y_3 + z_1z_3 = x_3x_2 + y_3y_2 + z_3z_2$$

is constant because the triangle is equilateral, the equation of the system of motion corresponding to $\ddot{y}_1$ takes the form

$$Kr_1(r_1^2 - 1)\omega^2 \sin\omega t = m_2 r_2 \sin(\omega t + a) + m_3 r_3 \sin(\omega t + b),$$

where K is a nonzero constant. For $t = 0$, this equation becomes

$$m_2 r_2 \sin a = -m_3 r_3 \sin b. \tag{13.4}$$

Dividing (13.3) by (13.4), we obtain that $z_2 = z_3$. Similarly, we can show that $z_1 = z_2 = z_3$, therefore the motion must take place in the same Euclidean plane on a circle orthogonal to the rotation axis. Proposition 7 then implies that $m_1 = m_2 = m_3$. $\square$

13.2 Negative Elliptic Lagrangian RE

Our next result is the analogue in $\mathbb{H}^2$ of Proposition 6.

Proposition 8. *Consider the curved* 3*-body problem with equal masses,* $m_1 = m_2 = m_3 =: m$*, in* $\mathbb{H}^2$*. Then for any* $m > 0$ *and* $z > 1$*, there are a positive and a negative* ω *that produce negative elliptic RE in which the bodies are at the vertices of an equilateral triangle that rotates in the plane* $z =$ *constant. Moreover, for every* $\omega^2/m > 0$ *there is a unique* $z > 1$ *that satisfies this property.*

Proof. Substituting in system (3.16) a solution of the form

$$x_i = \rho\cos(\omega t + \alpha_i), \quad y_i = \rho\sin(\omega t + \alpha_i), \quad z_i = z, \tag{13.5}$$

with $z = \sqrt{\rho^2 + 1}$, $\alpha_1 = 0, \alpha_2 = 2\pi/3, \alpha_3 = 4\pi/3$, we are led to the equation

$$\frac{8}{\sqrt{3}(3z^4 - 2z^2 - 1)^{3/2}} = \frac{\omega^2}{m}. \tag{13.6}$$

The left hand side is positive for $z > 1$, tends to infinity when $z \to 1$, and tends to zero when $z \to \infty$. So for any z in this interval and $m > 0$, there are a positive and a negative ω for which the above equation is satisfied. □

As we previously proved, an equilateral triangle rotating in its own plane forms a positive elliptic RE in $\mathbb{S}^2$ only if the 3 masses lying at its vertices are equal. The same result is true in $\mathbb{H}^2$, as we will further show.

Proposition 9. *In the curved* 3*-body problem in* $\mathbb{H}^2$*, if the bodies* $m_1, m_2, m_3 > 0$ *are initially at the vertices of an equilateral triangle in the plane* $z =$ *constant for some* $z > 1$*, then there are initial velocities that lead to a negative elliptic RE in which the triangle rotates in its own plane if and only if* $m_1 = m_2 = m_3$*.*

Proof. The implication which shows that if $m_1 = m_2 = m_3$, the rotating equilateral triangle is a negative elliptic RE, follows from Theorem 7. To prove the other implication, we substitute into equations (3.16) a solution of the form (13.5) with $i = 1, 2, 3,\ \rho := \rho_1, \rho_2, \rho_3,\ z := z_1 = z_2 = z_3 = (\rho^2 + 1)^{1/2}$, and $\alpha_1 = 0, \alpha_2 = 2\pi/3, \alpha_3 = 4\pi/3$. The computations then lead to the system

$$\begin{cases} m_1 + m_2 = \zeta\omega^2 \\ m_2 + m_3 = \zeta\omega^2 \\ m_3 + m_1 = \zeta\omega^2, \end{cases} \tag{13.7}$$

where $\zeta = \sqrt{3}(3z^4 - 2z^2 - 1)^{3/2}/4$. But for any $z =$ constant with $z > 1$, the above system has a solution only for $m_1 = m_2 = m_3 = \zeta\omega^2/2$. Therefore the masses must be equal. □

The following result perfectly resembles Theorem 8. The proof works the same way, by just replacing the circular trigonometric functions with hyperbolic ones and changing the signs to reflect the equations of motion in $\mathbb{H}^2$. This result finalizes our argument that space is Euclidean for distances comparable to those of our solar system.

Theorem 9. *For all negative elliptic Lagrangian RE of the curved* 3*-body problem in* $\mathbb{H}^2$*, the masses* $m_1, m_2, m_3 > 0$ *have to rotate on the same circle, whose plane must be orthogonal to the rotation axis, and therefore* $m_1 = m_2 = m_3$*.*

13.3 Positive Elliptic Eulerian RE

It is now natural to ask whether positive elliptic RE in which the bodies lie on a rotating geodesic exist in $\mathbb{S}^2$, since—as Theorem 5 shows—they cannot be generated from regular N-gons. The answer in the case $N = 3$ of equal masses is given by the following result.

