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Near-Collision Dynamics for Particle Systems with Quasihomogeneous Potentials

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We study collision and ejection orbits of 3-particle systems having the potential $W = U + V$, where U and V are homogeneous functions of degree $-a$ and $-b$, respectively, with $1 \leqslant a < b$. We show that for $b \neq 2$, collision and ejection orbits tend to form asymptotically a central configuration. For the case $b = 2$, which corresponds to Maneff's gravitational law, we find a set of collision and ejection orbits reaching the triple collision manifold without asymptotic phase. This set contains an uncountable union of manifolds and has positive measure within the set of all rectilinear solutions.

1. INTRODUCTION

In a famous paper published in the second decade of our century, Karl Sundman [Su, 1912] proved that triple-collision-ejection orbits of the classical 3-body problem of celestial mechanics, tend to form a central configuration in the neighborhood of the collision-ejection. This property was further shown to be true for partial and simultaneous partial collisions-ejections in the n-body problem [Sp, 1970], [Sa, 1980, 1984], [El, 1990], [D, 1992a]. It was also proved recently that, for the Mücket–Treder gravitational law with logarithmic correction term, triple-collision-ejection orbits also tend to a Newtonian central configuration [D, 1992b].

The goal of this paper is to study collision-ejection orbits of 3-particle systems with potential functions of the form $W = U + V$, where U and V are homogeneous functions of degree $-a$ and $-b$ respectively, with $1 \leqslant a < b$. We call a function W of this kind *quasihomogeneous*. The McGehee transformation technique [M, 1974] will be used to blow-up the triple-collision-ejection singularity and to paste instead a *collision-ejection manifold* to the phase space. We show that for $b \neq 2$ the flow on the collision-ejection manifold is *gradient-like*. Further we define the notion of *central configuration* and see that there exist two triangular and three collinear central configurations for the 3-particle system. The case when the force is directly proportional with the product of the masses is the only one for

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which the triangular configurations are equilateral triangles. Using the gradient-like property we prove that for $b \neq 2$, collision-ejection orbits tend to form a central configuration near collision-ejection.

The case $b = 2$ is treated separately and seems to be the most interesting one. It covers, in particular, the so-called *Maneff* gravitational model given by a nonrelativistic, post-Newtonian law. This model is able to explain with a very good approximation the perihelion advance of the inner planets as well as that of the moon's perigee [Ma, 1924], [Ma, 1925], [Ma, 1930a], [Ma, 1930b]. Newton himself considered (in *Principia*: *Book I*, Prop. XLIV, Corollary 2, as well as in the *Portsmouth Collection* of—during his lifetime—unpublished manuscripts) the central force problem given by this law, in an attempt to clarify the apsidal motion of the moon. This couldn't be satisfactorily explained within the (today classical) model given by a force proportional with the inverse square of the distance.

We consider here the rectilinear problem for $b = 2$, and after regularizing binary collisions we show that the flow on the collision-ejection manifold (which is topologically equivalent to a sphere minus 4 points) has 2 rest points, each having two homoclinic orbits. All the other orbits on the collision-ejection manifold are periodic. Similarly as for $b \neq 2$, there exist triple collision-ejection orbits tending to form asymptotically a central configuration, but besides these we put into the evidence a large class of solutions which tend to the triple collision-ejection after infinitely many binary collisions. They reach the collision-ejection manifold without asymptotic phase, by tending to the periodic orbits of this manifold. Each periodic orbit contains a two-dimensional local stable/unstable manifold of such solutions. (Consequently, the set of initial data leading to rectlinear collisions-ejections has positive measure within the set of initial data of rectilinear solutions.) Among quasihomogeneous potential laws, the case $b = 2$ is the only one having this remarkable property.

2. Equations of Motion and Transformations

Consider 3 particles of masses $m_i > 0$ in the Euclidean space $\mathfrak{R}^3$, having coordinates $\mathbf{q_i} = (q_i^1, q_i^2, q_i^3)$, $i = 1, 2, 3$, in an absolute reference system. Let $\mathbf{q} = (\mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3}) \in \mathfrak{R}^9$ be the *configuration* of the system of particles and define the *quasihomogeneous* potential $W = U + V$, where

$$U\colon \mathfrak{R}^9 \backslash \Delta \to \mathfrak{R}_+, \; U(\mathbf{q}) = \sum_{1 \leqslant i < j \leqslant 3} \alpha(m_i, m_j)\, q_{ij}^{-a},$$

$$V\colon \mathfrak{R}^9 \backslash \Delta \to \mathfrak{R}_+, \; V(\mathbf{q}) = \sum_{1 \leqslant i < j \leqslant 3} \beta(m_i, m_j)\, q_{ij}^{-b}$$

are homogeneous functions of degree $-a$ and $-b$ respectively, with $1 \leqslant a \leqslant b$, $q_{ij} = |\mathbf{q_i} - \mathbf{q_j}|$ is the Euclidean distance between particles i and j, Δ denotes the collision-ejection set

$$\Delta = \bigcup_{1 \leqslant i < j \leqslant 3} \{\mathbf{q} \mid \mathbf{q_i} = \mathbf{q_j}\},$$

and α, β are symmetric positive functions of the masses, i.e. such that $\alpha(m_i, m_j) = \alpha(m_j, m_i) > 0$ and likewise for β.

The equations of motion are given by the system

$$\begin{cases} \dot{\mathbf{q}} = M^{-1}\mathbf{p} \\ \dot{\mathbf{p}} = \nabla W(\mathbf{q}), \end{cases} \tag{2.1}$$

where $M = \operatorname{diag}(m_1, m_1, m_1, m_2, m_2, m_2, m_3, m_3, m_3)$, $\nabla = (\partial_1, \partial_2, \partial_3)$ is the gradient operator and $\mathbf{p} = M\dot{\mathbf{q}}$ denotes the *momentum* of the system. In case $a = b = 1$ and $\alpha(m_i, m_j) = \beta(m_i, m_j) = (G/2)\, m_i m_j$, where G is the gravitational constant, we are in the classical Newtonian 3-body problem. We will therefore be interested in values of a and b with $1 \leqslant a < b$.

