

Stability of fixed points and associated relative equilibria of the 3-body problem on \mathbb{S}^1 and \mathbb{S}^2

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Abstract

We prove that the fixed points of the curved 3-body problem and their associated relative equilibria are Lyapunov stable if the solutions are restricted to \mathbb{S}^1 , but their linear stability depends on the angular velocity if the bodies are considered on \mathbb{S}^2 . More precisely, the associated relative equilibria are linearly stable if and only if the angular velocity is greater than a certain critical value.

Key words: celestial mechanics; curved 3-body problem; fixed-point solutions; relative equilibria; stability.

1 Introduction

The curved 3-body problem is a natural extension of the classical 3-body problem to spaces of constant nonzero Gaussian curvature. Its history, which started with Bolyai and Lobachevsky, is outlined in [1], where it is also shown that, in the 2-dimensional case, the study of the problem can be reduced to the unit sphere, \mathbb{S}^2 , and the unit hyperbolic sphere, \mathbb{H}^2 . The equations of motion can be written as a Hamiltonian system in extrinsic coordinates with holonomic constraints in the Euclidean space, for positive curvature, but in the Minkowski space, for negative curvature. This formulation led to fruitful results, especially in the study of symmetric motions, [1, 2, 3, 4, 5, 6, 7, 8, 10, 17].

One class of solutions are the fixed points, which occur only on spheres, [1, 2]. We study here their stability as well as that of their associated relative equilibria for the 3-body problem on \mathbb{S}^1 and \mathbb{S}^2 . Fixed points are critical points of the force function that defines the system on the configuration space. For the bodies m_1, m_2, m_3 , they give rise to periodic relative equilibria, which in spherical coordinates (φ, θ) have the form

$$\theta_i(t) = \theta_i, \quad \varphi_i(t) = \varphi_i + \omega t, \quad \dot{\theta}_i = 0, \quad \dot{\varphi}_i = \omega, \quad i \in \{1, 2, 3\},$$

for any $\omega \neq 0$. There have been some previous studies of the stability of orbits in the curved 3- and 4-body problem, [7, 10], but none of them considered fixed points, so this appears to be a first attempt in this direction.

For fixed points of the 3-body problem on \mathbb{S}^2 , the masses must lie on a great circle of \mathbb{S}^2 at the vertices of an acute triangle, [1]. In Section 2, we prove a general criterion for the existence of fixed points for $n > 2$ masses. It is known that not all masses can form fixed points, [4], and in Section 3 we find all mass triples that do so. In Section 4, we investigate the stability of fixed points and of their associated relative equilibria on \mathbb{S}^1 . Since the associated relative equilibria are periodic orbits, their stability is defined as the stability of the corresponding rest points of the flow in the reduced phase space, [15]. To perform the reduction, we use coordinates similar to the Jacobi coordinates of the classical n -body problem. By showing that the corresponding rest points are maxima of the force function, we prove that the fixed points and the associated relative equilibria are Lyapunov stable on \mathbb{S}^1 . In Section 5, we study the linear stability of these solutions on \mathbb{S}^2 . Their linear stability is defined as the stability of the corresponding rest points of the linearized system restricted to proper linear subspaces. Using an idea from the work of Rick Moeckel, [12], we determine proper linear subspaces to study the fixed-point solutions and the relative equilibria. Then we show that their linear stability depends on the angular velocity ω . More precisely, the associated relative equilibria are linearly stable if and only if the angular velocity is greater than a certain value determined by the configuration.

2 Existence of fixed points on \mathbb{S}^1

The goal of this section is to introduce the curved n -body problem on \mathbb{S}^2 , define a class of orbits we call fixed points, and provide a criterion for the existence of these solutions for any number $n > 2$ of point masses.

A Hamiltonian system is given by the triple (H, M, w) , where M is an even dimensional manifold, w is a symplectic form on M , and H is an infinitely differentiable, real-valued function on M . The curved n -body problem on \mathbb{S}^2 , $n \geq 2$, is given by a Hamiltonian system (H, M, w) as follows. Let m_1, \dots, m_n be a system of n positive point masses (bodies), whose configuration is given by the vector

$$\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_n) \in (\mathbb{S}^2)^n,$$

with $\mathbf{q}_i = (x_i, y_i, z_i) = (\sin \theta_i \cos \varphi_i, \sin \theta_i \sin \varphi_i, \cos \theta_i)$ in spherical coordinates (φ_i, θ_i) , $i \in \{1, \dots, n\}$. We denote by d_{ij} the geodesic distance between the point masses m_i and m_j , $i, j \in \{1, \dots, n\}$. The kinetic energy has the form

$$T(\varphi_1, \theta_1, \dots, \varphi_n, \theta_n) := \sum_{i=1}^n \frac{m_i}{2} \left(\dot{\theta}_i^2 + \sin^2 \theta_i \dot{\varphi}_i^2 \right),$$

and the conjugate momenta are given by

$$p_{\theta_i} = m_i \dot{\theta}_i, \quad p_{\varphi_i} = m_i \sin^2 \theta_i \dot{\varphi}_i, \quad i \in \{1, \dots, n\}.$$

The configuration space is $W = (\mathbb{S}^2)^n \setminus \Delta$, where Δ is the singularity set,

$$\Delta := \{\mathbf{q} \in (\mathbb{S}^2)^n : \exists(i, j) \text{ with } i \neq j \text{ such that } d_{ij} = 0 \text{ or } d_{ij} = \pi\}.$$

The cotangent potential on W , which extends the Newtonian potential to the sphere, is given by

$$U(\varphi_1, \theta_1, \dots, \varphi_n, \theta_n) := - \sum_{1 \leq i < j \leq n} m_i m_j \cot d_{ij}.$$

Let d denote the differential, $V = -U$ be the force function, and T^*W represent the cotangent bundle of W . Then the curved n -body problem on \mathbb{S}^2 is described by the Hamiltonian system

$$H = T + U = T - V, \quad M = T^*W, \quad w = d \left(\sum_i p_{\theta_i} d\theta_i + p_{\varphi_i} d\varphi_i \right). \quad (1)$$

Definition 1. A configuration $\mathbf{q} \in W = (\mathbb{S}^2)^n \setminus \Delta$ is called a fixed point if it is a critical point of the force function V .

The terminology is justified since, once placed at such an initial configuration in \mathbb{S}^2 with zero initial velocities, the bodies don't move, so we have a rest point of the system. Therefore a configuration of this kind can be also thought as a solution of (1) with zero momenta.

Since in this paper we are mainly interested in fixed points of the 3-body problem, we must note that the three bodies always lie on a geodesic \mathbb{S}^1 , as shown in [1]. So for our further purposes we will assume that all n bodies lie on \mathbb{S}^1 . Without loss of generality, we can take \mathbb{S}^1 to be the equator, so a fixed-point solution has the form

$$\theta_i(t) = \frac{\pi}{2}, \quad \varphi_i(t) = \varphi_i \text{ (constant)}, \quad p_{\theta_i} = p_{\varphi_i} = 0, \quad i \in \{1, \dots, n\}, \quad (2)$$

If we rotate a fixed point uniformly about the z -axis with any angular velocity $\omega \neq 0$, we obtain an associated relative equilibrium:

$$\theta_i(t) = \frac{\pi}{2}, \quad \varphi_i(t) = a_i + \omega t, \quad p_{\theta_i} = 0, \quad p_{\varphi_i} = m_i \omega, \quad i \in \{1, \dots, n\}, \quad (3)$$

where a_1, \dots, a_n are constants.

It has been known that fixed points cannot lie within one hemisphere, [1]. We briefly sketch the proof here. We use the Cartesian coordinates of \mathbb{S}^2 , $\mathbf{q}_i = (x_i, y_i, z_i)$. Using the fact that $x_i^2 + y_i^2 + z_i^2 = 1$ and $\cos d_{ij} = \mathbf{q}_i \cdot \mathbf{q}_j$, direct computation shows $\frac{\partial d_{ij}}{\partial z_i} = \frac{z_j - \cos d_{ij} z_i}{-\sin d_{ij}}$. Suppose that \mathbf{q} is a configuration with all particles on the north hemisphere and q_1 is the lowest, i.e., $z_i \geq z_1 \geq 0$ for all i , and that not all particles are on the equator. Then

$$\frac{\partial d_{1j}}{\partial z_1} = \frac{z_j - \cos d_{1j} z_1}{-\sin d_{1j}} \leq \frac{z_1(1 - \cos d_{1j})}{-\sin d_{1j}} < 0.$$

Then $\frac{\partial V}{\partial z_1} = \sum_{j=1, j \neq i}^n \frac{-m_i m_j}{\sin^2 d_{1j}} \frac{\partial d_{1j}}{\partial z_1} > 0$. Hence \mathbf{q} could not be a critical point of V .