Theorem 10. *Consider the curved* 3*-body problem in* $\mathbb{S}^2$ *with equal masses,* $m_1 = m_2 = m_3 =: m$*. Fix the body* m_1 *at* $(0, 0, 1)$ *and the bodies* m_2 *and* m_3 *at the opposite ends of a diameter on the circle* $z =$ *constant. Then, for any* $m > 0$ *and* $z \in (-0.5, 0) \cup (0, 1)$*, there are a positive and a negative* ω *that produce positive elliptic Eulerian RE.*

Proof. Substituting into the equations of motion (3.15) a solution of the form

$$x_1 = 0, \ y_1 = 0, \ z_1 = 1,$$

$$x_2 = r \cos \omega t, \ y_2 = r \sin \omega t, z_2 = z,$$

$$x_3 = r \cos(\omega t + \pi), \ y_3 = r \sin(\omega t + \pi), z_3 = z,$$

with $r \geq 0$ and z constants satisfying $r^2 + z^2 = 1$, leads either to identities or to the algebraic equation

$$\frac{4z + |z|^{-1}}{4z^2(1 - z^2)^{3/2}} = \frac{\omega^2}{m}. \tag{13.8}$$

The function on the left hand side is negative for $z \in (-1, -0.5)$, takes the value 0 at $z = -0.5$, is positive for $z \in (-0.5, 0) \cup (0, 1)$, and undefined at $z = 0$. Therefore, for every $m > 0$ and $z \in (-0.5, 0) \cup (0, 1)$, there are a positive and a negative ω that lead to an Eulerian RE. For $z = -0.5$, we recover the equilateral FP. The sign of ω determines the sense of rotation. $\square$

Remark 14. A qualitative argument shows that for every $\omega^2/m \in (0, 64\sqrt{15}/45)$, there are three values of z that satisfy relation (13.8): one in the interval $(-0.5, 0)$ and two in the interval $(0, 1)$ (see Figure 13.2).

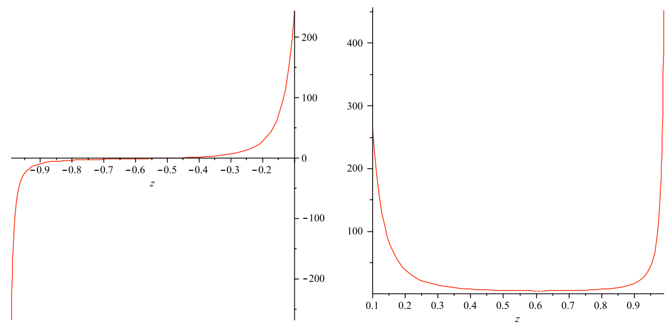

Figure 13.2: The graph of the function $f(z) = \frac{4z+|z|^{-1}}{4z^2(1-z^2)^{3/2}}$ in the intervals $(-1, 0)$ and $(0, 1)$, respectively.

Remark 15. If in Theorem 10 we take the masses $m_1 =: m$ and $m_2 = m_3 =: M$, the analogue of equation (13.8) is

$$\frac{4mz + M|z|^{-1}}{4z^2(1-z^2)^{3/2}} = \frac{\omega^2}{m}.$$

Then solutions exist for any $z \in (-\sqrt{M/m}/2, 0) \cup (0, 1)$. This means that there are no FP for $M \geq 4m$, a fact that agrees with Theorem 1 of [36], so positive elliptic Eulerian RE exist for such masses for all $z \in (-1, 0) \cup (0, 1)$.

13.4 Negative Elliptic Eulerian RE

We will further prove the $\mathbb{H}^2$ analogue of Theorem 10.

Theorem 11. *Consider the curved* 3*-body problem in* $\mathbb{H}^2$ *with masses* $m_1 = m_2 = m_3 =: m$*. Initially fix the body* m_1 *at the "north pole"* $(0, 0, 1)$ *and the bodies* m_2 *and* m_3 *at the opposite ends of a diameter on the circle* $z =$ *constant. Then, for any* $m > 0$ *and* $z > 1$*, there are a positive and a negative* ω*, which produce negative elliptic Eulerian RE that rotate around the* z *axis.*

Proof. Substituting into the equations of motion (3.16) a solution of the form

$$\begin{aligned}
x_1 &= 0, & y_1 &= 0, & z_1 &= 1,\\
x_2 &= \rho\cos\omega t, & y_2 &= \rho\sin\omega t, & z_2 &= z,\\
x_3 &= \rho\cos(\omega t + \pi), & y_3 &= \rho\sin(\omega t + \pi), & z_3 &= z,
\end{aligned}$$

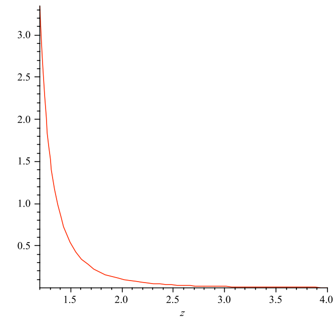

Figure 13.3: The graph of the function $f(z) = \frac{4z^2+1}{4z^3(z^2-1)^{3/2}}$ for $z > 1$.

where $\rho \geq 0$ and $z \geq 1$ are constants satisfying $z^2 = \rho^2 + 1$, leads either to identities or to the algebraic equation

$$\frac{4z^2+1}{4z^3(z^2-1)^{3/2}} = \frac{\omega^2}{m}. \tag{13.9}$$

The function on the left hand side is positive for $z > 1$. Therefore, for every $m > 0$ and $z > 1$, there are a positive and a negative ω that lead to a negative elliptic Eulerian RE. The sign of ω determines the sense of rotation. $\square$

Remark 16. A qualitative argument shows that for every $\omega^2/m > 0$, there is exactly one $z > 1$ that satisfies equation (13.9) (see Figure 13.3).

13.5 Negative Hyperbolic RE

In this section, we will prove a negative result concerning the existence of negative hyperbolic RE moving along geodesics of $\mathbb{H}^2$. More precisely, we will show that, unlike in $\mathbb{S}^2$, there are no orbits for which the bodies chase each other along a geodesic and maintain the same initial distances for all times.