Analogous to the Newtonian case, there exist 10 uniform first integrals. Those of the *momentum* and *center of mass* imply that the set $\mathbf{Q} \times \mathbf{P}$ is invariant for the equations (2.1), where

$$\mathbf{Q} = \left\{\mathbf{q} \,\middle|\, \sum m_i \mathbf{q_i} = \mathbf{0}\right\} \quad \text{and} \quad \mathbf{P} = \left\{\mathbf{p} \,\middle|\, \sum \mathbf{p_i} = \mathbf{0}\right\}.$$

From now on we will restrict the equations of motion to the above invariant set, which physically means that the motion is regarded with respect to the center of mass of the 3 particles. We will also use the integral of energy

$$T(\mathbf{p}(t)) - W(\mathbf{q}(t)) = h,$$

where $T\colon \mathfrak{R}^9 \to [0, \infty)$, $T(\mathbf{p}) = \frac{1}{2}\sum_{i=1}^{3} m_i^{-1}\, |\mathbf{p_i}|^2$ is the *kinetic energy* and h is the *energy constant*.

Standard results of the differential equations theory ensure, for given initial data $(\mathbf{q}, \mathbf{p})(0) \in (\mathfrak{R}^9 \backslash \Delta) \times \mathfrak{R}^9$, the existence and uniqueness of an analytic solution $(\mathbf{q}, \mathbf{p})$ of the Eq. (2.1), defined on a maximal interval $[0, t^*)$, $0 < t^* \leqslant \infty$. Analogously one can work with intervals of the form $(t^*, 0]$. In case t^* is finite, the solution is said to experience a *singularity*.

Since our goal is to understand the behavior of triple-collision-ejection and near-triple-collision-ejection solutions of the particle system, we will use McGehee transformations (for details see [M, 1974]) to blow-up the triple collision-ejection singularity. Consider transformations of the form

$$\begin{cases} r=(\mathbf{q}^T M\mathbf{q})^{1/2} \\ \mathbf{s}=r^{-1}\mathbf{q} \\ y=\mathbf{p}^T\mathbf{s} \\ \mathbf{x}=\mathbf{p}-yM\mathbf{s}. \end{cases} \tag{2.2}$$

Notice that $\mathbf{s}^T M\mathbf{s}=1$ and $\mathbf{x}^T\mathbf{s}=0$. Compose further (2.2) with the transformations

$$\begin{cases} v=r^{b/2}y \\ \mathbf{u}=r^{b/2}\mathbf{x} \end{cases} \tag{2.3}$$

and define along a triple-collision-ejection solution the time transformation

$$d\tau=r^{-1-b/2}\,dt. \tag{2.4}$$

Under the transformations (2.2), (2.3), (2.4), which are analytic diffeomorphisms, the equations of motion (2.1) become

$$\begin{cases} r'=rv \\ \mathbf{s}'=M^{-1}\mathbf{u} \\ v'=\dfrac{b}{2}v^2+\mathbf{u}^T M^{-1}\mathbf{u}-r^{b-a}U(\mathbf{s})-bV(\mathbf{s}) \\ \mathbf{u}'=\left(1-\dfrac{b}{2}\right)\mathbf{u}v-(\mathbf{u}^T M^{-1}\mathbf{u})\,M\mathbf{s}+r^{b-a}[\,U(\mathbf{s})\,M\mathbf{s}+\nabla U(\mathbf{s})] \\ \qquad +bV(\mathbf{s})\,M\mathbf{s}+\nabla V(\mathbf{s}), \end{cases} \tag{2.5}$$

where, by abuse, we have maintained the same notations for the new variables. The prime denotes differentiation with respect to the new (fictitious) time variable τ. The integral of energy takes the form

$$\tfrac{1}{2}(\mathbf{u}^T M^{-1}\mathbf{u}+v^2)-r^{b-a}U(\mathbf{s})-V(\mathbf{s})=r^b h. \tag{2.6}$$

Also notice that the sets $\{(r,\mathbf{s},v,\mathbf{u})\mid r=0\}$ and $\{(r,\mathbf{s},v,\mathbf{u})\mid r>0\}$ are invariant manifolds for the Eq. (2.6). We call the set

$$C=\{(r,\mathbf{s},v,\mathbf{u})\mid r=0 \text{ and equation (2.6) holds}\}$$

the *triple-collision-ejection manifold.* Notice that C is pasted to the phase space to replace the triple-collision-ejection singularity and though it is fictitious, the behavior of the flow on C gives information about near-triple-collision-ejection solutions.

3. Central Configurations

Before starting to analyse the new equations of motion, we will deal with the notion of *central configuration.* Consider a potential function of the form

$$\Gamma\colon \mathfrak{R}^9 \to \mathfrak{R}_+, \Gamma(\mathbf{q}) = \sum_{1 \leqslant i < j \leqslant 3} \gamma(m_i, m_j)\, q_{ij}^{-d}, \tag{3.1}$$

where $\gamma(m_i, m_j) = \gamma(m_j, m_i) > 0$, and for our purpose is enough to take $d \geqslant 1$.

A *central configuration* for the particles of masses $m_1, m_2, m_3 > 0$, is a solution $\mathbf{q}_0$ of the equations

$$\nabla\Gamma(\mathbf{q}) = \sigma M \mathbf{q}, \tag{3.2}$$

where $\sigma \neq 0$ is a constant and M is the matrix defined in the previous section. Since homothetic transformations and rotations of the geometric configuration given by $\mathbf{q}_0$ are also central configurations, factorize the set of central configurations to the equivalence relations given by homotheties and rotations. Thus, by a central configuration we usually understand a representative of one such class.

Central configurations play an important role in the study of the classical n-body problem, $n \geqslant 3$. It is known that in the Newtonian 3-body case, there exist exactly five central configurations. Three of them correspond to collinear configurations (one for each ordering of three particles on a non-oriented line) and two correspond to equilateral configurations (one for each possible orientation of a triangle in the plane). We prove here the following generalization of this fact.

Theorem 1. *In case*

$$(\gamma(m_i, m_j)\, m_k)^{1/(d+2)} < (\gamma(m_i, m_k)\, m_j)^{1/(d+2)} + (\gamma(m_k, m_j)\, m_i)^{1/(d+2)} \tag{3.3}$$

for all choices of mutually distinct indices $i, j, k \in \{1, 2, 3\}$, the set of central configurations corresponding to the potential Γ in (3.1) is formed by three collinear configurations and two triangular configurations. Otherwise, it is formed only by the three collinear configurations. Moreover, the case $\gamma(m_i, m_j) = k m_i m_j$, where $k > 0$ is a constant, is the only one giving rise to equilateral configurations.

Proof. As it can be seen from the above statement, the existence of the collinear central configurations doesn't depend on the inequalities (3.3). The proof concerning these configurations is using an idea in [Sa, 1980].