Let us now provide a general criterion for the existence of fixed points on \mathbb{S}^1 for any number $n > 2$ masses. Because of the $SO(2)$ -symmetry that acts on the bodies, we can assume without loss of generality that

$$0 = \varphi_1 < \varphi_2 < \dots < \varphi_n < 2\pi,$$

and $\varphi_{i+1} - \varphi_i < \pi$, $i \in \{1, \dots, n-1\}$ and $\varphi_n - \varphi_1 > \pi$.

Proposition 2. A configuration \mathbf{q} on $(\mathbb{S}^1)^n \setminus \Delta$ is a fixed point if and only if

$$\sum_{i=1, i \neq k}^N \frac{m_k m_i \sin(\varphi_k - \varphi_i)}{\sin^3 d_{ki}} = 0, \quad (4)$$

for each $k \in \{1, \dots, n\}$.

Proof. Fixed points are critical points of the force function $V = \sum_{1 \leq i < j \leq n} m_i m_j \cot d_{ij}$. Notice that $d_{ki} \in (0, \pi)$ could be either $|\varphi_k - \varphi_i|$ or $2\pi - |\varphi_k - \varphi_i|$. Then $\frac{\partial d_{ki}}{\partial \varphi_k}$ takes the values ± 1 . These facts are summarized in the following table.

$\varphi_k - \varphi_i$	d_{ki}	$\frac{\partial d_{ki}}{\partial \varphi_k}$
$(-2\pi, -\pi)$	$2\pi + \varphi_k - \varphi_i$	1
$(-\pi, 0)$	$\varphi_i - \varphi_k$	-1
$(0, \pi)$	$\varphi_k - \varphi_i$	1
$(\pi, 2\pi)$	$2\pi - \varphi_k + \varphi_i$	-1

Then

$$\frac{\partial d_{ki}}{\partial \varphi_k} = \text{sgn}(\sin(\varphi_k - \varphi_i)) = \frac{\sin(\varphi_k - \varphi_i)}{|\sin(\varphi_k - \varphi_i)|} = \frac{\sin(\varphi_k - \varphi_i)}{\sin d_{ki}}.$$

Taking the partial derivatives of V respect to φ_k , $k \in \{1, \dots, n\}$, we obtain that

$$0 = \sum_{i=1, i \neq k}^n \frac{m_k m_i}{\sin^2 d_{ki}} \frac{\partial d_{ki}}{\partial \varphi_k} = \sum_{i=1, i \neq k}^n \frac{m_k m_i \sin(\varphi_k - \varphi_i)}{\sin^3 d_{ki}}$$

for each $k \in \{1, \dots, n\}$, a remark that completes the proof. \square

3 Triangular fixed points

In this section we focus on the case $n = 3$. We set $\alpha = \varphi_2 - \varphi_1$, $\beta = \varphi_3 - \varphi_2$. Then, since $\varphi_{i+1} - \varphi_i < \pi$, we obtain

$$0 < \alpha < \pi, \quad 0 < \beta < \pi, \quad \pi < \alpha + \beta < 2\pi \quad (5)$$

and

$$d_{12} = \alpha, \quad d_{23} = \beta, \quad d_{13} = 2\pi - (\alpha + \beta).$$

From the point of view of Euclidean geometry, a fixed point for three masses forms a triangle in the xy -plane. We have $\angle \mathbf{q}_2 \mathbf{q}_1 \mathbf{q}_3 = \frac{d_{23}}{2} < \frac{\pi}{2}$, and, similarly, the two other angles are acute. Thus each fixed point must be an acute triangle (see Figure 1).

Equations (4) above can be written in the form

$$\frac{m_2}{\sin^2 \alpha} = \frac{m_3}{\sin^2(\alpha + \beta)}, \quad \frac{m_1}{\sin^2 \alpha} = \frac{m_3}{\sin^2 \beta}, \quad \frac{m_2}{\sin^2 \beta} = \frac{m_1}{\sin^2(\alpha + \beta)}. \quad (6)$$

With the help of these equations, we can now find all fixed points. It is known on one hand that, for any acute triangle configuration, there are mass triples (m_1, m_2, m_3) that generate fixed points, [4, 17]. On the other hand, not all mass triples can form

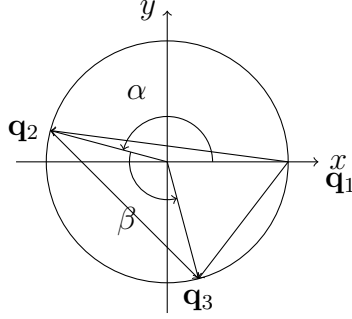


Figure 1: An acute triangle fixed-point configuration

fixed points. So for what mass triples, are there fixed points and how many such fixed points occur in each case? We count these configurations by fixing

$$\varphi_1 = 0 < \varphi_2 < \varphi_3 < 2\pi,$$

since any other configuration differs from this one by an isometry. It is easy to see that for any acute triangle there are infinitely many admissible mass triples,

$$(m_1, m_2, m_3) = m_3 \left(\frac{\sin^2 \alpha}{\sin^2 \beta}, \frac{\sin^2 \alpha}{\sin^2(\alpha + \beta)}, 1 \right). \quad (7)$$

Hence we normalize the masses by taking $m_1 + m_2 + m_3 = 1$. Then the admissible masses can be represented by $(m_1, m_2) \in (0, 1) \times (0, 1)$. Let us denote by \mathcal{U} the open region in \mathbb{R}^2 defined by (5), i.e., $0 < \alpha < \pi, 0 < \beta < \pi, \pi < \alpha + \beta < 2\pi$. Then (7) defines a mapping,

$$\begin{aligned} \chi : \mathcal{U} &\rightarrow (0, 1) \times (0, 1), \\ (\alpha, \beta) &\mapsto (m_1, m_2) = \frac{1}{\tau} \left(\frac{\sin^2 \alpha}{\sin^2 \beta}, \frac{\sin^2 \alpha}{\sin^2(\alpha + \beta)} \right), \end{aligned}$$

where $\tau = \frac{\sin^2 \alpha}{\sin^2 \beta} + \frac{\sin^2 \alpha}{\sin^2(\alpha + \beta)} + 1$.

Then we can formulate the question related to the admissible masses in more precise terms: What is the image of χ ? Is χ injective? Or, in other words, for what pairs (m_1, m_2) can we find an acute triangle fixed-point configuration? Is such configuration unique? The following result provides the answers to these questions.

Theorem 3. *Let*

$$\mathcal{V} := \{(m_1, m_2) \in (0, 1) \times (0, 1) : m_1^2 m_2^2 + m_1^2 m_3^2 + m_2^2 m_3^2 - 2m_1 m_2 m_3 < 0\},$$

where $m_3 = 1 - (m_1 + m_2)$ (see Figure 2). Then there exists an acute triangle fixed-point configuration only for mass triples $(m_1, m_2, m_3 = 1 - m_1 - m_2)$ such that $(m_1, m_2) \in \mathcal{V}$. Furthermore, such a configuration is unique.

Proof. Our approach is to find the inverse mapping $(\alpha, \beta) = \chi^{-1}(m_1, m_2)$. Then inequalities (5) would yield inequalities for (m_1, m_2) , which would give $\mathcal{V} := \chi(\mathcal{U})$, and since the inverse mapping exists, $\chi : \mathcal{U} \rightarrow \mathcal{V}$ must be bijective, so the configuration is unique.