Theorem 12. *Along any fixed geodesic, the curved N-body problem in $\mathbb{H}^2$ has no negative hyperbolic RE.*

Proof. Without loss of generality, we can restrict our proof to the geodesic $x = 0$. We will show that there are no $m_1, m_2, \ldots, m_N > 0$ such that system (3.16) has solutions of the form

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N), \ \mathbf{q}_i = (x_i, y_i, z_i), \tag{13.10}$$

$$x_i = 0, \quad y_i = \sinh(\omega t + \alpha_i), \quad z_i = \cosh(\omega t + \alpha_i), \quad \alpha_i \in \mathbb{R}, \quad i = 1, 2, \ldots, N.$$

After substitution, the equation corresponding to the y_i coordinate becomes

$$\sum_{j=1, j\neq i}^{n} \frac{m_j[\sinh(\omega t + \alpha_j) - \cosh(\alpha_i - \alpha_j)\sinh(\omega t + \alpha_i)]}{|\sinh(\alpha_i - \alpha_j)|^3} = 0. \tag{13.11}$$

Assume that $\alpha_i > \alpha_j$ for all $j \neq i$ and $\omega > 0$. Let $\alpha_{M(i)}$ be the maximum of all α_j with $j \neq i$. Then for $t \in (-\alpha_{M(i)}/\omega, -\alpha_i/\omega)$, we have that $\sinh(\alpha t + \alpha_j) < 0$ for all $j \neq i$ and $\sinh(\alpha t + \alpha_i) > 0$. Therefore the left hand side of equation (13.11) is negative in this interval, so the identity cannot take place for all $t \in \mathbb{R}$. It follows that a necessary condition to satisfy equation (13.11) is that $\alpha_{M(i)} \geq \alpha_i$. But this inequality must be verified for all $i = 1, 2, \ldots, N$, a fact that can be written as:

$$\begin{gathered}
\alpha_1 \geq \alpha_2 \ \text{ or } \ \alpha_1 \geq \alpha_3 \ \text{ or } \ \ldots \ \text{ or } \ \alpha_1 \geq \alpha_N, \\
\alpha_2 \geq \alpha_1 \ \text{ or } \ \alpha_2 \geq \alpha_3 \ \text{ or } \ \ldots \ \text{ or } \ \alpha_2 \geq \alpha_N, \\
\ldots \\
\alpha_N \geq \alpha_1 \ \text{ or } \ \alpha_N \geq \alpha_2 \ \text{ or } \ \ldots \ \text{ or } \ \alpha_N \geq \alpha_{N-1}.
\end{gathered}$$

The constants $\alpha_1, \alpha_2, \ldots, \alpha_N$ must satisfy one inequality from each of the above lines. But every possible choice implies the existence of at least one i and one j with $i \neq j$ and $\alpha_i = \alpha_j$. For those i and j, we have $\sinh(\alpha_i - \alpha_j) = 0$, so equation (13.11) is undefined, therefore equations (3.16) cannot have solutions of the form (13.10). Consequently, negative hyperbolic RE do not exist along the geodesic $x = 0$. □

13.6 Negative hyperbolic Eulerian RE

Theorem 12 raises the question whether negative hyperbolic RE do exist at all in $\mathbb{H}^2$ (their existence in $\mathbb{H}^3$ was proved in Chapter 11). For 3 equal masses, the answer is given by the following result, which shows that, in $\mathbb{H}^2$, 3 bodies can move along hyperbolas lying in parallel planes of $\mathbb{R}^3$, maintaining the initial distances among themselves and remaining on the same geodesic (which rotates hyperbolically). Such orbits resemble fighter planes flying in formation, rather than celestial bodies moving under the action of gravity alone.

Theorem 13. *In the curved 3-body problem of equal masses, $m_1 = m_2 = m_3 =: m$, in $\mathbb{H}^2$, for any given $m > 0$ and $x \neq 0$, there exist a positive and a negative ω that lead to negative hyperbolic Eulerian RE.*

Proof. We will show that $\mathbf{q}_i(t) = (x_i(t), y_i(t), z_i(t)), i = 1, 2, 3$, is a negative hyperbolic RE of system (3.16) for

$$\begin{aligned} x_1 &= 0, & y_1 &= \sinh \omega t, & z_1 &= \cosh \omega t, \\ x_2 &= x, & y_2 &= \rho \sinh \omega t, & z_2 &= \rho \cosh \omega t, \\ x_3 &= -x, & y_3 &= \rho \sinh \omega t, & z_3 &= \rho \cosh \omega t, \end{aligned}$$

where $\rho = (1 + x^2)^{1/2}$. Notice first that

$$x_1 x_2 + y_1 y_2 - z_1 z_2 = x_1 x_3 + y_1 y_3 - z_1 z_3 = -\rho,$$
$$x_2 x_3 + y_2 y_3 - z_2 z_3 = -2x^2 - 1,$$
$$\dot{x}_1^2 + \dot{y}_1^2 - \dot{z}_1^2 = \omega^2, \quad \dot{x}_2^2 + \dot{y}_2^2 - \dot{z}_2^2 = \dot{x}_3^2 + \dot{y}_3^2 - \dot{z}_3^2 = \rho^2 \omega^2.$$

Substituting the above coordinates and expressions into equations (3.16), we are led either to identities or to the equation

$$\frac{4x^2 + 5}{4x^2|x|(x^2 + 1)^{3/2}} = \frac{\omega^2}{m}, \tag{13.12}$$

from which the statement of the theorem follows. □

Remark 17. The left hand side of equation (13.12) is undefined for $x = 0$, but it tends to infinity when $x \to 0$ and to 0 when $x \to \pm\infty$. This means that for each $\omega^2/m > 0$ there are exactly one positive and one negative x (equal in absolute value), which satisfy the equation.

Remark 18. Theorem 13 is also true if, say, $m_1 =: m > 0$ and $m_2 = m_3 =: M > 0$, with $m \neq M$. Then the analogue of equation (13.12) is

$$\frac{m}{x^2|x|(x^2 + 1)^{1/2}} + \frac{M}{4x^2|x|(x^2 + 1)^{3/2}} = \omega^2,$$

and it is obvious that for any $m, M > 0$ and $x \neq 0$, there are a positive and negative ω satisfying the above equation.