The planar noncollinear case is treated using a different idea. Notice first that conditions (3.2) can be written as

$$\nabla(J\Gamma^{2/d})(\mathbf{q}) = \mathbf{0}, \tag{3.4}$$

where $J(\mathbf{q}) = \frac{1}{2}\sum_{i=1}^{3} m_i\,|\mathbf{q}_i|^2$. However, due to the fact that the configurations are restricted to the invariant set $\mathbf{Q}$ (see Section 2), we have $J(\mathbf{q}) = m^{-1}\sum_{1\leqslant i<j\leqslant 3} m_i m_j q_{ij}^2$, where $m = m_1 + m_2 + m_3$. Consider first the collinear case. Define δ: $\mathbf{Q} \to \mathfrak{R}^3$ by $\delta(\mathbf{q}) = \tilde{\mathbf{q}} = (q_{12}, q_{13}, q_{23})$. Taking $J = \tilde{J}\circ\delta$ and $\Gamma = \tilde{\Gamma}\circ\delta$ one has

$$\nabla(J\Gamma^{2/d})(\mathbf{q}) = \nabla(\tilde{J}\tilde{\Gamma}^{2/d})(\tilde{\mathbf{q}})\cdot D\delta(\mathbf{q}),$$

where $D\delta(\mathbf{q})$ is the Jacobian matrix. This implies that $\mathbf{q}$ is a central configuration if and only if

$$\nabla(\tilde{J}\tilde{\Gamma}^{2/d})(\tilde{\mathbf{q}})\cdot D\delta(\mathbf{q}) = \mathbf{0}. \tag{3.5}$$

We further treat the case when the ordering of the particles on the line is m_1, m_2, m_3. The other two cases are treated in the same way by circular permutations. A simple computation shows that

$$D\delta(\mathbf{q}) = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

In order to be collinear, a central configuration is also subject to the constraint

$$q_{13} = q_{12} + q_{23}. \tag{3.6}$$

Since $\tilde{J}\tilde{\Gamma}^{2/d}$ is a homogeneous function of degree 0, one can fix the scale by setting $\tilde{J} = 1$. This defines a sphere in $\mathfrak{R}^3$ on which every class of central configurations (obtained after factorizing through homotheties and rotations) has a representative. Thus, equations (3.5) can be written as

$$\nabla(\tilde{\Gamma}^{2/d})(\tilde{\mathbf{q}})\cdot D\delta(\mathbf{q}) = \mathbf{0}. \tag{3.7}$$

Observe now that $\tilde{\Gamma}^{2/d}$ is concave up and has a unique minimum point if restricted to the sphere $\tilde{J} = 1$. Writing Eq. (3.7) explicitly and using (3.6), one obtains that the only possible solution for (3.7) is at the minimum of $\tilde{\Gamma}^{2/d}$, i.e. for $\nabla\tilde{\Gamma}^{2/d} = \mathbf{0}$. This implies that for the given ordering of the particles we have a unique collinear central configuration.

Regarding the planar case one can see that conditions (3.4) are now free of geometrical constraints. Since q_{12}, q_{13}, q_{23} are geometrically independent (see [W, 1941], pp. 274–277) one can replace conditions (3.4) by

$$(\nabla \tilde{J}\tilde{\Gamma}^{2/d})(\tilde{\mathbf{q}}) = \mathbf{0}.$$

Computing each condition separately we get

$$q_{ij}^{d+2} = \frac{\gamma(m_i, m_j)}{m_i m_j} K, \quad 1 \leqslant i < j \leqslant 3,$$

where $K = m\tilde{J}(\tilde{\mathbf{q}})/\tilde{\Gamma}(\tilde{\mathbf{q}})$. For having configurations which do not degenerate into a segment, the triangle inequalities $q_{ij} < q_{ik} + q_{kj}$ have to be fulfilled for all choices of mutually distinct indices $i, j, k \in \{1, 2, 3\}$, thus the necessity of relations (3.3) follows. Notice that these relations are not automatically fulfilled. Take, for example, $\gamma(m_i, m_j) = (m_i m_j)^{d+3}$ and $m_1 = 1$, $m_2 = 2$, $m_3 = 7$; this choice will not verify one of the inequalities.

Let us now prove that Eq. (3.5) will always have a unique solution in the noncollinear case. For simplicity change the notations and take $x = q_{12}$, $y = q_{13}$, $z = q_{23}$, $a = \gamma(m_1, m_2)$, $b = \gamma(m_1, m_3)$, $c = \gamma(m_2, m_3)$. Eq. (3.5) are then equivalent to the system

$$\begin{cases} x^{d+2} = \dfrac{a(m_1 m_3 y^2 + m_2 m_3 z^2)}{m_1 m_2 (b y^{-d} + c z^{-d})} \\[2ex] y^{d+2} = \dfrac{b(m_1 m_2 x^2 + m_2 m_3 z^2)}{m_1 m_3 (a x^{-d} + c z^{-d})} \\[2ex] z^{d+2} = \dfrac{c(m_1 m_2 x^2 + m_1 m_3 y^2)}{m_2 m_3 (a x^{-d} + b y^{-d})}. \end{cases} \tag{3.8}$$

Since each equation of this system can be written as $F(x, y, z) = 1$, where F is a homogeneous function of degree 0, look for solutions of the form $(x, y, z) = (x, \mu x, \nu x)$, $\mu, \nu > 0$. This makes system (3.8) equivalent to the system in the unknowns μ, ν:

$$\begin{cases} \zeta_1(\mu) + \zeta_2(\nu) = 0 \\ m_1 m_3 \mu^{d+2}(a + c\nu^{-d}) = b m_2 (m_1 + m_3 \nu^2) \\ m_2 m_3 \nu^{d+2}(a + b\mu^{-d}) = c m_1 (m_2 + m_3 \mu^2), \end{cases} \tag{3.9}$$

where $\zeta_1(\mu) = -m_1 m_2 b \mu^{-d} + a m_3 m_1 \mu^2$ and $\zeta_2(\nu) = -m_1 m_2 c \nu^{-d} + a m_3 m_2 \nu^2$. This system has the unique solution

$$\mu = (b a^{-1} m_2 m_3^{-1})^{1/(d+2)}, \qquad \nu = (c a^{-1} m_1 m_3^{-1})^{1/(d+2)}.$$

In order to prove that it is unique, notice first that ζ_1 and ζ_2 are increasing functions. From the second equation in (3.9) read that

$$\mu(v) = [bm_2(m_1 + m_3 v^2)/m_1 m_3(a + cv^{-d})]^{1/(d+2)},$$

so μ is an increasing function of v. Therefore, $\zeta_1 \circ \mu + \zeta_2$ is an increasing function of v. Since the first equation in (3.9) can now be written as $(\zeta_1 \circ \mu)(v) + \zeta_2(v) = 0$, it follows that system (3.9) cannot have more than one solution. Thus, Eqs. (3.8) have the family of solutions $(x, y, z) = (\lambda, \lambda\mu, \lambda v)$, $\lambda > 0$. The fact that conditions (3.3) must be fulfilled by the solution in order to form a triangle, can now be recovered from the above formulas. In particular, for $\gamma(m_i, m_j) = km_i m_j$, $k \neq 0$, the system has the family of solutions $x = y = z$ and this is the only case when the triangular configurations are equilateral. This completes the proof.

