Chapter 1

Logic

1.1 Statements and Compound Statements

A *statement* or *proposition* is an assertion which is either true or false, though you may not know which. That is, a statement is something that has a *truth value*.

Here are some examples of statements.

- There are no integers a and b so that $\sqrt{2} = \frac{a}{b}$. (True.)
- For all integers $n \geq 0$, the number $n^2 - n + 41$ is prime. (False.)
- Every even positive integer except 2 is the sum of two prime numbers. (Goldbach's Conjecture: unknown.)

We usually use letters to denote statements. A good way to think of these letters is as variables that can take the values "true" and "false". Variables that can take two possible values are sometimes called *Boolean variables*, after the British logician George Boole.

A *compound statement* is one formed by joining other statements together with *logical connectives*. Several such connectives are defined below. The statements that are joined together can themselves be compound statements.

Let p and q be statements.

The **conjunction of** p **and** q (read: p **and** q) is the statement $p \wedge q$ which asserts that p and q are both true. Notice that the wedge symbol looks vaguely like the letter "n" in and.

The **disjunction of** p **and** q (read: p **or** q) is the statement $p \vee q$ which asserts that either p is true, or q is true, or both are true. Notice that this is the inclusive sense of the word "or". Also, the vee symbol looks vaguely like the letter "r" in or.

The **implication** $p \to q$ (read: p **implies** q, or *if* p *then* q) is the statement which asserts that *if* p is true, *then* q is also true. We agree that $p \to q$ is true when p is false. The statement p is called the *hypothesis* of the implication, and the statement q is called the *conclusion* of the implication.

The **biconditional** or **double implication** $p \leftrightarrow q$ (read: p **if and only if** q) is the statement which asserts that p and q if p is true, then q is true, and if q is true then p is true. Put differently, $p \leftrightarrow q$ asserts that p and q have the same truth value.

A *truth table* gives the truth values of a statement for all possible combinations of truth values of the other statements from which it is made. Here and elsewhere, 0 and 1 will represent the truth values "false" and "true", respectively.

p	q	$p \wedge q$	$p \vee q$	$p \to q$	$p \leftrightarrow q$
0	0	0	0	1	1
0	1	0	1	1	0
1	0	0	1	0	0
1	1	1	1	1	1

Examples.

- "$(e < \pi) \wedge (57 \textit{ is prime})$" is false because "$(57 \textit{ is prime})$" is false.
- "$(\sqrt{2} \textit{ is rational}) \vee (\frac{1+\sqrt{5}}{2} < 2)$" is true because "$(\frac{1+\sqrt{5}}{2} < 2)$" is true (the statement "$(\sqrt{2} \textit{ is rational})$" is false).
- "$(5^2 < 0) \to (1 < 2)$" is true because the hypothesis "$(5^2 < 0)$" is false.
- "$(1 < 2) \to (5^2 < 0)$" is false because the hypothesis is true and the conclusion is false. This example demonstrates that $p \to q$ and $q \to p$ are not the same.

- "$(1 = 2) \leftrightarrow$ (*the number of primes is finite*)" is true because both "$(1 = 2)$" and "(*the number of primes is finite*)" are false.

1.2 Negation of Statements

The **negation** of a statement p (read: **not** p) is the statement $\neg p$ which asserts that p is not true. Sometimes it is helpful to think of $\neg p$ as asserting "*it is not the case that p is true*". Thus, $\neg p$ is false when p is true, and true when p is false.

p	$\neg p$
0	1
1	0

Notice that "$\neg$" is not a logical connective. It does not join two statements together. Instead, it applies to a single (possibly compound) statement.

Negation has precedence over logical connectives. Thus $\neg p \vee q$ means $(\neg p) \vee q$.

The negation of $\neg p$ is the statement with the opposite truth value as $\neg p$, thus $\neg(\neg p)$ is just another name for p.

The negation of $p \wedge q$ asserts "it is not the case that p and q are both true". Thus, $\neg(p \wedge q)$ is true exactly when one or both of p and q is false, that is, when $\neg p \vee \neg q$ is true.

Similarly, $\neg(p \vee q)$ can be seen to the same as $\neg p \wedge \neg q$.

Our reasoning can be checked on the truth tables below. Observe that the pairs of statements in question have the same truth value given any combination of possible truth values of p and q.

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$	$p \vee q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
0	0	1	1	0	1	1	0	1	1
0	1	1	0	0	1	1	1	0	0
1	0	0	1	0	1	1	1	0	0
1	1	0	0	1	0	0	1	0	0

To find the negation of $p \rightarrow q$, we return to its description. The statement is false only when p is true and q is false. Therefore $\neg(p \vee q)$ is the same as

$p \wedge \neg q$. Using the same reasoning, or by negating the negation, we can see that $p \rightarrow q$ is the same as $\neg p \vee q$.