Remark 19. Theorem 13 also works for 2 bodies of equal masses, $m_1 = m_2 =: m > 0$, of coordinates

$$x_1 = -x_2 = x, \quad y_1 = y_2 = \rho \sinh \omega t, \quad z_1 = z_2 = \rho \cosh \omega t,$$

where x is a positive constant and $\rho = (x^2 + 1)^{3/2}$. Then the analogue of equation (13.12) is

$$\frac{1}{4x^2|x|(x^2 + 1)^{3/2}} = \frac{\omega^2}{m},$$

which obviously supports a statement similar to the one in Theorem 13.

Chapter 14

Saari's conjecture

In 1970, Don Saari conjectured that solutions of the classical N-body problem with constant moment of inertia are relative equilibria, [85], [86]. This statement is surprising since one does not expect that such a weak constraint would force the bodies to maintain constant mutual distances all along the motion. Perhaps this is also the reason why the conjecture led to several wrong attempts at proving it, some of which were even published, [79], [80].

The case $N = 3$ was finally solved by Rick Moeckel in 2005, [76], and the collinear case, when all the bodies are on a rotating line, for any potential that depends only on the mutual distances, was settled in [34] in 2006. The problem is open in general. A homographic version of the conjecture was also stated, part of which was solved in the case $N = 3$, [31]. A complete proof for $N = 3$ in the case of equal masses was recently announced, [47].

14.1 Extension of Saari's conjecture to $\mathbb{S}^2$ and $\mathbb{H}^2$

The moment of inertia is defined in classical Newtonian celestial mechanics as

$$\frac{1}{2}\sum_{i=1}^{N} m_i \mathbf{q}_i \cdot \mathbf{q}_i,$$

a function that gives a crude measure of the bodies' distribution in space. But this definition makes little sense in $\mathbb{S}^2$ and $\mathbb{H}^2$ because $\sigma \mathbf{q}_i \odot \mathbf{q}_i = 1$ for every $i = 1, 2, \ldots, N$. To avoid this problem, we adopt the standard point of view used in physics, and define the moment of inertia in $\mathbb{S}^2$ and $\mathbb{H}^2$ about the direction of the angular momentum. But while fixing an axis in $\mathbb{S}^2$ does not restrain generality, the symmetries of $\mathbb{H}^2$ makes us distinguish between two cases.

Indeed, in $\mathbb{S}^2$ we can assume that the rotation takes place around the z axis, and thus define the moment of inertia as

$$\mathbf{I} := \sum_{i=1}^{N} m_i(x_i^2 + y_i^2). \tag{14.1}$$

In $\mathbb{H}^2$, all possibilities can be reduced via suitable isometric transformations to:

(i) the symmetry about the z axis, when the moment of inertia takes the same form (14.1), and

(ii) the symmetry about the x axis, which corresponds to hyperbolic rotations, when—in agreement with the definition of the Lorentz product—we define the moment of inertia as

$$\mathbf{J} := \sum_{i=1}^{N} m_i(y_i^2 - z_i^2). \tag{14.2}$$

The case of the negative parabolic rotations will not be considered because there are no such RE.

These definitions allow us to formulate the following conjecture:

Saari's conjecture in $\mathbb{S}^2$ and $\mathbb{H}^2$. *For the curved N-body problem in $\mathbb{S}^2$ and $\mathbb{H}^2$, every solution that has a constant moment of inertia about the direction of the angular momentum is either an elliptic relative equilibrium, in $\mathbb{S}^2$ or $\mathbb{H}^2$, or a hyperbolic relative equilibrium in $\mathbb{H}^2$.*

14.2 The proof in the geodesic case

By generalizing an idea we used in the Euclidean case, we can now settle this conjecture when the bodies undergo another constraint. More precisely, we will prove the following result.

Theorem 14. *For the curved N-body problem in $\mathbb{S}^2$ and $\mathbb{H}^2$, every solution with constant moment of inertia about the direction of the angular momentum for which the bodies remain aligned along a geodesic that rotates elliptically in $\mathbb{S}^2$ or $\mathbb{H}^2$, or hyperbolically in $\mathbb{H}^2$, is either an elliptic relative equilibrium, in $\mathbb{S}^2$ or $\mathbb{H}^2$, or a hyperbolic relative equilibrium in $\mathbb{H}^2$.*

Proof. Let us first prove the case in which $\mathbf{I}$ is constant in $\mathbb{S}^2$ and $\mathbb{H}^2$, i.e. when the geodesic rotates elliptically. According to the above definition of $\mathbf{I}$, we can assume without loss of generality that the geodesic passes through the point $(0, 0, 1)$ and rotates about the z-axis with angular velocity $\omega(t) \neq 0$. The angular momentum of each body is $\mathbf{L}_i = m_i\mathbf{q}_i \otimes \dot{\mathbf{q}}_i$, so its derivative with respect to t takes the form

$$\dot{\mathbf{L}}_i = m_i\dot{\mathbf{q}}_i \otimes \dot{\mathbf{q}}_i + m_i\mathbf{q}_i \otimes \ddot{\mathbf{q}}_i = m_i\mathbf{q}_i \otimes \widetilde{\nabla}_{\mathbf{q}_i}U(\mathbf{q}) - m_i\dot{\mathbf{q}}_i^2\mathbf{q}_i \otimes \mathbf{q}_i =$$