Remark. The shape of both the collinear and triangular configurations depends on the values of d and on the form of γ (and implicitly on the values of the masses, with the exception of the equilateral case).

4. An Asymptotic Property

We return to the study of the Eqs. (2.5). Notice that restricting them to the collision-ejection manifold C, the equations become

$$\begin{cases} \mathbf{s}' = M^{-1}\mathbf{u} \\ v' = \dfrac{b}{2}\, v^2 + \mathbf{u}^T M^{-1}\mathbf{u} - bV(\mathbf{s}) \\ \mathbf{u}' = \left(1 - \dfrac{b}{2}\right) \mathbf{u}v - (\mathbf{u}^T M^{-1}\mathbf{u})\ M_{\mathbf{s}} + bV(\mathbf{s})\, M\mathbf{s} + \nabla V(\mathbf{s}). \end{cases} \tag{4.1}$$

The manifold C is obviously compact if restricted to the *configuration space* (since $\mathbf{s}^T M\mathbf{s} = 1$), but is unbounded in *phase space*. Orbits on C having the property that $\mathbf{s}_i = \mathbf{s}_j$ asymptotically, for some $i \neq j$, become unbounded and will run off the collision-ejection manifold in finite time. Denote the set of these orbits by B. They are in connection with solutions leading to binary collisions, and they will not be our object of study in this general setting. This seems to be a difficult task, so, in the last section of this paper, we will deal only with the particular case of the one-dimensional universe.

The goal of this section is to study only pure triple-collision-ejection orbits (i.e. those that do not encounter other singularities than the triple-collision one). This will make our work easier due to the structure of orbits on $C \backslash B$. The solutions of the equations (4.1) on $C \backslash B$ are globally defined.

Since $C \backslash B$ is an invariant manifold, any solution tending/ejecting to/from it, needs an infinite amount of time to reach $C \backslash B$. Thus, for pure triple collision-ejection solutions, the fictitious time variable τ has the property $|\tau| \to \infty$ when the triple approach is attained.

Notice further that the energy relation for the Eqs. (4.1) takes the form

$$v^2 + \mathbf{u}^T M^{-1} \mathbf{u} = 2V(\mathbf{s}). \tag{4.2}$$

Using (4.2) one obtains $v' = (1 - (b/2))\, \mathbf{u}^T M^{-1} \mathbf{u}$. Therefore, depending on the value of b, v' can be zero or maintain the same sign for all solutions. This observation and the following definition make us distinguish two cases.

DEFINITION. Let X be a metric space and φ: $X \times \mathfrak{R} \to X$ a flow on it. The flow will be called *gradient-like* if the following conditions are fulfilled

(i) The rest points are isolated.

(ii) There exists a continuous function g: $X \to \mathfrak{R}$ such that $g(\varphi(x, t)) < g(x)$ for all $t > 0$, unless x is a rest point.

Remark. Gradient-like flows do not have periodic orbits.

Returning to the Eqs. (4.1) observe that for $b \neq 2$ the flow is gradient-like (with respect to $g(\mathbf{s}, v, \mathbf{u}) = v$ if $b < 2$, and with respect to $g(\mathbf{s}, v, \mathbf{u}) = -v$ if $b > 2$). We will later see that the flow is not gradient-like for $b = 2$.

In this section we restrict the study to the case $b \neq 2$. Our aim is to prove the following result.

THEOREM 2. *If $b \neq 2$, any triple-collision-ejection solution tends to form asymptotically a central configuration.*

Theorem 2 states that if $b \neq 2$, then pure triple-collision orbits tend always to form a central configuration. It cannot happen, for example, that the triangle having the particles at its vertices, collapses to a point such that its sides do not have a limiting position (up to rotations within a central configuration class) when the collision takes place. In order to perform the proof of the theorem we need the following result due to McGehee [M, 1974].

THEOREM 3. *Let X be a locally compact metric space and φ a flow on it. Take $x_0 \in X$ such that its ω-limit, $\omega(x_0) = \bigcap_{t>0} cl\{\varphi(x, [t, \infty))\}$, is a non-empty compact set. If φ restricted to $\omega(x_0)$ is gradient-like, then $\omega(x_0)$ is a single point.*

Proof of Theorem 2. We perform the proof for the case $1 < b < 2$, the case $b > 2$ following with obvious changes. Notice first that the restpoints of the flow associated to the Eqs. (2.5) are characterized by the following conditions

$$\text{(i) } r = 0, \quad \text{(ii) } \mathbf{u} = \mathbf{0}, \quad \text{(iii) } v = \pm(2V(\mathbf{s}))^{1/2}, \quad \text{(iv) } \nabla V(\mathbf{s}) + bV(\mathbf{s})\, M\mathbf{s} = 0.$$

Using the transformations (2.2), (2.3), (2.4) one can see that equation (iv) is equivalent to the condition of being a central configuration (see (3.1)). Condition (i) shows that the rest points belong to the collision-ejection manifold C.