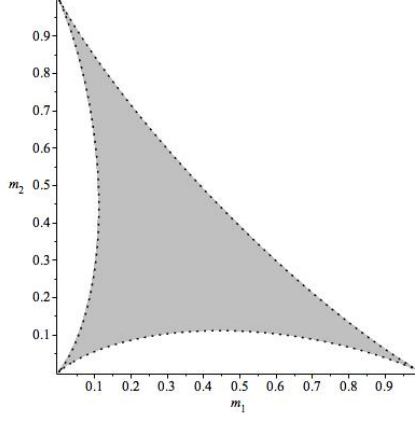


Figure 2: The region \mathcal{V}

By (6), we set $u = \sqrt{\frac{m_3}{m_1}} = \frac{\sin \beta}{\sin \alpha}$, $v = \sqrt{\frac{m_3}{m_2}} = -\frac{\sin(\alpha+\beta)}{\sin \alpha}$. Then

$$-v = \cos \beta + \frac{\sin \beta}{\sin \alpha} \cos \alpha = \pm \sqrt{1 - u^2 \sin^2 \alpha} + u \cos \alpha,$$

$$(-u \cos \alpha - v)^2 = u^2 \cos^2 \alpha + 2uv \cos \alpha + v^2 = 1 - u^2 \sin^2 \alpha,$$

and we obtain

$$\begin{aligned} \cos \alpha &= \frac{1 - u^2 - v^2}{2uv} = \frac{m_1 m_2 - m_3(m_1 + m_2)}{2m_3 \sqrt{m_1 m_2}} \\ \cos \beta &= -v - u \cos \alpha = \frac{u^2 - v^2 - 1}{2v} = \frac{m_3(m_2 - m_1) - m_1 m_2}{2m_1 \sqrt{m_2 m_3}}. \end{aligned} \quad (8)$$

Both $\cos \alpha$ and $\cos \beta$ are injective on \mathcal{U} , so the inverse mapping χ^{-1} is

$$\alpha = \cos^{-1} \frac{m_1 m_2 - m_3(m_1 + m_2)}{2m_3 \sqrt{m_1 m_2}}, \quad \beta = \cos^{-1} \frac{m_3(m_2 - m_1) - m_1 m_2}{2m_1 \sqrt{m_2 m_3}}.$$

To find $\chi(\mathcal{U})$, note that the condition (5) is equivalent to

$$\cos^2 \alpha < 1, \quad \cos^2 \beta < 1, \quad \sin(\alpha + \beta) < 0.$$

Substituting (8) into the above inequalities and using the normalization condition $m_1 + m_2 + m_3 = 1$, we find that the three inequalities lead to the same relationship:

$$m_1^2 m_2^2 + m_2^2 m_3^2 + m_1^2 m_3^2 - 2m_1 m_2 m_3 < 0,$$

which is the image of the mapping χ and it gives all admissible mass triples, a remark that completes the proof (see also Figure 2). \square

We end this section with a property of the acute isosceles triangle fixed-point configurations. The result below was first obtained in [4], so here we come up with a simpler proof.

Corollary 4. *For each acute isosceles triangle fixed-point configuration, if the masses m_1 and m_3 are at the vertices of the base, then $m_1 = m_3$ and $m_1 < 4m_2$. Reciprocally, if $m_1 = m_3$ and $m_1 < 4m_2$, then the only kind of fixed point these three masses can form is an acute isosceles triangle.*

Proof. Let the fixed-point configuration be an acute isosceles triangle, for instance, $\alpha = \beta$, then by (7), we have $m_1 = m_3$. Consequently

$$m_1^2 m_2^2 + m_2^2 m_3^2 + m_1^2 m_3^2 - 2m_1 m_2 m_3 = m_1^2 (m_1^2 + 2m_2^2 - 2m_2) < 0.$$

Therefore

$$m_1^2 < 2m_2(1 - m_2) = 2m_2 \cdot 2m_1,$$

which implies that

$$m_1 < 4m_2.$$

Reciprocally, if $m_1 = m_3$ and $m_1 < 4m_2$, then the bodies can form an acute isosceles triangle fixed-point configuration by the first part of this corollary. They can only form such configuration by Theorem 3. \square

4 Reduction and stability on \mathbb{S}^1

It is easy to see that if the initial velocity of each particle is confined to the tangent space $T(\mathbb{S}^1)$, then the motion takes place in $T^*(\mathbb{S}^1)^3$ forever, where $T^*(\mathbb{S}^1)$ is the cotangent bundle of \mathbb{S}^1 . In other words, $T^*(\mathbb{S}^1)^3$ is an invariant manifold of the equations of motion. In this section, we are going to study the stability of all acute triangle fixed-point solutions and their associated relative equilibria on this invariant manifold. We will first perform the reduction of the 3-body problem to $T^*(\mathbb{S}^1)^3$ and then prove that all these solutions are Lyapunov stable.

Confined to $T^*(\mathbb{S}^1)^3$, the Hamiltonian system take the form

$$H(\varphi_i, p_{\varphi_i}) = \sum_i \frac{1}{2} \frac{p_{\varphi_i}^2}{m_i} - \sum_{j \neq i} m_i m_j \cot d_{ij}, \quad T^*(\mathbb{S}^1)^3, \quad w = d \left(\sum_i p_{\varphi_i} d\varphi_i \right).$$

It is easy to see that in these coordinates, the conserved total angular momentum [1] is given by the expression

$$\mathbf{J}(\varphi_i, p_{\varphi_i}) = p_{\varphi_1} + p_{\varphi_2} + p_{\varphi_3}.$$

We further restrict the study of the stability of relative equilibria to the quotient space S'_p of $T^*(\mathbb{S}^1)^3$, which we define using

$$S_p := \{(\mathbf{q}, \mathbf{p}) \in T^*(\mathbb{S}^1)^3 : \mathbf{J} = p\},$$

and let S'_p be the quotient space under the Lie group $SO(2)$. For all values of p , these spaces are always smooth manifolds and have dimension 4. We can apply this procedure to the fixed-point solutions as well. The advantage of this approach is to eliminate the drift caused by the $SO(2)$ symmetry on the bodies. Indeed, by symmetry, if we perturb a fixed-point solution with some initial velocity $\dot{\varphi}_i = \epsilon$, then we obtain a relative equilibrium with angular velocity ϵ . We can therefore introduce the definition below.

Definition 5. A fixed-point solution (or its associated relative equilibrium) is Lyapunov stable on \mathbb{S}^1 if the corresponding rest point of the reduced Hamiltonian system on the quotient space S'_p is Lyapunov stable.

Thus we need to explicitly find the quotient manifold S'_p and the reduced Hamiltonian H_p (see [16] for a theoretical approach to this kind of problem). The angular momentum here behaves like the linear momentum of the classical n -body problem. Thus to perform the reduction, we introduce a type of Jacobi coordinates [11]:

$$\tilde{\varphi} = \frac{1}{\bar{m}}(m_1\varphi_1 + m_2\varphi_2 + m_3\varphi_3), \quad \phi_1 = \varphi_2 - \varphi_1, \quad \phi_2 = \varphi_3 - \nu_1\varphi_1 - \nu_2\varphi_2,$$

where

$$\bar{m} = m_1 + m_2 + m_3, \quad \nu_1 = \frac{m_1}{m_1 + m_2}, \quad \nu_2 = \frac{m_2}{m_1 + m_2}.$$

The corresponding conjugate momenta are then given by

$$p_{\tilde{\varphi}} = \bar{m}\dot{\tilde{\varphi}}, \quad p_{\phi_1} = \nu_3\dot{\phi}_1, \quad p_{\phi_2} = \nu_4\dot{\phi}_2,$$

where $\nu_3 = \frac{m_1m_2}{m_1+m_2}$ and $\nu_4 = \frac{(m_1+m_2)m_3}{\bar{m}}$. It is easy to verify that

$$p_{\tilde{\varphi}}d\tilde{\varphi} + p_{\phi_1}d\phi_1 + p_{\phi_2}d\phi_2 = p_{\varphi_1}d\varphi_1 + p_{\varphi_2}d\varphi_2 + p_{\varphi_3}d\varphi_3$$

and that the Hamiltonian is

$$H = \frac{1}{2} \left(\frac{p_{\tilde{\varphi}}^2}{\bar{m}} + \frac{p_{\phi_1}^2}{\nu_3} + \frac{p_{\phi_2}^2}{\nu_4} \right) - V(\phi_1, \phi_2),$$

where $V(\phi_1, \phi_2)$ is the force function in the new variables. Note that $\varphi_2 - \varphi_1 = \phi_1 = \alpha$, and

$$\beta = \varphi_3 - \varphi_2 = \phi_2 + \nu_1\varphi_1 + \nu_2\varphi_2 - \varphi_2 = \phi_2 - \nu_1\phi_1.$$