$$m_i\mathbf{q}_i \otimes \widetilde{\nabla}_{\mathbf{q}_i}U(\mathbf{q}),$$

with $\kappa = 1$ in $\mathbb{S}^2$ and $\kappa = -1$ in $\mathbb{H}^2$. Here $\otimes$ is a general notation for the cross product, which means the standard cross product in $\mathbb{R}^3$, for positive curvature, and the cross product of the Minkowski space $\mathbb{R}^{2,1}$ (i.e. the third component having the opposite sign of the standard cross product's third component), for negative curvature. Since $\mathbf{q}_i \odot \widetilde{\nabla}_{\mathbf{q}_i}U(\mathbf{q}) = 0$, it follows that $\widetilde{\nabla}_{\mathbf{q}_i}U_\kappa(\mathbf{q})$ is either zero or orthogonal to $\mathbf{q}_i$. (Recall that orthogonality here is meant in terms of the standard inner product because, both in $\mathbb{S}^2$ and $\mathbb{H}^2$, $\mathbf{q}_i \odot \widetilde{\nabla}_{\mathbf{q}_i}U(\mathbf{q}) = \mathbf{q}_i \cdot \nabla_{\mathbf{q}_i}U(\mathbf{q})$.) If $\widetilde{\nabla}_{\mathbf{q}_i}U(\mathbf{q}) = \mathbf{0}$, then $\dot{\mathbf{L}}_i = \mathbf{0}$, so $\dot{L}_i^z = 0$.

Assume now that $\widetilde{\nabla}_{\mathbf{q}_i}U(\mathbf{q})$ is orthogonal to $\mathbf{q}_i$. Since all the particles are on a geodesic, their corresponding position vectors are in the same plane, therefore any linear combination of them is in this plane, so $\widetilde{\nabla}_{\mathbf{q}_i}U(\mathbf{q})$ is in the same plane. Thus $\widetilde{\nabla}_{\mathbf{q}_i}U(\mathbf{q})$ and $\mathbf{q}_i$ are in a plane orthogonal to the xy plane. It follows that $\dot{\mathbf{L}}_i$ is parallel to the xy plane and orthogonal to the z axis. Thus the z component, $\dot{L}_i^z$, of $\dot{\mathbf{L}}_i$ is 0, the same conclusion we obtained in the case $\widetilde{\nabla}_{\mathbf{q}_i}U(\mathbf{q}) = \mathbf{0}$. Consequently, $L_i^z = c_i$, where c_i is a constant.

Let us also remark that since the angular momentum and angular velocity vectors are parallel to the z axis, $L_i^z = \mathbf{I}_i\omega(t)$, where $\mathbf{I}_i = m_i(x_i^2+y_i^2)$ is the moment of inertia of the body m_i about the z-axis. Since the total moment of inertia, $\mathbf{I}$, is constant, and $\omega(t)$ is the same for all bodies because they belong to the same rotating geodesic, it follows that

$$\sum_{i=1}^N \mathbf{I}_i\omega(t) = \mathbf{I}\omega(t) = c,$$

where c is a constant. Consequently, ω is a constant vector.

Moreover, since $L_i^z = c_i$, it follows that $\mathbf{I}_i\omega(t) = c_i$. Then every $\mathbf{I}_i$ is constant, and so is every z_i, $i = 1, 2, \dots, N$. Hence each body of mass m_i has a constant z_i-coordinate, and all bodies rotate with the same constant angular velocity around the z-axis, properties that agree with our definition of an elliptic RE.

We now prove the case $\mathbf{J} =$ constant, i.e. when the geodesic rotates hyperbolically in $\mathbb{H}^2$. According to the definition of $\mathbf{J}$, we can assume that the bodies are on a moving geodesic whose plane contains the x axis for all time and whose vertex slides along the geodesic hyperbola $x = 0$. (This moving geodesic hyperbola can be also visualized as the intersection between the sheet $z > 0$ of the hyperbolic sphere and the plane containing the x axis and rotating about it. For an instant, this plane also contains the z axis.)

The angular momentum of each body is $\mathbf{L}_i = m_i \mathbf{q}_i \boxtimes \dot{\mathbf{q}}_i$, where $\boxtimes$ denotes the cross product in the Minkowski space $\mathbb{R}^{2,1}$, so we can show as before that its derivative takes the form $\dot{\mathbf{L}}_i = m_i \mathbf{q}_i \boxtimes \overline{\nabla}_{\mathbf{q}_i} U(\mathbf{q})$, where $\overline{\nabla} = (\partial_x, \partial_y, -\partial_z)$. Again, $\overline{\nabla}_{\mathbf{q}_i} U(\mathbf{q})$ is either zero or orthogonal to $\mathbf{q}_i$. In the former case we can draw the same conclusion as earlier, that $\dot{\mathbf{L}}_i = \mathbf{0}$, so in particular $\dot{L}_i^x = 0$. In the latter case, $\mathbf{q}_i$ and $\overline{\nabla}_{\mathbf{q}_i} U(\mathbf{q})$ are in the plane of the moving hyperbola, so their cross product, $\mathbf{q}_i \boxtimes \overline{\nabla}_{\mathbf{q}_i} U(\mathbf{q})$, is orthogonal to the x axis, and therefore $\dot{L}_i^x = 0$. Thus $\dot{L}_i^x = 0$ in either case.

From here the proof proceeds as before by replacing $\mathbf{I}$ with $\mathbf{J}$ and the z axis with the x axis, and noticing that $L_i^x = \mathbf{J}_i \omega(t)$, to show that every m_i has a constant x_i coordinate. In other words, each body is moving along a (in general non-geodesic) hyperbola given by the intersection of the hyperboloid with a plane orthogonal to the x axis. These facts, in combination with the sliding of the moving geodesic hyperbola along the fixed geodesic hyperbola $x = 0$, are in agreement with our definition of a hyperbolic relative equilibrium. □

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Bibliography

[1] R. Abraham and J.E. Marsden, *Foundations of Mechanics*, 2nd ed., Benjamin-Cummings, Reading MA, 1978.