Therefore, in order to prove Theorem 2, it is enough for us to show that triple-collision-ejection solutions will reach $C \backslash B$ with asymptotic phase through the equilibrium solutions of system (2.5). By Theorem 1 and the fact that the flow on $C \backslash B$ is gradient-like, it is enough to show that the ω-limit set of a triple-collision-ejection orbit is a nonempty compact set. To prove this we use an idea of McGehee [M, 1974]. Denote first by $\varphi = (r, \mathbf{s}, v, \mathbf{u})$ a solution of the Eqs. (2.5) ending/beginning in a triple collision-ejection and define the following sets

$$S = \{\phi \mid r \leqslant \varepsilon\} \cap \{\phi \mid (1/2)(\mathbf{u}^T M^{-1} \mathbf{u} + v^2) - r^{b-a} U(\mathbf{s}) - V(\mathbf{s}) = r^b h\},$$
$$G = \{\phi \in S \mid |v| \leqslant \mu\}, \quad G^+ = \{\phi \in S \mid v \geqslant \mu\}, \quad G^- = \{\phi \in S \mid v \leqslant -\mu\}$$
$$\gamma^\pm = \{\phi \in G^\pm \mid r = \varepsilon\}, \quad \sigma^\pm = \{\phi \in S \mid v = \pm\mu\},$$

where $\varepsilon > 0$, $\mu > 0$ are constants. Let $[\tau_1, \tau_2]$, with $\tau_1 < \tau_2$, be a closed interval of the fictitious time variable τ. We call $\phi([\tau_1, \tau_2])$ an *orbit segment*. The orbit segment is said to be maximal in a closed set K, if $\phi([\tau_1, \tau_2]) \subset K$ but $\phi(I) \not\subset K$, for any interval I containing $[\tau_1, \tau_2]$ but larger than it.

The following statements are true:

(i) For ε suitably chosen, if $\phi([\tau_1, \tau_2])$ is a maximal orbit segment in G^+, then $\phi(\tau_1) \in \sigma^+$ and $\phi(\tau_2) \in \gamma^+$.

(ii) For the same ε in (i), if $\phi([\tau_1, \tau_2])$ is a maximal orbit segment in G^-, then $\phi(\tau_1) \in \gamma^-$ and $\phi(\tau_2) \in \sigma^-$.

Let us prove (i). The proof of (ii) works in a similar way. If $\phi(\tau) \in \sigma^+$, then $v = \mu > 0$ and $r \leqslant \varepsilon$. Due to the continuity of the solutions with respect to initial data, since $v' > 0$ on C on nonequilibrium orbits (this happens because $\mathbf{u} = \mathbf{0}$ only at equilibria), it follows that for r sufficiently small, $v' > 0$ for the Eqs. (2.5), on any compact interval of time. On the other hand, for the same equations, $r' = rv > 0$ if $\phi(\tau) \in \sigma^+$. These imply that the points on σ^+ are entering G^+, so $\phi(\tau_2) \in \gamma^+$. Also, points in γ^+ are leaving G^+, so $\phi(\tau_1) \in \sigma^+$. Statement (i) is thus proved.

Recall that for the orbit ϕ leading to a total collapse, $r(\tau) \to 0$ when $\tau \to \infty$. Note that since $v'(\tau) > 0$ for τ finite, we can always choose the value of μ such that ϕ enters G at some finite moment of time. We show further that for the above choice of ε, and for μ chosen such that $\phi(\tau_2) \in G$, it follows that $\phi(\tau) \in G$ for all $\tau \geqslant \tau_2$. Suppose this is not true, i.e. there exits a $\tau_3 \geqslant \tau_2$ such that $\phi(\tau_3) \in G^+$. Then, since $r(\tau) < \varepsilon$ and $r(\tau) \to 0$, it follows that r can not take the value ε for τ large enough. This means that ϕ can never reach γ^+. Thus, by (i), we obtain that $\phi(\tau) \in G^+$ for all $\tau \geqslant \tau_3$. But $v > 0$ in G^+, so $r' = rv > 0$ for $\tau \geqslant \tau_3$. This contradicts the fact that $r(\tau) \to 0$ for $\tau \to \infty$ and consequently $\phi(\tau)$ can not be in G^+ for $\tau \geqslant \tau_2$. In the same way, by (ii), $\phi(\tau)$ can not be in G^- for $\tau \geqslant \tau_2$. Consequently $\phi(\tau) \in G$ for $\tau \geqslant \tau_2$. Since G is compact and any ω-limit set is closed, it follows that the ω-limit set is nonempty and compact. This completes the proof of Theorem 2.

5. Maneff's Gravitational Law and the Case $b = 2$

Between 1924 and 1930, G. Maneff [Ma, 1924], [Ma, 1925], [Ma, 1930a], [Ma, 1930b] proposed and analysed a nonrelativistic gravitational law that can explain very well not only the perihelion advance of the inner planets but is also in agreement with the observations of the moon's perigee. Most of the nonrelativistic models (including the Newtonian one) fail to explain simultaneously these issues [H, 1975, pp. 238–239]. The potential has the form

$$\sum_{i<j} \left[\frac{Gm_i m_j}{q_{ij}} \left(1 + \frac{3G(m_i + m_j)}{2c^2 q_{ij}} \right) \right],$$

where G is the gravitational constant and c is the speed of light. By taking $\alpha(m_i, m_j) = Gm_i m_j$, $\beta(m_i, m_j) = (3G^2/2c^2) m_i m_j (m_i + m_j)$, $a = 1$, and $b = 2$ in the definition of W in Section 2, we have a practical example of a potential with $b = 2$.

We further restrict our study to the rectilinear problem and, for simplicity, we also take $a = 1$. But the results of this section are true for any a, $1 \leqslant a < 2$. The Eqs. (2.5) are thus 8-dimensional and the variables are constrained by 5 equations: the integral of energy (2.6), the relations $\mathbf{s}^T M \mathbf{s} = 1$ and $\mathbf{u}^T \mathbf{s} = 0$, derived from the way the McGehee's transformations have been defined, and the relations $\sum m_i s_i = 0$ and $\sum u_i = 0$, obtained through transformations from the equations defining the invariant set $\mathbf{Q} \times \mathbf{P}$. McGehee's technique will further offer us an idea of how to regularize binary-collision-ejection singularities. For this, take $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ to be points on the sphere $\mathbf{S} = \{\mathbf{s} \mid \mathbf{s}^T M \mathbf{s} = 1\}$ with

