Celestial mechanics

Encyclopedia of Nonlinear Science

by Florin Diacu

Although its origins can be traced back in antiquity to the first attempts of explaining the apparently irregular wandering of the planets, celestial mechanics was born in 1687 with the release of Newton's *Principia.* In 1799 Laplace introduced the term *mécanique céleste* [L], which was adopted to describe the branch of astronomy that studies the motion of celestial bodies under the influence of gravity. Celestial mechanics is researched and developed by astronomers and mathematicians, the methods used to investigate it including numerical analysis, the theory of dynamical systems, perturbation theory, the quantitative and qualitative theory of differential equations, topology, the theory of probabilities, differential and algebraic geometry, combinatorics, etc.

Ptolemy's idea of the epicycles—according to which planets are orbiting on small circles, whose centers move on larger circles, whose centers move on even larger circles around the earth—dominated the antiquity and the Middle Ages. In 1543, after working for more than 30 years on a new theory, Copernicus finished writing *De revolutionibus*, a book in which he expressed the motion of the planets with respect to a heliocentric reference system, i.e. one with the sun at its origin. This allowed Kepler to use the existing observations and formulate three laws, published in 1609 in *Astronomia Nova*:

1. *The law of motion*: every planet moves on an ellipse having the sun at one of its foci.

2. *The law of areas*: every planet moves such that the segment planet-sun sweeps equal areas in equal intervals of time.

3. *The harmonic law*: the squares of the periods of any two planets are to each other as the cubes of their mean distances from the sun.

But all these achievements were empirical, based on observations, not on deductions obtained from a more general physical law.

In 1666 Newton came up with the idea that the attractive force responsible for the free fall of objects might be the same as the one keeping the moon on its orbit. He conjectured that the expression of this force is directly proportional with the product of the masses and inversely proportional with the square of the distance between bodies. The tools of calculus, which he had invented independently of—and at about the same time as—Leibniz, allowed him to proceed with the computations. Two decades later, in *Principia*, he proved the correctness of his theory. Kepler's laws follow as consequences. They are obtained from the differential equations of the Newtonian two-body problem (also called the Kepler problem) given by a potential energy of the form $U(r) = -Gm_1m_2/r$, where G is the gravitational constant and r is the distance between the bodies of masses m_1 and m_2.

After Newton, mathematicians like Johann Bernoulli, Clairaut, Euler, d'Alembert, Laplace, Lagrange, Poisson, Jacobi, Weierstrass, Haretu, and many others attacked various theoretical questions of celestial mechanics (e.g. the 2- and the 3-body problem, the lunar problem, the motion of Jupiter's satellites, the stability of the solar system, etc.) mostly with the quantitative tools of analysis, algebra, and the theory of differential equations. On the practical side, the first resounding success in the field was the prediction of Halley's comet return, which occurred in 1758—as the calculations had shown. An even more spectacular achievement came in 1846 with the discovery of the planet Neptune on the basis of perturbation theory through computations independently performed by Adams and Le Verrier. Having its origin in one of Euler's papers, which applied the calculus of trigonometric functions to the 3-body problem, perturbation theory is now an independent branch of mathematics (see e.g. [V], [GH]) that is often used in celestial mechanics.

An important theoretical advance was achieved by Poincaré towards the end of the 19th century, when the questions of celestial mechanics—especially those concerning the Newtonian 3-body problem—received a lot of attention. While working on this problem, Poincaré understood that the quantitative methods of obtaining explicit solutions for differential equations are not strong enough to help him make significant progress, so he tried to describe the qualitative behaviour of orbits (e.g. stability, the motion in the neighbourhood of collisions and at infinity, existence of periodic solutions, etc.) even when their expressions are too complicated or impossible to derive, which is the case in general. His ideas led to the birth of several branches of mathematics, including the theory of dynamical systems, nonlinear analysis, chaos, stability, and algebraic topology [B], [DH].

Today's astronomers working in celestial mechanics are primarily interested in questions directly related to the solar system, like the accurate prediction of eclipses, orbits of comets and asteroids, the motion of Jovian moons, Saturn's rings, artificial satellites, etc. The invention of the electronic computer had a significant impact on the practical aspects of the field. The development of numerical methods allowed researchers to obtain good approximations of the planets' motion for long intervals of time. This type of results are also used in astronautics. No space mission, from the Sputnik, the Apollo, and the Pioneer programmes to the space shuttle, the Hubble telescope launch, and the recent international space collaboration projects, could have been possible without the contributions of celestial mechanics.

Contemporary mathematicians active in the field are mostly dealing with theoretical issues, as for example the study of the general N-body problem and its particular cases [W] ($N = 2, 3, 4$, the collinear, isosceles, rhomboidal, Sitnikov, and planetary problems, central configurations, etc.) trying to answer questions regarding motion near singularities and at infinity, periodic orbits, stability and chaos, oscillatory behaviour, Arnold diffusion, etc. Some researchers also study alternative gravitational forces like the one of Manev [DM], [H], [M], which offers a good relativistic approximation at the level of the solar system.

Celestial mechanics and mathematics have always influenced each other's development, a trend that is far from slowing down today. The contemporary needs of space science bring a new wave of interest in the theoretical and practical aspects of celestial mechanics, making its connections with mathematics stronger than ever before.

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