[2] A. Albouy and A. Chenciner, Le problème des N corps et les distances mutuelles, *Inventiones math.* **131** (1998), 151-184.

[3] A. Albouy and V. Kaloshin, Finiteness of central configurations for five bodies in the plane, *Ann. of Math.* **176** (2012), 535-588.

[4] A. Baker, *Matrix Groups: An Introduction to Lie Group Theory*, Springer Verlag, London, 2002.

[5] K. Becker, M. Becker, and J.H. Schwarz, *String Theory and M-Theory: A Modern Introduction*, Cambridge University Press, 2007.

[6] P. de Bernardis et. al., A flat Universe from high-resolution maps of the cosmic microwave background radiation, *Nature* **404**, 6781 (2000), 955-959.

[7] J. Bertrand, Théorème relatif au mouvement d'un point attiré vers un center fixe, C. R. Acad. Sci. **77** (1873), 849-853.

[8] W. Bolyai and J. Bolyai, *Geometrische Untersuchungen*, Hrsg. P. Stäckel, Teubner, Leipzig-Berlin, 1913.

[9] R. Bonola, *Non-Euclidean Geometry*, Dover, New York, 1955.

[10] H. Bruns, Über die Integrale des Vielkörper-Problems, *Acta Math.* **11** (1887), 25-96.

[11] D.M. Burton, *The History of Mathematics, an Introduction*, Allyn and Bacon, Inc., Newton, Massachusetts, 1985.

[12] H. Cabral and F. Diacu, *Classical and Celestial Mechanics: The Recife Lectures*, Princeton University Press, Princeton, N.J., 2002.

[13] J.F. Cariñena, M.F. Rañada, and M. Santander, Central potentials on spaces of constant curvature: The Kepler problem on the two-dimensional sphere $\mathbb{S}^2$ and the hyperbolic plane $\mathbb{H}^2$, *J. Math. Phys.* **46** (2005), 052702.

[14] J.F. Cariñena, M.F. Rañada, and M. Santander, The Kepler problem and the Laplace-Runge-Lenz vector on spaces of constant curvature and arbitrary signature, *Qual. Th. Dyn. Syst.* **7** (2008), 87-99.

[15] J. Chazy, *La théorie de la relativité et la mécanique céleste*, Gauthier-Villars, Paris, 1930.

[16] A. Chenciner and R. Montgomery, A remarkable periodic solution of the three-body problem in the case of equal masses, *Ann. of Math.* **152** (2000), 881-901.

[17] A. Chenciner and H. Jiménez-Pérez, Angular momentum and Horn's problem, arXiv:1110.5030v2.

[18] H.S.M. Coxeter, *Regular polytopes*, Methuen & Co., London, 1948.

[19] H.S.M. Coxeter, *Introduction to Geometry*, 2nd edition, John Wiley & Sons, New York, 1969.

[20] T. Damour, M. Soffel, and C. Xu, General-relativistic celestial mechanics. I. Method and definition of reference systems, *Phys. Rev. D* **43**, 10 (1991), 3273-3307.

[21] T. Damour, M. Soffel, and C. Xu, General-relativistic celestial mechanics. II. Translational equations of motion, *Phys. Rev. D* **45**, 4 (1992), 1017-1044.

[22] T. Damour, M. Soffel, and C. Xu, General-relativistic celestial mechanics. III. Rotational equations of motion, *Phys. Rev. D* **47**, 8 (1993), 3124-3135.

[23] T. Damour, M. Soffel, and C. Xu, General-relativistic celestial mechanics. IV. Theory of satellite motion, *Phys. Rev. D* **49**, 2 (1994), 618-635.

[24] T. Dénes, Real face of János Bolyai, *Notices Amer. Math. Soc.* **58**, 1 (2011), 41-51.

[25] F. Diacu, Near-collision dynamics for particle systems with quasihomogeneous potentials, *J. Differential Equations* **128** (1996), 58-77.

[26] F. Diacu, Orbital anomalies, *Math. Intelligencer* **31**, 2 (2009), 45-49.

[27] F. Diacu, On the singularities of the curved N-body problem, *Trans. Amer. Math. Soc.* **363**, 4 (2011), 2249-2264.

[28] F. Diacu, Polygonal homographic orbits of the curved 3-body problem, *Trans. Amer. Math. Soc.* **364** (2012), 2783-2802.

[29] F. Diacu, Relative equilibria of the 3-dimensional curved n-body problem, arXiv:1108.1229, 102 pages.

[30] F. Diacu, The non-existence of the center-of-mass and the linear-momentum integrals in the curved N-body problem, arXiv:1202.4739, 12 pages.

[31] F. Diacu, T. Fujiwara, E. Pérez-Chavela, and M. Santoprete, Saari's homographic conjecture of the 3-body problem, *Trans. Amer. Math. Soc.* **360**, 12 (2008), 6447-6473.

[32] F. Diacu, R. Martínez, E. Pérez-Chavela, and C. Simó, On the stability of tetrahedral relative equilibria in the positively curved 4-body problem, in preparation.

[33] F. Diacu and E. Pérez-Chavela, Homographic solutions of the curved 3-body problem, *J. Differential Equations* **250** (2011), 340-366.

[34] F. Diacu, E. Pérez-Chavela, and M. Santoprete, Saari's conjecture for the collinear N-body problem, *Trans. Amer. Math. Soc.* **357**, 10 (2005), 4215-4223.

[35] F. Diacu, E. Pérez-Chavela, and M. Santoprete, The N-body problem in spaces of constant curvature. Part I: Relative equilibria, *J. Nonlinear Sci.* **22**, 2 (2012), 247-266, DOI: 10.1007/s00332-011-9116-z.