$a_1 = a_2 < a_3$ and $b_1 < b_2 = b_3$, in the sense of the ordering of particles on the line ($s_1 \leqslant s_2 \leqslant s_3$). Note that

$$0 < \mathbf{a}^T M \mathbf{b} < 1. \tag{5.1}$$

Choose λ to be the smallest positive number such that

$$\cos 2\lambda = \mathbf{a}^T M \mathbf{b}. \tag{5.2}$$

Consider now the function $S(s) = (\sin 2\lambda)^{-1} [\mathbf{a} \sin(\lambda(1-s)) + \mathbf{b} \sin(\lambda(1+s))]$ and define the transformation

$$\begin{cases} s = S^{-1}(\mathbf{s}) \\ u = \mathbf{s}^T X^T \mathbf{u}, \end{cases} \tag{5.3}$$

where $X\colon \mathbf{Q} \to \mathbf{Q}$, $X = X_1 M/(m_1 + m_2 + m_3) + (m_1 m_2 m_3 / m_1 + m_2 + m_3)^{1/2} M^{-1} X_2$, $M = \operatorname{diag}(m_1, m_2, m_3)$,

$$X_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

Notice that $d/ds\, S(s) = \lambda X S(s)$. Take $\tilde{V} = V \circ S$, $\tilde{U} = U \circ S$ and compute $d/ds\, \tilde{V}(s) = \lambda DV(S(s))\, XS(s)$, $(d/ds)\, \tilde{U}(s) = \lambda DU(S(s))\, XS(s)$, where $DV = (\nabla V)^T$ and $DU = (\nabla U)^T$. Also observe that we have $X^T M X = M$, $\mathbf{a}^T X^T M \mathbf{b} > 0$ and for $\mathbf{s} \in \mathbf{Q}$, $\mathbf{s}^T M X \mathbf{s} = 0$ and $X^2 \mathbf{s} = -\mathbf{s}$. Using the transformations (5.3) and the above relations, Eqs. (2.5) turn into the 4-dimensional system

$$\begin{cases} r' = rv \\ v' = v^2 + u^2 - r\tilde{U}(s) - 2\tilde{V}(s) \\ s' = \lambda^{-1} u \\ u' = r\lambda^{-1} \dfrac{d}{ds} \tilde{U}(s) + \lambda^{-1} \dfrac{d}{ds} \tilde{V}(s), \end{cases} \tag{5.4}$$

having the energy relation

$$\tfrac{1}{2}(u^2 + v^2 - \tilde{V}(s)) = r(rh + \tilde{U}(s)),$$

where

$$\tilde{U}(s) = A \sin 2\lambda \left[\frac{\alpha(m_1, m_2)}{(b_2 - b_1) \sin \lambda(1+s)} + \frac{\alpha(m_2, m_3)}{(a_3 - a_2) \sin \lambda(1-s)} + \frac{\alpha(m_1, m_3)}{(a_3 - a_2) \sin \lambda(1-s) + (b_2 - b_1) \sin \lambda(1+s)} \right],$$

$$\tilde{V}(s) = B \sin^2 2\lambda \left[\frac{\beta(m_1, m_2)}{(b_2 - b_1)^2 \sin^2 \lambda(1+s)} + \frac{\beta(m_2, m_3)}{(a_3 - a_2)^2 \sin^2 \lambda(1-s)} \right.$$
$$\left. + \frac{\beta(m_1, m_3)}{[(a_3 - a_2) \sin \lambda(1-s) + (b_2 - b_1) \sin \lambda(1+s)]^2} \right].$$

Also notice that $s \in (-1, 1)$. Take $R(s) = (1-s^2)^2 \ \tilde{V}(s)$ and observe that this function can be analytically extended to the interval $[-1, 1]$. We continue to denote this extension by R. Consider the transformation

$$\tilde{w} = (1-s^2)^2 \ R^{-1/2}(s) u, \tag{5.5}$$

which is an analytic diffeomorphism, and compose it with the time transformation

$$d\tau = (1-s^2)^2 \ R^{-1/2}(s) \ d\xi, \tag{5.6}$$

where ξ is the new time variable. Rescaling $\tilde{w}$ by $w = \lambda \tilde{w}$ and using the energy relation, the equations of motion (5.4) become

$$\begin{cases} r' = rv \dfrac{(1-s^2)^2}{R^{1/2}(s)} \\[2ex] v' = r(2rh + \tilde{U}(s)) \dfrac{(1-s^2)^2}{R^{1/2}(s)} \\[2ex] s' = \lambda^{-2} w \\[2ex] w' = 4s \left[\dfrac{(1-s^2)^2}{R(s)} v^2 - 1 - s^2 - 2r(rh + \tilde{U}(s)) \dfrac{(1-s^2)^2}{R(s)} \right] \\[2ex] \qquad + \dfrac{d/ds \ R(s)}{R(s)} \left[(1-s^2)^2 - \dfrac{w^2}{2\lambda^2} \right] + r \dfrac{(1-s^2)^4}{R(s)} \cdot \dfrac{d}{ds} R(s). \end{cases} \tag{5.7}$$

Notice that these equations are regular at double collisions-ejections. This becomes clear by observing that the energy relation takes the form

$$w^2 + (1-s^2)^4 \ R^{-1}(s) \ v^2 - 2(1-s^2)^2 = 2r(rh + \tilde{U}(s))(1-s^2)^4 \ R^{-1}(s). \tag{5.8}$$

The collision-ejection manifold (further denoted by C) will also be the intersection of the invariant set $\{r = 0\}$ with the surface given by the energy relation (5.8). In order to reach C, one needs $|\xi| \to \infty$. This is now obvious

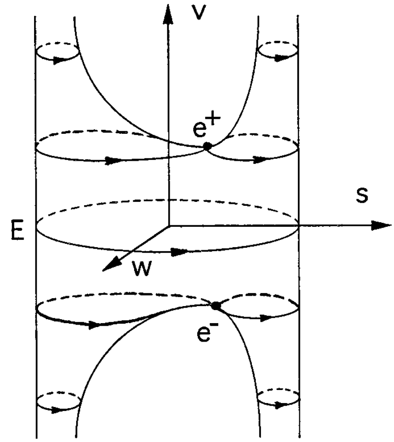

FIG. 1. The flow on the collision manifold for the case $b = 2$.

since all orbits on C are globally defined. Indeed, the flow on C will be given by

$$\begin{cases} v' = 0 \\ s' = \lambda^{-2} w \\ w' = 4s\left[\dfrac{(1-s^2)^2}{R(s)} v^2 - 1 - s^2\right] + \dfrac{d/ds\, R(s)}{R(s)}\left[(1-s^2)^2 - \dfrac{w^2}{\lambda^2}\right], \end{cases} \tag{5.9}$$

and it is regular.

Since the energy relation can be written as

$$\frac{w^2}{2(1-s^2)^2} + \frac{v^2}{2R(s)/(1-s^2)^2} = 1,$$

C is topologically equivalent to a sphere minus four points, like the one in Fig. 1.