Recall that $d_{12} = \alpha, d_{23} = \beta, d_{13} = 2\pi - (\alpha + \beta)$. Thus

$$\begin{aligned} V(\phi_1, \phi_2) &= m_1m_2 \cot \alpha + m_2m_3 \cot \beta - m_1m_3 \cot(\alpha + \beta) \\ &= m_1m_2 \cot \phi_1 + m_2m_3 \cot(\phi_2 - \nu_1\phi_1) - m_1m_3 \cot(\phi_2 + \nu_2\phi_1). \end{aligned}$$

We now study the stability of the fixed-point solution. In this case $\dot{\varphi}_1 = \dot{\varphi}_2 = \dot{\varphi}_3 = 0$, we have $\bar{m}\dot{\tilde{\varphi}} = 0$, and $\mathbf{J} = 0$. So the quotient manifold is S'_0 . On this quotient manifold, $p_{\tilde{\varphi}} = 0$ and we can set $\tilde{\varphi} = 0$ since we identify all points that differ by a rotation. Thus we use $\phi_1, \phi_2, p_{\phi_1}, p_{\phi_2}$ as the coordinates of the 4-dimensional manifold S'_0 . Under these coordinates, we have the reduced Hamiltonian function

$$H_0 = \frac{1}{2} \left(\frac{p_{\phi_1}^2}{\nu_3} + \frac{p_{\phi_2}^2}{\nu_4} \right) - V(\phi_1, \phi_2).$$

Suppose \mathbf{q} is a fixed-point configuration on \mathbb{S}^1 of three masses m_1, m_2, m_3 . Their positions are given by

$$\varphi_1 = 0, \quad \varphi_2 = \alpha, \quad \varphi_3 = \alpha + \beta,$$

(see Figure 1), where α, β are in region (5). By Section 3, the masses and the two angles are related by equations (6). Then the fixed-point solution $(0, \alpha, \alpha + \beta, 0, 0, 0)$

we want to study in $T^*(\mathbb{S}^1)^3$ becomes a rest point of the reduced Hamiltonian system on S'_0 :

$$\phi_1 = \alpha, \quad \phi_2 = \beta + \nu_1 \alpha, \quad p_{\phi_1} = 0, \quad p_{\phi_2} = 0.$$

Let us denote this rest point by X . Our goal is to study the stability of X , a property given by the following result.

Theorem 6. *Every acute triangle fixed-point solution is Lyapunov stable on \mathbb{S}^1 .*

Proof. We will show that the rest point X is a local minimum of the Hamiltonian H_0 . Since the Hamiltonian is preserved during any motion, we can then conclude that the rest point X is stable. The rest point X is obviously a minimum of the kinetic energy. We will show that it is also a local maximum of V by studying its Hessian matrix.

We use the variables (α, β) instead of (ϕ_1, ϕ_2) . Then

$$V(\alpha, \beta) = m_1 m_2 \cot \alpha + m_2 m_3 \cot \beta - m_1 m_3 \cot(\alpha + \beta),$$

and the Hessian matrix $H(V)$ of V at X has the form

$$H(V) = 2 \begin{bmatrix} \frac{m_1 m_2 \cos \alpha}{\sin^3 \alpha} - \frac{m_1 m_3 \cos(\alpha + \beta)}{\sin^3(\alpha + \beta)} & -\frac{m_1 m_3 \cos(\alpha + \beta)}{\sin^3(\alpha + \beta)} \\ -\frac{m_1 m_3 \cos(\alpha + \beta)}{\sin^3(\alpha + \beta)} & \frac{m_2 m_3 \cos \beta}{\sin^3 \beta} - \frac{m_1 m_3 \cos(\alpha + \beta)}{\sin^3(\alpha + \beta)} \end{bmatrix}.$$

Recall that equations (6) are

$$\frac{m_2}{\sin^2 \alpha} = \frac{m_3}{\sin^2(\alpha + \beta)}, \quad \frac{m_2}{\sin^2 \beta} = \frac{m_1}{\sin^2(\alpha + \beta)}, \quad \frac{m_1}{\sin^2 \alpha} = \frac{m_3}{\sin^2 \beta}.$$

By direct computation, we find that the diagonal entries of $\frac{1}{2}H(V)$ are

$$\begin{aligned} \frac{m_1 m_2 \cos \alpha}{\sin^3 \alpha} - \frac{m_1 m_3 \cos(\alpha + \beta)}{\sin^3(\alpha + \beta)} &= \frac{m_1 m_2}{\sin^2 \alpha} \frac{\sin \beta}{\sin(\alpha + \beta) \sin \alpha} < 0, \\ \frac{m_2 m_3 \cos \beta}{\sin^3 \beta} - \frac{m_1 m_3 \cos(\alpha + \beta)}{\sin^3(\alpha + \beta)} &= \frac{m_1 m_3}{\sin^2(\alpha + \beta)} \frac{\sin \alpha}{\sin(\alpha + \beta) \sin \beta} < 0. \end{aligned}$$

Hence the trace of $H(V)$ is negative. Here we use the fact that (α, β) is in region (5), thus $\sin \alpha > 0$, $\sin \beta > 0$, and $\sin(\alpha + \beta) < 0$. Further straightforward computations show that the determinant of $\frac{1}{2}H(V)$ is

$$\begin{aligned} & \frac{m_1 m_2 \cos \alpha}{\sin^3 \alpha} \frac{m_2 m_3 \cos \beta}{\sin^3 \beta} - \left(\frac{m_1 m_2 \cos \alpha}{\sin^3 \alpha} + \frac{m_2 m_3 \cos \beta}{\sin^3 \beta} \right) \frac{m_1 m_3 \cos(\alpha + \beta)}{\sin^3(\alpha + \beta)} \\ &= \frac{m_1 m_3 m_2^2 \cos \alpha \cos \beta}{\sin^3 \alpha \sin^3 \beta} - \frac{m_1 m_3}{\sin^2(\alpha + \beta)} \frac{\sin(\alpha + \beta)}{\sin \alpha \sin \beta} \frac{m_1 m_3 \cos(\alpha + \beta)}{\sin^3(\alpha + \beta)} \\ &= \frac{m_1 m_3 m_2^2}{\sin^3 \alpha \sin^3 \beta} (\cos \alpha \cos \beta - \cos(\alpha + \beta)) = \frac{m_1 m_3 m_2^2}{\sin^2 \alpha \sin^2 \beta} > 0. \end{aligned}$$

These two facts imply that the two eigenvalues of $H(V)$ are both negative. Then the fixed-point configuration is a local maximum of V . We can thus conclude that the rest point $X = (\alpha, \beta + \nu_1 \alpha, 0, 0)$ is a local minimum of $H = T - V$. This remark completes the proof. \square

We further study the stability of the associated relative equilibria. In this case $\dot{\varphi}_1 = \dot{\varphi}_2 = \dot{\varphi}_3 = \omega$, we have $\bar{m}\dot{\varphi} = \bar{m}\omega$, and $\mathbf{J} = \bar{m}\omega$. So the quotient manifold is $S'_{\bar{m}\omega}$. On this quotient manifold, $p_{\tilde{\varphi}} = \bar{m}\omega$ and we can set $\tilde{\varphi} = 0$ since we identify all points that differ by a rotation. Thus we use $\phi_1, \phi_2, p_{\phi_1}, p_{\phi_2}$ as the coordinates of the 4-dimensional manifold $S'_{\bar{m}\omega}$. Under these coordinates, we have the reduced Hamiltonian

$$H_{\bar{m}\omega} = \frac{1}{2} \left(\frac{p_{\phi_1}^2}{\nu_3} + \frac{p_{\phi_2}^2}{\nu_4} \right) - V(\phi_1, \phi_2) + \frac{\bar{m}\omega^2}{2}.$$

Then the relative equilibrium (3) in $T^*(\mathbb{S}^1)^3$ becomes a rest point in $S'_{\bar{m}\omega}$,

$$\phi_1 = \alpha, \quad \phi_2 = \beta + \nu_1\alpha, \quad p_{\phi_1} = 0, \quad p_{\phi_2} = 0.$$

Let us denote this rest point by X_1 . Note that H_0 and $H_{\bar{m}\omega}$ are the same up to a constant. Then we can conclude that X_1 is also a local minimum of $H_{\bar{m}\omega}$. So we have proved the following result.