[36] F. Diacu, E. Pérez-Chavela, and M. Santoprete, The N-body problem in spaces of constant curvature. Part II: Singularities, *J. Nonlinear Sci.* **22**, 2 (2012), 267-275, DOI: 10.1007/s00332-011-9117-y.

[37] F. Diacu, E. Pérez-Chavela, and J. Guadalupe Reyes Victoria, An intrinsic approach in the curved N-body problem. The negative curvature case, *J. Differential Equations* **252** (2012), 4529-4562.

[38] F. Dillen and W. Kühnel, Ruled Weingarten surfaces in Minkowski 3-space, *Manuscripta Math.* **98** (1999), 307-320.

[39] C. Doran and A. Lasenby, Geometric Algebra for Physicists, Cambridge Univ. Press, 2003.

[40] A. Eddington and G.L. Clark, The problem of n bodies in general relativity theory, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **166** (1938), 465-475.

[41] A. Einstein, *The Meaning of Relativity*, 5th edition, Princeton University Press, Princeton, N.J., 2005.

[42] A. Einstein, L. Infeld, and B. Hoffmann, The gravitational equations and the problem of motion, *Ann. of Math.* **39**, 1 (1938), 65-100.

[43] B. Elmabsout, Sur l'existence de certain configurations d'équilibre relatif dans le problème des N corps, *Celest. Mech. Dyn. Astr.* **41** (1988), 131-151.

[44] L. Euler, *Institutiones calculi differentialis cum eius usu in analysi finitorum ac doctrina serierum*, published as a book in 1755, *Opera Omnia*, Ser. 1, vol. 10, (ed. G. Kowalewski), Birkhäuser, 1980.

[45] R. L. Faber, *Foundations of Euclidean and Non-Euclidean Geometry*, Marcel Dekker, New York, 1983.

[46] V. A. Fock, Sur le mouvement des masses finie d'après la théorie de gravitation einsteinienne, *J. Phys. Acad. Sci. USSR* **1** (1939), 81-116.

[47] T. Fujiwara, H. Fukuda, H. Ozaki, and T. Taniguchi, Saari's homographic conjecture for the planar equal-mass three-body problem in Newton gravity, arXiv:1202.0893, 11 pages.

[48] I.M. Gelfand and S.V. Fomin, *Calculus of Variations*, Prentice-Hall, Englewood Cliffs, New Jersey, 1963.

[49] G. Goe, Comments on Miller's "The myth of Gauss's experiment on the Euclidean nature of physical space," *Isis* **65**, 1 (1974), 83-87.

[50] L. García Gutiérrez and M. Santander, Levi-Civita regularization and geodesic flows for the 'curved' Kepler problem, arXiv:0707.3810v2, 2007.

[51] G.B. Halsted, Gauss and non-Euclidean geometry, *Amer. Math. Monthly* **7**, 11 (1900), 247-252.

[52] J. Hano and K. Nomizu, On isometric immersions of the hyperbolic plane into the Lorentz-Minkowski space and the Monge-Ampère equation of certain type, *Math. Ann.* **262** (1983), 245-253.

[53] H. Hopf, Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche, *Math. Ann.* **104** (1931), 637-665.

[54] L. Infeld, On a new treatment of some eigenvalue problems, *Phys. Rev.* **59** (1941), 737-747.

[55] L. Infeld, *Quest—The evolution of a scientist*, Readers Union Ltd., London, 1942.

[56] L. Infeld and A. Schild, A note on the Kepler problem in a space of constant negative curvature, *Phys. Rev.* **67** (1945), 121-122.

[57] J. Jost, *Riemannian Geometry and Geometric Analysis*, Springer Verlag, 2005.

[58] W. Killing, Die Rechnung in den nichteuklidischen Raumformen, *J. Reine Angew. Math.* **89** (1880), 265-287.

[59] W. Killing, Die Mechanik in den nichteuklidischen Raumformen, *J. Reine Angew. Math.* **98** (1885), 1-48.

[60] W. Killing, *Die Nicht-Euklidischen Raumformen in Analytischer Behandlung*, Teubner, Leipzig, 1885.

[61] V. V. Kozlov and A. O. Harin, Kepler's problem in constant curvature spaces, *Celestial Mech. Dynam. Astronom* **54** (1992), 393-399.

[62] T. Levi-Civita, The relativistic problem of several bodies, *Amer. J. Math.* **59**, 1 (1937), 9-22.

[63] T. Levi-Civita, *Le problème des n corps en relativité générale*, Gauthier-Villars, Paris, 1950; or the English translation: *The N-body problem in general relativity*, D. Reidel, Dordrecht, 1964.

[64] D. Lewis and T. Ratiu, Rotating n-gon/kn-gon vortex configurations, *J. Nonlinear Sci.* **6** (1996), 285-414.

[65] H. Liebmann, Die Kegelschnitte und die Planetenbewegung im nichteuklidischen Raum, *Berichte Königl. Sächsischen Gesell. Wiss., Math. Phys. Klasse* **54** (1902), 393-423.

[66] H. Liebmann, Über die Zentralbewegung in der nichteuklidische Geometrie, *Berichte Königl. Sächsischen Gesell. Wiss., Math. Phys. Klasse* **55** (1903), 146-153.

[67] H. Liebmann, *Nichteuklidische Geometrie*, G. J. Göschen, Leipzig, 1905; 2nd ed. 1912; 3rd ed. Walter de Gruyter, Berlin, Leipzig, 1923.

[68] R. Lipschitz, Untersuchung eines Problems der Variationsrechnung, in welchem das Problem der Mechanik enthalten ist, *J. Reine Angew. Math.* **74** (1872), 116-149.