Let us see in more detail how the flow on the collision-ejection manifold looks like. In order to have rest points, one necessarily needs $w = 0$. This implies that the equation

$$\frac{4s(1-s^2)^2}{R(s)} k - 4s(1+s^2) + \frac{d/ds\, R(s)}{R(s)} (1-s^2)^2 = 0$$

has to be fulfilled, where $k = v^2 > 0$. In terms of $\tilde{V}$, the above equation becomes

$$\frac{d}{ds} \tilde{V}(s) = \frac{8s}{(1-s^2)^2} \tilde{V}(s) - \frac{4s}{(1-s^2)^2} k, \tag{5.10}$$

which is affine and has the solution $\tilde{V}(s) = k/2$. This implies that the other condition of having a rest point is

$$v = \pm(2\tilde{V}(s))^{1/2}.$$

Introducing this back into (5.10) we get that at a rest point $d/ds\, \tilde{V}(s) = 0$, which, by the above inverse transformations, is equivalent to the condition of being a central configuration. This shows that with respect to the Eqs. (5.7), the only rest points of the flow are at points (r, v, s, w) with $r = 0$, $w = 0$, and $d/ds\, \tilde{V}(s) = 0$, i.e. the points e^+ and e^- on the manifold (see Figure 1). Computing the eigenvalues, one sees that both e^+ and e^- are not hyperbolic, e^+ has a 2-dimensional unstable manifold and a 1-dimensional stable manifold, while e^- has a 2-dimensional stable manifold and a 1-dimensional unstable manifold. Both e^+ and e^- have two homoclinic orbits at the constant level of v corresponding to the $\pm$ minimum value of $\tilde{V}$. Due to the equation $v' = 0$ and the lack of other restpoints on C, the remaining orbits on the collision-ejection manifold are cycles.

This shows that the set Λ of collision-ejection orbits contains a 2-dimensional manifold Λ_1. Orbits in Λ_1 tend/eject to/from the corresponding collinear central configurations e^- and e^+, respectively.

The following natural question arises. Is $\Lambda = \Lambda_1$ as it happens for $b \neq 2$? We will see that this is the case for pure triple-collision-ejection solutions of the one-dimensional universe, but is not true if binary collisions are taken into consideration. The last assertion is obtained by proving the existence of a set of orbits tending/ejecting to/from the periodic orbits of C. More precisely (see Figure 2), there exists a 2-dimensional manifold of solutions ejecting from every periodic orbit north of the equator E (including E), and

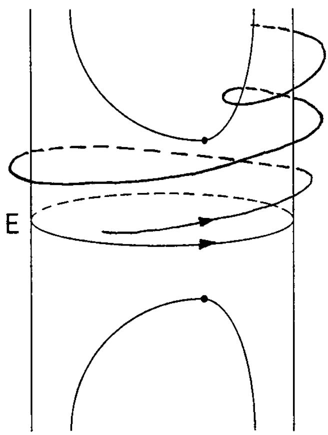

FIG. 2. Orbits ejecting from the equator.

there exists a 2-dimensional manifold of solutions tending to every periodic orbit south of the equator E (including E again). Notice that E is defined as the set $\{r=0\} \cap \{v=0\}$. So, E will be the only periodic orbit to which solutions tend and from which solutions eject. Let us now summarize the results and complete their proofs.

THEOREM 4. *The set Λ of collision-ejection orbits of the rectilinear 3-body problem with $b=2$, and in particular for Maneff's gravitational law, is of the form $\Lambda = \Lambda_1 \cup \Lambda_2$ with $\Lambda_1 \cap \Lambda_2 = \emptyset$. The set Λ_1 contains a 2-dimensional manifold and is formed by all the orbits tending/ejecting to/from an equilibrium (which corresponds to a central configuration). On the other hand $\Lambda_2 = \bigcup_{\mu \in \Re} Z_\eta$, where each Z_η contains a 2-dimensional manifold of solutions tending to the periodic orbit $\{r=0, v=\eta\}$ if $\eta \leqslant 0$, and ejecting from it if $\eta \geqslant 0$.*

Proof. The statement concerning Λ_1 is summarizing the previous remarks of this section. Let us now prove the part refering to Λ_2. For this we use the analyticity of the solutions of (5.7) to describe the first return (Poincaré) map associated to every periodic orbit $\{r=0, v=\eta\}$. Notice first that the last equation in (5.7) can be eliminated due to the energy relation (5.8), and the variable w can be expressed in terms of r, v, and s. Thus, the autonomous system (5.7) is equivalent to the nonautonomous system

$$\begin{cases} \dfrac{dr}{ds} = \lambda^2 r v f(r, v, s) \\ \dfrac{dv}{ds} = \lambda^2 r(2hr + \tilde{U}(s)) f(r, v, s), \end{cases} \tag{5.11}$$

where $f(r, v, s) = [2hr^2 + 2r\tilde{U}(s) - v^2 + 2R(s)/(1-s^2)]^{-1/2}$. The vector field is well defined since the denominator cancels only when r tends to 0 and v^2 tends to $2\tilde{V}(s)$, which happens only when a solution tends to one of the equilibria of the collision-ejection manifold. Excluding this case (which has been previously discussed) we see that the vector field defining (5.11) is regular for all s in $[-1, 1]$. This can be seen by multiplying the numerator and the denominator of each component of the vector field by $1-s^2$. The vector field is also analytic in r, v, s. The solutions of the Eqs. (5.11) are globally defined and periodic in s. The periodicity follows by the way R, $\tilde{U}$, and $\tilde{V}$ are defined, the period being $\Theta = 2\pi/\lambda$.

Developing f in a Taylor series in (r, v) around a point $(0, \eta, s)$, the Eqs. (5.11) become

$$\begin{cases} \dfrac{dr}{ds} = A(s)\, rv + krvF(v-\eta, s) + r^2 Q_1(r, v, s) \\ \dfrac{dv}{ds} = A(s)\, \tilde{U}(s) r + \lambda^2 \tilde{U}(s)\, rF(v-\eta, s) + r^2 Q_2(r, v, s), \end{cases} \tag{5.12}$$

where A, B are periodic in s but are constant with respect to r and v, k is a constant, Q_1, Q_2 are bounded in (r, v) and periodic in s, and $F(v-\eta, s) = \sum_{k=1}^{\infty} (1/k!)\, D_v^k f(0, \eta, s)(v-\eta)^k$. Here, F represents the part of the Taylor expansion free of terms containing r.