Theorem 7. *Every relative equilibrium associated to an acute triangle fixed-point configuration is Lyapunov stable on \mathbb{S}^1 .*

Remark 8. *The method of proving the stability of relative equilibria by showing that the rest points are minima of the Hamiltonian never works in the classical n -body problem. The obstacle lies in the fact that planar central configurations of the classical n -body problem are never maxima of the force function, [12].*

5 Stability on \mathbb{S}^2

In this section we study the linear stability of the above solutions on \mathbb{S}^2 . Different from the previous case, their stability depends on the angular velocity ω . We first introduce rotating coordinates to treat a general relative equilibrium on \mathbb{S}^2 as rest point, and obtain the linearized system $\dot{v} = L(v - X_\omega)$. We then compute L for relative equilibria associated with fixed-point configurations on the equator. As in the classical n -body problem, we study the stability of the rest points on a proper subspace. Inspired by the work of Moeckel [12], we find the proper linear subspace. In the end, we show that these solutions are linearly stable if ω^2 is greater than a critical value.

So consider a general relative equilibrium on \mathbb{S}^2 with angular velocity ω and introduce the rotating coordinates

$$\bar{\theta}_i = \theta_i, \quad \dot{\bar{\theta}}_i = \dot{\theta}_i, \quad \bar{\varphi}_i = \varphi_i - \omega t, \quad \dot{\bar{\varphi}}_i = \dot{\varphi}_i - \omega, \quad \bar{p}_{\theta_i} = p_{\theta_i}, \quad \bar{p}_{\varphi_i} = p_{\varphi_i}, \quad i \in \{1, 2, \dots, n\}.$$

In these new coordinates, the original Hamiltonian system

$$\begin{aligned} \dot{\theta}_i &= \frac{\partial H}{\partial p_{\theta_i}} = \frac{p_{\theta_i}}{m_i}, & \dot{p}_{\theta_i} &= -\frac{\partial H}{\partial \theta_i} = \frac{p_{\varphi_i}^2 \cos \theta_i}{m_i \sin^3 \theta_i} - \frac{\partial U}{\partial \theta_i}, \\ \dot{\varphi}_i &= \frac{\partial H}{\partial p_{\varphi_i}} = \frac{p_{\varphi_i}}{m_i \sin^2 \theta_i}, & \dot{p}_{\varphi_i} &= -\frac{\partial H}{\partial \varphi_i} = -\frac{\partial U}{\partial \varphi_i}, \quad i \in \{1, 2, \dots, n\}, \end{aligned}$$

becomes

$$\begin{aligned}\dot{\bar{\theta}}_i &= \frac{\bar{p}_{\theta_i}}{m_i}, & \dot{\bar{p}}_{\theta_i} &= \frac{\bar{p}_{\varphi_i}^2 \cos \bar{\theta}_i}{m_i \sin^3 \bar{\theta}_i} - \frac{\partial U}{\partial \bar{\theta}_i}, \\ \dot{\bar{\varphi}}_i &= \frac{\bar{p}_{\varphi_i}}{m_i \sin^2 \bar{\theta}_i} - \omega, & \dot{\bar{p}}_{\varphi_i} &= -\frac{\partial U}{\partial \varphi_i}, \quad i \in \{1, 2, \dots, n\}.\end{aligned}\quad (9)$$

Thus system (9) is Hamiltonian with

$$H = \sum_{i=1}^n \frac{1}{2} \left(\frac{\bar{p}_{\theta_i}^2}{m_i} + \frac{\bar{p}_{\varphi_i}^2}{m_i \sin^2 \bar{\theta}_i} - 2\bar{p}_{\varphi_i} \omega \right) + U$$

and symplectic form $w = d \left(\sum_{i=1}^n \bar{p}_{\theta_i} d\bar{\theta}_i + \bar{p}_{\varphi_i} d\bar{\varphi}_i \right)$.

We will use $\theta_i, \varphi_i, p_{\theta_i}, p_{\varphi_i}$ instead of $\bar{\theta}_i, \bar{\varphi}_i, \bar{p}_{\theta_i}, \bar{p}_{\varphi_i}$ if no further confusion arises. Denote by X_ω the rest point in system (9) corresponding to a relative equilibrium (3) with angular velocity ω . Then X_ω is

$$\theta_i(t) = \theta_i(0), \quad \varphi_i(t) = \varphi_i(0), \quad p_{\theta_i} = 0, \quad p_{\varphi_i} = \omega m_i \sin^2 \theta_i,$$

and we are going to study the stability of X_ω for the linearized system

$$\dot{v} = L(v - X_\omega), \quad v = (\theta_1, \dots, \theta_n, \varphi_1, \dots, \varphi_n, p_{\theta_1}, \dots, p_{\theta_n}, p_{\varphi_1}, \dots, p_{\varphi_n}). \quad (10)$$

By straightforward computation, we get .

$$\begin{aligned}L &= \begin{bmatrix} 0 & 0 & M^{-1} & 0 \\ K & 0 & 0 & M^{-1}C^{-1} \\ \frac{\partial^2 V}{\partial \theta_i \partial \theta_j} + N & \frac{\partial^2 V}{\partial \theta_i \partial \varphi_j} & 0 & -K^T \\ \frac{\partial^2 V}{\partial \varphi_i \partial \theta_j} & \frac{\partial^2 V}{\partial \varphi_i \partial \varphi_j} & 0 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} \frac{-p_{\varphi_1}^2 (1+2 \cos^2 \theta_1)}{m_1 \sin^4 \theta_1} & \dots & 0 \\ 0 & \frac{-p_{\varphi_2}^2 (1+2 \cos^2 \theta_2)}{m_2 \sin^4 \theta_2} & 0 \\ \vdots & \dots & \vdots \\ 0 & \dots & \frac{-p_{\varphi_n}^2 (1+2 \cos^2 \theta_n)}{m_n \sin^4 \theta_n} \end{bmatrix}, \\ K &= \begin{bmatrix} \frac{-2p_{\varphi_1} \cos \theta_1}{m_1 \sin^3 \theta_1} & \dots & 0 \\ 0 & \frac{-2p_{\varphi_2} \cos \theta_2}{m_2 \sin^3 \theta_2} & 0 \\ \vdots & \dots & \vdots \\ 0 & \dots & \frac{-2p_{\varphi_n} \cos \theta_n}{m_n \sin^3 \theta_n} \end{bmatrix}, \quad M^{-1} = \begin{bmatrix} \frac{1}{m_1} & \dots & 0 \\ 0 & \frac{1}{m_2} & 0 \\ \vdots & \dots & \vdots \\ 0 & \dots & \frac{1}{m_n} \end{bmatrix}, \quad C^{-1} = \begin{bmatrix} \frac{1}{\sin^2 \theta_1} & \dots & 0 \\ 0 & \frac{1}{\sin^2 \theta_2} & 0 \\ \vdots & \dots & \vdots \\ 0 & \dots & \frac{1}{\sin^2 \theta_n} \end{bmatrix},\end{aligned}$$

where L is a $4n \times 4n$ matrix, whereas N, K, M^{-1} , and C^{-1} are $n \times n$ matrices.

It is generally difficult to find the normal form of L . However, for relative equilibria associated to a fixed-point configuration on the equator, things are easier.