[69] R. Lipschitz, Extension of the planet-problem to a space of n dimensions and constant integral curvature, *Quart. J. Pure Appl. Math.* **12** (1873), 349-370.

[70] N. I. Lobachevsky, The new foundations of geometry with full theory of parallels [in Russian], 1835-1838, In Collected Works, V. 2, GITTL, Moscow, 1949, p. 159.

[71] D.W. Lyons, An elementary introduction to the Hopf fibration, Math. Mag. **76**, 2 (2003), 87-98.

[72] J.E. Marsden and T.S. Ratiu, Introduction to Mechanics and Symmetry, (2nd ed.) Springer Verlag, 1999.

[73] R. Martínez and C. Simó, On the stability of the Lagrangian homographic solutions in a curved three-body problem on $\mathbb{S}^2$, *Discrete Contin. Dyn. Syst. Ser. B* (to appear).

[74] R. McGehee, Triple collision in the collinear three-body problem, *Inventiones math.* **27** (1974), 191-227.

[75] A. I. Miller, The myth of Gauss's experiment on the Euclidean nature of physical space, *Isis* **63**, 3 (1972), 345-348.

[76] R. Moeckel, A computer-assisted proof of Saari's conjecture for the planar three-body problem, *Trans. Amer. Math. Soc.* **357**, (2005) 3105-3117.

[77] R. Mosseri and R. Dandoloff, Geometry of entangled states, Bloch spheres and Hopf fibrations, *J. Phys. A* **34** (2001), 10243-10252.

[78] M. Nakahara, *Geometry, Topology and Physics*, Institute of Physics Publishing, Philadelphia, 1990.

[79] J.I. Palmore, Relative equilibria and the virial theorem, *Celestial Mech.* **19** (1979), 167-171.

[80] J.I. Palmore, Saari's conjecture revisited, *Celestial Mech.* **25** (1981), 79-80.

[81] E. Pérez-Chavela and J.G. Reyes Victoria, An intrinsic approach in the curved N-body problem. The positive curvature case, *Trans. Amer. Math. Soc.* **364**, 7 (2012), 3805-3827.

[82] L. Perko and E. Walter, Regular polygon solutions of the N-body problem, *Proc. Amer. Math. Soc.* **94** (1985), 301-309.

[83] H. Poincaré, *The Value of Science: Essential Writings of Henri Poincaré*, The Modern Library, New York, 2001.

[84] W.F. Reynolds, Hyperbolic geometry on a hyperboloid, *Amer. Math. Monthly* **100**, 5 (1993), 442-455.

[85] D. Saari, On bounded solutions of the n-body problem, in *Periodic Orbits, Stability and Resonances*, G.E.O. Gicaglia, ed., pp. 76-81, D. Reidel, Dordrecht, 1970.

[86] D. Saari, *Collisions, Rings, and Other Newtonian N-body Problems*, American Mathematical Society, Regional Conference Series in Mathematics, No. 104, Providence, RI, 2005.

[87] E. Schering, Die Schwerkraft im Gaussischen Raume, *Nachr. Königl. Gesell. Wiss. Göttingen* 13 July, **15** (1870), 311-321.

[88] E. Schering, Die Schwerkraft in mehrfach ausgedehnten Gaussischen und riemannschen Räumen, *Nachr. Königl. Gesell. Wiss. Göttingen* 26 Febr., **6** (1873), 149-159.

[89] E. Schrödinger, A method for determining quantum-mechanical eigenvalues and eigenfunctions, *Proc. Royal Irish Acad., Sect. A* **46** (1940), 9-16.

[90] P.J. Serret, *Théorie nouvelle géométrique et mécanique des lignes a double courbure*, Mallet-Bachelier, Paris, 1860.

[91] S. Sgries, Erinnerung an "dunkle Seiten," Gedenkkolloquium für den jüdischen Mathematiker Heinrich Liebmann, *Heidelberger Nachrichten*, June 19, 2008.

[92] A.V. Shchepetilov, Comment on "Central potentials on spaces of constant curvature: The Kepler problem on the two-dimensional sphere $\mathbb{S}^2$ and the hyperbolic plane $\mathbb{H}^2$," [J. Math. Phys. **46** (2005), 052702], *J. Math. Phys.* **46** (2005), 114101.

[93] A.V. Shchepetilov, Reduction of the two-body problem with central interaction on simply connected spaces of constant sectional curvature, *J. Phys. A: Math. Gen.* **31** (1998), 6279-6291.

[94] A.V. Shchepetilov, Nonintegrability of the two-body problem in constant curvature spaces, *J. Phys. A: Math. Gen.* V. 39 (2006), 5787-5806; corrected version at math.DS/0601382.

[95] A.V. Shchepetilov, *Calculus and mechanics on two-point homogenous Riemannian spaces*, Lecture notes in physics, vol. 707, Springer Verlag, 2006.

[96] S. Smale, Mathematical problems for the next century, *Mathematics: frontiers and perspectives*, Amer. Math. Society, Providence, RI, pp. 271-294.

[97] A. F. Stevenson, Note on the Kepler problem in a spherical space, and the factorization method for solving eigenvalue problems, *Phys. Rev.* **59** (1941), 842-843.

[98] J.R. Weeks, *The Shape of Space*, Marcel Dekker, New York, 2002.

[99] H. Weyl, *Space, Time, Matter*, 4th ed., Methuen & comp., London, 1952.

[100] A. Wintner, *The Analytical Foundations of Celestial Mechanics*, Princeton University Press, Princeton, N.J., 1941.

[101] H.E. Wolfe, *Introduction to Non-Euclidean Geometry*, Holt, Rineheart and Winston, New York, 1945.

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