Since the Eqs. (5.12) are analytic with respect to the initial data, the first return map around $\{r=0, v=\eta\}$, for a Poincaré section fixed at $s=s_0$, with $-1<s_0<1$, is well defined and is given by

$$\Psi: \begin{pmatrix} r \\ v \end{pmatrix} \to \begin{pmatrix} re^{\Theta A v} + rv \sum_{j=1}^{\infty} \alpha_j (v-\eta)^j\, \Theta^j + r^2 O(r, v) \\ v + ABre^{\Theta A v} + r \sum_{j=1}^{\infty} \beta_j (v-\eta)^j\, \Theta^j + r^2 O(r, v) \end{pmatrix},$$

where Θ is the above computed period of the vector field with respect to s; the quantities $\alpha_j, \beta_j, j \geqslant 1$, and $A, B>0$, are constants. Due to the fact that the solutions of the Eqs. (5.12) are globally defined and analytic in s, and analytic in (r, v) around $(0, \eta)$ for any η, the Poincaré map is well defined for any choice of s_0 as defined above. Notice that the exponential occurs by summing up the series having r as a factor. Observe that $\Psi(0, \eta) = (0, \eta)$ for all real η, so all $(r, v) = (0, \eta)$ are fixed points of the first return map. The linearization of Ψ at $(0, \eta)$ is obtained by computing

$$D\Psi(0, \eta) = \begin{pmatrix} e^{\Theta A \eta} & 0 \\ ABe^{\Theta A \eta} & 1 \end{pmatrix}.$$

This shows that for $\eta>0$, Ψ contains a 1-dimensional unstable manifold, and for $\eta<0$ it contains a 1-dimensional stable manifold. Therefore the corresponding sets Z_η contain 2-dimensional unstable/stable manifolds.

The only completely degenerate case occurs for the fixed point $(0, 0)$, when both eigenvalues of $D\Psi(0, 0)$ are 1. Here we will apply the following result due to Casasayas, Fontich, and Nunez [CFN, Thm. 2.1 & 3.1], which we state in the special case needed here. Let $F=(F_1, F_2)$ be an analytic function defined on some open neighborhood of $(0, 0)$ in $\Re^2$, with values in $\Re^2$, such that: (i) $F(0, y) = (0, y)$; (ii) $DF(0, 0) = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$; (iii) $D_x D_y F_1(0, 0) > 0$. Then there exist stable and unstable invariant manifolds which locally are graphs of analytic functions. More precisely, there exist $\varphi^s: (-\delta, 0] \to [0, \infty)$ and $\varphi^u: [0, \delta) \to [0, \infty)$, defining $W^s_{loc}(\delta) = \{(\varphi^s(y), y), y \in (-\delta, 0]\}$ and $W^u_{loc}(\delta) = \{(\varphi^u(y), y), y \in [0, \delta)\}$. Moreover $\varphi^{s,u}(y) \sim \alpha(y^2/2)$, where $\alpha = D_x D_y F_1(0, 0)$.

This result applies directly to the fixed point $(0, 0)$ of the first return map, if we notice that for a suitably chosen Poincaré section (i.e. for a value of s_0), the quantity $ABe^{\Theta A \eta}$ can be made 1. Thus Ψ has a local stable and a local unstable manifold behaving like a parabola in the neighborhood of a triple collision. This shows that the set Z_0 corresponding to the equator of the collision manifold is composed by a 2-dimensional stable and a 2-dimensional unstable manifold. This completes the proof of the theorem.

COROLLARY 5. *Within the set of initial data of rectilinear solutions, the subset of those leading to triple collisions has positive Lebesgue measure.*

Proof. This is obvious by Theorem 4 since Λ_2 is the uncountable union of sets containing 2-dimensional manifolds. Thus Λ_2 has dimension 3, the same as the phase space of the rectilinear problem.

This shows a significant difference between the Maneff and the Newtonian rectilinear 3-body problem (in the classical case the set of regularized solutions leading to triple collisions is a 2-dimensional manifold, so the set of initial data leading to them has zero measure).

Let us finally give the physical interpretation of triple-collision orbits in Λ_2. We take first a look at orbits tending to the equator E. Supposing we have the ordering $q_1 < q_2 < q_3$, the particles m_1 and m_3 go eventually to the common center of mass of the system while m_2 bounces back and forth between m_1 and m_3. Due to the regularization, every binary collision is elastic, in the sense that it is analytically continued by a binary ejection. The 3 particles approach the triple collision-ejection manifold without asymptotic phase, since m_2 collides infinitely often with both m_1 and m_3. For ejection solutions this scenario is reversed. The same interpretation is given for orbits tending/ejecting to/from periodic orbits in between the equilibria.

Let us see what happens with a solution tending to a periodic orbit on the left lower horn of the collision manifold. In this case the particles m_1 and m_3 also go to the common center of mass, but m_2 encounters infinitely many binary collisions only with m_1 and no collisions at all with m_3. For orbits tending/ejecting to/from periodic orbits of the other horns, the physical interpretation becomes obvious.

Due to the gradient-like property of the flows on C for $b \neq 2$, remark the structurally unstable character of the flow on the collision-ejection manifold for Maneff's gravitational law with respect to the parameter b. This means that the set Λ_2 occurs only for the case $b = 2$ and orbits of this kind are unlikely to occur for gravitational laws obtained by perturbing the Newtonian one with homogeneous functions. Small perturbations within this class break the periodic orbits and make the flow on C increasing or decreasing with respect to the variable v.

Since the whole dynamics of orbits in Λ_2 depends on the regularization technique for binary collisions obtained through the Eqs. (5.7), a last remark is necessary in this sense. In [M, 1981] it is shown that for negative energy levels, the planar 2-body problem defined by the inverse square potential law, doesn't admit regularization with respect to the initial data for binary collision orbits. This property is also true for Maneff's 2-body problem, and it will be proved somewhere else. Binary collisions cannot be regularized because they also occur for nonzero angular momenta, i.e. the point masses can spin around each other before the collision.

Nevertheless, the above regularization technique makes sense within the invariant set of rectilinear solutions. Even from the physical point of view it is plausible to imagine that particles moving on a line will maintain a rectilinear motion after a binary collision. Moreover, binary-collision-ejection orbits can be continuously extended with respect to the initial data within the invariant set of rectilinear solutions, as we have seen above. Unfortunately this cannot be done outside this invariant set, due to the spiraling property mentioned above. This limitation will therefore not allow an extension of the above technique to the planar or the spatial case, so the existence of triple-collision-ejection orbits occuring after infinitely many collisions in the general planar or spatial case looks pointless.

There is, however, another invariant set where the regularization of binary collisions-ejections can be done, namely that of planar isosceles orbits, as it has been shown in [D, 1993]. Let us mention that, due to an insufficiently detailed analysis of the Poincaré map, the existence of 2-dimensional manifolds contained in the set of orbits leading to each periodic orbit on the collision-ejection manifold has not been recognized there. But the method developed in this section also applies to the isosceles case.

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