Lemma 9. *For fixed-point configurations on the equator, $L = \begin{bmatrix} 0 & 0 & M^{-1} & 0 \\ 0 & 0 & 0 & M^{-1} \\ H_\omega & 0 & 0 & 0 \\ 0 & G & 0 & 0 \end{bmatrix}$,*

where $H_\omega = H - \omega^2 M = \left[\frac{\partial^2 V}{\partial \theta_i \partial \theta_j} \right] - \omega^2 M$ and $G = \left[\frac{\partial^2 V}{\partial \varphi_i \partial \varphi_j} \right]$. And the elements of H and G are

$$H_{ij} = \frac{m_i m_j}{\sin^3 d_{ij}}, \quad H_{ii} = - \sum_{j \neq i, j=1}^n H_{ij} \cos d_{ij}, \quad G_{ij} = \frac{-2m_i m_j \cos d_{ij}}{\sin^3 d_{ij}}, \quad G_{ii} = - \sum_{j \neq i, j=1}^n G_{ij}.$$

Proof. In this case $\theta_i = \frac{\pi}{2}$, so $K = 0$, $C^{-1} = I_n$. Since that $d_{ij} = \cos^{-1} \mathbf{q}_i \cdot \mathbf{q}_j$, we have

$$\frac{\partial d_{ij}}{\partial \theta_i} = \frac{-1}{\sin d_{ij}} \left(\mathbf{q}_j \cdot \frac{\partial \mathbf{q}_i}{\partial \theta_i} \right), \quad \frac{\partial d_{ij}}{\partial \varphi_i} = \frac{-1}{\sin d_{ij}} \left(\mathbf{q}_j \cdot \frac{\partial \mathbf{q}_i}{\partial \varphi_i} \right),$$

and

$$\begin{aligned} \frac{\partial^2 V}{\partial \theta_i \partial \theta_j} &= m_i m_j \left[\frac{3 \mathbf{q}_i \cdot \mathbf{q}_j}{\sin^5 d_{ij}} \left(\mathbf{q}_j \cdot \frac{\partial \mathbf{q}_i}{\partial \theta_i} \right) \left(\mathbf{q}_i \cdot \frac{\partial \mathbf{q}_j}{\partial \theta_j} \right) + \frac{1}{\sin^3 d_{ij}} \frac{\partial \mathbf{q}_j}{\partial \theta_j} \cdot \frac{\partial \mathbf{q}_i}{\partial \theta_i} \right], \\ \frac{\partial^2 V}{\partial \theta_i \partial \theta_i} &= \sum_{j \neq i, j=1}^n m_i m_j \left[\frac{3 \mathbf{q}_i \cdot \mathbf{q}_j}{\sin^5 d_{ij}} \left(\mathbf{q}_j \cdot \frac{\partial \mathbf{q}_i}{\partial \theta_i} \right)^2 + \frac{1}{\sin^3 d_{ij}} \mathbf{q}_j \cdot \frac{\partial^2 \mathbf{q}_i}{\partial \theta_i^2} \right], \\ \frac{\partial^2 V}{\partial \varphi_i \partial \varphi_j} &= m_i m_j \left[\frac{3 \mathbf{q}_i \cdot \mathbf{q}_j}{\sin^5 d_{ij}} \left(\mathbf{q}_j \cdot \frac{\partial \mathbf{q}_i}{\partial \varphi_i} \right) \left(\mathbf{q}_i \cdot \frac{\partial \mathbf{q}_j}{\partial \varphi_j} \right) + \frac{1}{\sin^3 d_{ij}} \frac{\partial \mathbf{q}_j}{\partial \varphi_j} \cdot \frac{\partial \mathbf{q}_i}{\partial \varphi_i} \right], \\ \frac{\partial^2 V}{\partial \varphi_i \partial \varphi_i} &= \sum_{j \neq i, j=1}^n m_i m_j \left[\frac{3 \mathbf{q}_i \cdot \mathbf{q}_j}{\sin^5 d_{ij}} \left(\mathbf{q}_j \cdot \frac{\partial \mathbf{q}_i}{\partial \varphi_i} \right)^2 + \frac{1}{\sin^3 d_{ij}} \mathbf{q}_j \cdot \frac{\partial^2 \mathbf{q}_i}{\partial \varphi_i^2} \right], \\ \frac{\partial^2 V}{\partial \theta_i \partial \varphi_j} &= m_i m_j \left[\frac{3 \mathbf{q}_i \cdot \mathbf{q}_j}{\sin^5 d_{ij}} \left(\mathbf{q}_j \cdot \frac{\partial \mathbf{q}_i}{\partial \theta_i} \right) \left(\mathbf{q}_i \cdot \frac{\partial \mathbf{q}_j}{\partial \varphi_j} \right) + \frac{1}{\sin^3 d_{ij}} \frac{\partial \mathbf{q}_j}{\partial \varphi_j} \cdot \frac{\partial \mathbf{q}_i}{\partial \theta_i} \right], \\ \frac{\partial^2 V}{\partial \theta_i \partial \varphi_i} &= \sum_{j \neq i, j=1}^n m_i m_j \left[\frac{3 \mathbf{q}_i \cdot \mathbf{q}_j}{\sin^5 d_{ij}} \left(\mathbf{q}_j \cdot \frac{\partial \mathbf{q}_i}{\partial \varphi_i} \right) \left(\mathbf{q}_j \cdot \frac{\partial \mathbf{q}_i}{\partial \theta_i} \right) + \frac{1}{\sin^3 d_{ij}} \mathbf{q}_j \cdot \frac{\partial^2 \mathbf{q}_i}{\partial \varphi_i \partial \theta_i} \right]. \end{aligned}$$

Note that

$$\frac{\partial \mathbf{q}_i}{\partial \theta_i} = \frac{\partial}{\partial \theta_i} (\sin \theta_i \cos \varphi_i, \sin \theta_i \sin \varphi_i, \cos \theta_i) = (\cos \theta_i \cos \varphi_i, \cos \theta_i \sin \varphi_i, -\sin \theta_i) = (0, 0, -1).$$

Similarly we obtain

$$\begin{aligned} \frac{\partial \mathbf{q}_i}{\partial \varphi_i} &= (-\sin \varphi_i, \cos \varphi_i, 0), & \frac{\partial^2 \mathbf{q}_i}{\partial \theta_i \partial \varphi_i} &= (0, 0, 0), \\ \frac{\partial^2 \mathbf{q}_i}{\partial \theta_i^2} &= -(\cos \varphi_i, \sin \varphi_i, 0), & \frac{\partial^2 \mathbf{q}_i}{\partial \varphi_i^2} &= -(\cos \varphi_i, \sin \varphi_i, 0). \end{aligned}$$

Thus we have

$$\begin{aligned} \mathbf{q}_j \cdot \frac{\partial \mathbf{q}_i}{\partial \theta_i} &= 0, & \frac{\partial \mathbf{q}_j}{\partial \theta_j} \cdot \frac{\partial \mathbf{q}_i}{\partial \theta_i} &= 1, & \mathbf{q}_j \cdot \frac{\partial^2 \mathbf{q}_i}{\partial \theta_i^2} &= -\cos d_{ij}, & \mathbf{q}_j \cdot \frac{\partial \mathbf{q}_i}{\partial \varphi_i} &= \pm \sin d_{ij}, \\ \frac{\partial \mathbf{q}_j}{\partial \varphi_j} \cdot \frac{\partial \mathbf{q}_i}{\partial \theta_i} &= 0, & \mathbf{q}_j \cdot \frac{\partial^2 \mathbf{q}_i}{\partial \theta_i \partial \varphi_i} &= 0, & \frac{\partial \mathbf{q}_j}{\partial \varphi_j} \cdot \frac{\partial \mathbf{q}_i}{\partial \varphi_i} &= \cos d_{ij}, & \mathbf{q}_j \cdot \frac{\partial^2 \mathbf{q}_i}{\partial \varphi_i^2} &= -\cos d_{ij}. \end{aligned}$$

Then straightforward computation shows that the block $\left[\frac{\partial^2 V}{\partial \theta_i \partial \varphi_j} \right]$ is zero, and

$$H_{ij} = \frac{m_i m_j}{\sin^3 d_{ij}}, \quad H_{ii} = - \sum_{j \neq i, j=1}^n H_{ij} \cos d_{ij}, \quad G_{ij} = \frac{-2 m_i m_j \cos d_{ij}}{\sin^3 d_{ij}}, \quad G_{ii} = - \sum_{j \neq i, j=1}^n G_{ij},$$

a remark that completes the proof. \square

We can now find the normal form of L by computing the normal forms of $H_\omega M^{-1}$ and GM^{-1} .

Lemma 10. $H_\omega M^{-1}$ and GM^{-1} are diagonalizable. If $u \in \mathbb{C}^n$ is an eigenvector of $H_\omega M^{-1}$ (GM^{-1}) with eigenvalue $\lambda \neq 0$, then there exist a two dimensional invariant subspace of L in \mathbb{C}^{4n} on which L is $\begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & -\sqrt{\lambda} \end{bmatrix}$. If $u \in \mathbb{C}^n$ is an eigenvector of $H_\omega M^{-1}$ (GM^{-1}) with eigenvalue 0, then there exist a two dimensional invariant subspace of L in \mathbb{C}^{4n} on which L is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Proof. It is enough to prove this for $H_\omega M^{-1}$. Note H_ω is symmetric, thus $H_\omega M^{-1}$ is symmetric with respect to the inner product $\langle v, w \rangle = v^T M^{-1} w$ so it is diagonalizable with respect to some M^{-1} orthogonal basis. Now suppose $H_\omega M^{-1} u = \lambda u$, $\lambda \neq 0$. Then .

$$\begin{bmatrix} 0 & 0 & M^{-1} & 0 \\ 0 & 0 & 0 & M^{-1} \\ H_\omega & 0 & 0 & 0 \\ 0 & G & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{M^{-1}u}{\sqrt{\lambda}} & \frac{M^{-1}u}{-\sqrt{\lambda}} \\ 0 & 0 \\ u & u \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} M^{-1}u & M^{-1}u \\ 0 & 0 \\ \sqrt{\lambda}u & -\sqrt{\lambda}u \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{M^{-1}u}{\sqrt{\lambda}} & \frac{M^{-1}u}{-\sqrt{\lambda}} \\ 0 & 0 \\ u & u \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & -\sqrt{\lambda} \end{bmatrix}.$$

Similarly, if $H_\omega M^{-1} u = 0$, then

$$\begin{bmatrix} 0 & 0 & M^{-1} & 0 \\ 0 & 0 & 0 & M^{-1} \\ H_\omega & 0 & 0 & 0 \\ 0 & G & 0 & 0 \end{bmatrix} \begin{bmatrix} M^{-1}u & 0 \\ 0 & 0 \\ 0 & u \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & M^{-1}u \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} M^{-1}u & 0 \\ 0 & 0 \\ 0 & u \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

This completes the proof. \square

Normally, a rest point X_ω is called linearly stable if X_ω is a stable rest point of the linearized system (10). However, as for relative equilibria of the classical n -body problem, the symmetries and integrals of the problem make it impossible to satisfy this condition.

Note that V is invariant under the $SO(3)$ action, which implies that any fixed-point configuration remains a fixed-point configuration after any rotation. We can find the three vectors corresponding to the three rotations:

$$v_1 = \begin{bmatrix} \cos d_{11} \\ \vdots \\ \cos d_{1n} \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad v_2 = \begin{bmatrix} \sin(\varphi_1 - \varphi_1) \\ \vdots \\ \sin(\varphi_n - \varphi_1) \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}. \quad (11)$$

Lemma 11. Let H and G be the matrices defined in Lemma 9 and v_1, v_2, v_3 the vectors defined in (11). Then

$$Hv_1 = 0, \quad Hv_2 = 0, \quad Gv_3 = 0.$$

Proof. Note that

$$\begin{aligned} \cos d_{1i} &= \cos(\varphi_1 - \varphi_k + \varphi_k - \varphi_i) = \cos d_{1k} \cos d_{ki} - \sin(\varphi_1 - \varphi_k) \sin(\varphi_k - \varphi_i), \\ \sin(\varphi_i - \varphi_1) &= \sin(\varphi_i - \varphi_k + \varphi_k - \varphi_1) = \sin(\varphi_k - \varphi_1) \cos d_{ki} - \cos d_{1k} \sin(\varphi_k - \varphi_i). \end{aligned}$$

Then the k -th entry of Hv_1 is

$$\begin{aligned}
& H_{k1} \cos d_{11} + \cdots + \left(- \sum_{i \neq k} H_{ki} \cos d_{ki} \right) \cos d_{1k} + \cdots + H_{kn} \cos d_{1n} \\
&= \frac{m_k m_1 \cos d_{11}}{\sin^3 d_{k1}} + \cdots - \left(\sum_{i \neq k} \frac{m_k m_i \cos d_{1k} \cos d_{ki}}{\sin^3 d_{ki}} \right) + \cdots + \frac{m_k m_n \cos d_{1n}}{\sin^3 d_{kn}} \\
&= \sum_{i \neq k} m_k m_i \frac{\cos d_{1i} - \cos d_{1k} \cos d_{ki}}{\sin^3 d_{ki}} = -\sin(\varphi_1 - \varphi_k) \sum_{i \neq k} m_k m_i \frac{\sin(\varphi_k - \varphi_i)}{\sin^3 d_{ki}}.
\end{aligned}$$

And the k -th entry of Hv_2 is

$$\begin{aligned}
& H_{k1} \sin(\varphi_1 - \varphi_1) + \cdots + \left(- \sum_{i \neq k} H_{ki} \cos d_{ki} \right) \sin(\varphi_k - \varphi_1) + \cdots + H_{kn} \sin(\varphi_n - \varphi_1) \\
&= \sum_{i \neq k} m_k m_i \frac{\sin(\varphi_i - \varphi_1) - \sin(\varphi_k - \varphi_1) \cos d_{ki}}{\sin^3 d_{ki}} = -\cos d_{k1} \sum_{i \neq k} m_k m_i \frac{\sin(\varphi_k - \varphi_i)}{\sin^3 d_{ki}}.
\end{aligned}$$

Using criterion (4) of fixed-point configurations for n masses,

$$\sum_{i, i \neq k} \frac{m_k m_i \sin(\varphi_k - \varphi_i)}{\sin^3 d_{ki}} = 0,$$

we find that $Hv_1 = Hv_2 = 0$. Note that $\sum_{i=1}^n G_{ik} = 0$ for all k . Hence $Gv_3 = 0$, a remark that completes the proof. \square

Now we consider the stability of the fixed-point solutions, i.e, X_0 . In this case, $H_\omega = H$. Consider the 6-dimensional subspace E_1 of \mathbb{C}^{4n} spanned by the vectors

$$\begin{bmatrix} v_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ Mv_1 \\ 0 \end{bmatrix}, \begin{bmatrix} v_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ Mv_2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ v_3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ Mv_3 \end{bmatrix}.$$

Then using Lemma 11, we obtain that E_1 is an invariant subspace for L . The matrix of $L|_{E_1}$ in this basis is

$$L|_{E_1} = \text{diag} \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}.$$

Though all eigenvalues on E_1 are 0, there are three nontrivial Jordan blocks, a fact which implies that the rest point is not linearly stable in the conventional sense. This instability is trivial as a natural effect of the symmetry of this Hamiltonian system. Indeed, we can perturb a fixed-point solution into a relative equilibrium by any rotation in $SO(3)$. Then the angular positions of these orbits drift away from each other, a property mathematically reflected by the nontrivial Jordan blocks, as remarked in [13].

It is traditional in celestial mechanics to view the drifts in this subspace as harmless. Indeed, they can be eliminated by fixing the angular momentum and passing to

a quotient manifold under the action of the rotational symmetry group, [13]. Thus it is reasonable to formulate a definition of linear stability based on the behaviour of L in a complementary subspace, [12]. To define such a subspace, it is necessary to introduce the *skew inner product* of two complex vectors $v, w \in \mathbb{C}^{4n}$:

$$\Omega(v, w) = v^T J w, \quad J_{4n \times 4n} = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}.$$

Using the fact that $L = JS$, $S^T = S$, where S is the Hessian matrix of the Hamiltonian at X_0 , we obtain that

$$\Omega(v, Lw) = -\Omega(Lv, w).$$

With the help of this property it is easy to show that the skew-orthogonal complement of an invariant subspace of L is again invariant. Indeed, let E denote the skew orthogonal complement in \mathbb{C}^{4n} of E_1 , that is,

$$E = \{v \in \mathbb{C}^{4n} : \Omega(v, w) = 0 \text{ for all } w \in E_1\}.$$

Then E is an L invariant subspace of dimension $4n - 6$.

Definition 12. A fixed-point solutions X_0 associated with a fixed-point configuration on the equator is called linearly stable if X_0 is a stable rest point of the restriction of the linearized equation (10) to E .

For $n = 3$, i.e., the relative equilibria associated with acute triangle fixed-point configurations, we have $E = \mathbb{C}^6$, and we can find the normal form of $L|_E$.

Theorem 13. For each acute triangle fixed-point solution, $L|_E$ is diagonalizable and in properly chosen basis,

$$L|_E = \text{diag}\left\{\sqrt{\lambda_1}, -\sqrt{\lambda_1}, i\sqrt{|\lambda_2|}, -i\sqrt{|\lambda_2|}, i\sqrt{|\lambda_3|}, -i\sqrt{|\lambda_3|}\right\},$$

where $0, 0, \lambda_1 > 0$ are the eigenvalues of HM^{-1} , and $0, \lambda_2 < 0, \lambda_3 < 0$ are the eigenvalues of GM^{-1} . All acute triangle fixed-point solutions on the equator are unstable on \mathbb{S}^2 .

Proof. We first find the eigenvalues of HM^{-1} and GM^{-1} . Recall that $d_{12} = \alpha, d_{23} = \beta, d_{13} = 2\pi - (\alpha + \beta)$. Using α, β , by direct computation, we obtain that

$$HM^{-1} = \begin{bmatrix} \frac{-m_2 \cos \alpha}{\sin^3 \alpha} + \frac{m_3 \cos(\alpha + \beta)}{\sin^3(\alpha + \beta)} & \frac{m_1}{\sin^3 \alpha} & \frac{-m_1}{\sin^3(\alpha + \beta)} \\ \frac{m_2}{\sin^3 \alpha} & \frac{-m_1 \cos \alpha}{\sin^3 \alpha} + \frac{-m_3 \cos \beta}{\sin^3 \beta} & \frac{m_2}{\sin^3 \beta} \\ \frac{-m_3}{\sin^3(\alpha + \beta)} & \frac{m_3}{\sin^3 \beta} & \frac{m_1 \cos(\alpha + \beta)}{\sin^3(\alpha + \beta)} + \frac{-m_2 \cos \beta}{\sin^3 \beta} \end{bmatrix}.$$

Lemma 11 implies that Mv_1 and Mv_2 are two eigenvectors of HM^{-1} with eigenvalue 0. Thus the other eigenvalue λ_1 equals the trace of the matrix. Using the same idea as in the proof of Theorem 6, we find that the second diagonal entry is

$$\frac{-m_1 \cos \alpha}{\sin^3 \alpha} + \frac{-m_3 \cos \beta}{\sin^3 \beta} = -\frac{m_3}{\sin^2 \beta} \frac{\sin(\alpha + \beta)}{\sin \alpha \sin \beta} > 0.$$

The first one and the third one are just opposite to diagonal entries of the matrix in the proof of Theorem 6, so they are positive. Hence λ_1 is positive since it equals the trace.

Lemma 11 also implies that G has one eigenvalue 0. Note that the proof of Theorem 6 implies that fixed-point configurations are local maxima of V on $(\mathbb{S}^1)^3 \setminus \Delta$. Thus the two other eigenvalues of $G = \left[\frac{\partial^2 V}{\partial \varphi_i \partial \varphi_j} \right]$ are both negative. Note that $M^{-\frac{1}{2}}$ is well defined. Then G is congruent to $G_1 := (M^{-\frac{1}{2}})^T G M^{-\frac{1}{2}}$, which is similar to $M^{\frac{1}{2}} G_1 M^{-\frac{1}{2}} = G M^{-1}$. By Sylvester's law of inertia, [9], we have

$$n_-(G M^{-1}) = n_-(G) = 2, \quad n_0(G M^{-1}) = n_0(G) = 1,$$

where $n_0(A)$ is the number of zero eigenvalues and $n_-(A)$ is the number of negative eigenvalues of matrix A . This proves the eigenvalues of $H M^{-1}$ are $0, 0, \lambda_1 > 0$, and the eigenvalues of $G M^{-1}$ are $0, \lambda_2 < 0, \lambda_3 < 0$.

By Lemma 10, we see that L on \mathbb{C}^{12} is similar to

$$\text{diag} \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \sqrt{\lambda_1}, -\sqrt{\lambda_1}, i\sqrt{|\lambda_2|}, -i\sqrt{|\lambda_2|}, i\sqrt{|\lambda_3|}, -i\sqrt{|\lambda_3|} \right\}.$$

Recall that the normal form of $L|_{E_1}$ is given by the first three nontrivial Jordan blocks. We thus obtain that $L|_E$ is similar to

$$\text{diag} \left\{ \sqrt{\lambda_1}, -\sqrt{\lambda_1}, i\sqrt{|\lambda_2|}, -i\sqrt{|\lambda_2|}, i\sqrt{|\lambda_3|}, -i\sqrt{|\lambda_3|} \right\}.$$

Then the positive eigenvalue $\sqrt{\lambda_1}$ indicates that all acute triangle fixed-point solutions and all associated relative equilibria are unstable on \mathbb{S}^2 , a remark that completes the proof. \square

Now we study the linear stability of the associated relative equilibria. Define E_2 as the two-dimensional subspace spanned by

$$[0, v_3^T, 0, 0]^T, \quad [0, 0, 0, (M v_3)^T]^T.$$

Then L has one Jordan block on E_2 . By the same reason, it is reasonable to define stability based on the behaviour of the system on the complementary space, that is

$$\tilde{E} = \{v \in \mathbb{C}^{4n} : \Omega(v, w) = 0 \text{ for all } w \in E_2\}.$$

Then \tilde{E} is an L invariant subspace of dimension $4n - 2$.

Definition 14. A relative equilibrium X_ω associated with a fixed-point configuration on the equator is called linearly stable if X_ω is a stable rest point of the restriction of the linearized equation (10) to \tilde{E} .

Theorem 15. Let X_ω be a relative equilibria associated with an acute triangle fixed-point configuration on the equator. Then it is unstable on \mathbb{S}^2 if and only if $0 < \omega^2 \leq \lambda_1$, and it is linearly stable if and only $\lambda_1 < \omega^2$, where

$$\lambda_1 = -\frac{m_2}{\sin^2 \alpha} \frac{\sin \beta}{\sin(\alpha + \beta) \sin \alpha} - \frac{m_3}{\sin^2 \beta} \frac{\sin \alpha}{\sin(\alpha + \beta) \sin \beta} - \frac{m_3}{\sin^2 \beta} \frac{\sin(\alpha + \beta)}{\sin \alpha \sin \beta}.$$

Proof. By Lemma 10 and the proof of the above theorem, we only need to find the eigenvalues of $H M^{-1} - \omega^2$. Note that $H M^{-1}$ has three eigenvectors, with two eigenvalues being zero. Thus there exists an invertible matrix P such that

$$H M^{-1} = P \text{diag}\{\lambda_1, 0, 0\} P^{-1}.$$

Therefore

$$HM^{-1} - \omega^2 = P \text{diag}\{\lambda_1 - \omega^2, -\omega^2, -\omega^2\} P^{-1}.$$

Thus the eigenvalues of L restricted to \tilde{E} are all purely complex if and only if $\lambda_1 < \omega^2$. Straightforward computations lead to the value of λ_1 . This remark completes the proof. \square

It is interesting to compare the fixed points on \mathbb{S}^1 and the collinear central configurations of the classical n -body problem. The fixed points of three particles are local maxima of the potential restricted on the equator, while the collinear central configurations are local minima of the potential restricted on a line. For each collinear configuration of the classical n -body problem, there is only one angular velocity to make a circular motion, while any angular velocity leads to circular motion in our case. And it is very interesting to notice that the stability depends on the velocity.

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