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg(p \rightarrow q)$	$p \wedge \neg q$
0	0	1	1	1	0	0
0	1	1	0	1	0	0
1	0	0	1	0	1	1
1	1	0	0	1	0	0

Finally, the statement $p \leftrightarrow q$ asserts that p and q have the same truth value. Hence $\neg(p \leftrightarrow q)$ asserts that p and q have different truth values. This happens when p is true and q is false, or when p is false and q is true. Thus, $\neg(p \leftrightarrow q)$ is the same as $(p \wedge \neg q) \vee (\neg p \wedge q)$.

p	q	$\neg p$	$\neg q$	$p \leftrightarrow q$	$\neg(p \leftrightarrow q)$	$p \wedge \neg q$	$p \wedge \neg q$	$(p \wedge \neg q) \vee (\neg p \wedge q)$
0	0	1	0	0	1	0	0	0
0	1	1	0	1	0	0	1	1
1	0	0	1	1	0	1	0	1
1	1	0	0	0	1	0	0	0

1.3 Making and Using Truth Tables

Recall that the truth values of a statement can be summarized in a truth table. Also recall that "$\neg$" has precedence over logical connectives, and otherwise there is no implied order. This means that $p \vee \neg q$ is $p \vee (\neg q)$, and that $p \vee q \rightarrow r$ is actually $(p \vee q) \rightarrow r$, though it is far better to simply regard the statement as ambiguous and insist on proper bracketing.

To make a truth table, start with columns corresponding to the most basic statements (usually represented by letters). If there are k of these you will need 2^k rows to list all possible combinations of truth values. Then, working with what's inside the brackets first (just like algebra!), add a new column for each connective in the expression, and fill in the truth values using the definitions from before.

The truth table for $(\neg p \rightarrow r) \rightarrow (q \vee \neg r)$ will have 8 rows. Starting with the collection of truth possible values for p, q and r, we add columns to obtain the truth values of $\neg p$, $(\neg p \rightarrow r)$, $\neg r$, $(q \vee \neg r)$, and then, finally, the entire statement we want.

p	q	r	$\neg p$	$\neg p \rightarrow r$	$\neg r$	$q \vee \neg r$	$(\neg p \rightarrow r) \rightarrow (q \vee \neg r)$
0	0	0	1	0	1	1	1
0	0	1	1	1	0	0	0
0	1	0	1	0	1	1	1
0	1	1	1	1	0	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	0	0	0
1	1	0	0	1	1	1	1
1	1	1	0	1	1	0	0

Sometimes only part of the truth table needs to be made. For example, suppose it is given a and b are false, and c is true. Then the truth value of $\neg a \vee (b \vee \neg c)$ can be found by completing the single row of the truth table where a, b and c have the given truth values.

If we are given that p is false and q is true, then we can find all possible truth values of $\neg(p \leftrightarrow r) \rightarrow (q \rightarrow s)$ by completing the four rows of the truth table where p and q have the truth values given, and all possible truth values for r and s occur.

Sometimes information about truth values can be given a more indirectly. Suppose we're given that $\neg a \rightarrow (b \leftrightarrow \neg c)$ is false, and asked to determine all possible truth values of $(a \vee b) \wedge (\neg b \vee \neg c)$. The information that given implication is false, lets us conclude that its hypothesis, $\neg a$, is true (so a is false), and its conclusion, $(b \leftrightarrow \neg c)$, is false (so b and $\neg c$ have different truth values, that is, b and c have the same truth value. Hence we need a truth table with only two rows:

a	b	c	$\neg b$	$\neg c$	$a \vee b$	$\neg b \vee \neg c$	$(a \vee b) \wedge (b \vee \neg c)$
0	0	0	1	1	0	1	0
0	1	1	0	0	1	0	0

Therefore, if $\neg a \rightarrow (b \leftrightarrow \neg c)$ is false, so is $(a \vee b) \wedge (\neg b \vee \neg c)$.

1.4 Converse and Contrapositive

The *converse* of the implication $p \to q$ is $q \to p$. The example above shows that an implication and its converse can have different truth values, and therefore can not be regarded as the same.

The *contrapositive* of the implication $p \to q$ is $\neg q \to \neg p$.

For example, the contrapositive of "*if a and b are odd integers then the integer ab is odd*" is "*if the integer ab is even then it is not the case that the integers a and b are odd*", or equivalently "*if the integer ab is even then the integer a is even or the integer b is even*".

				implication		contrapositive	converse
p	q	$\neg p$	$\neg q$	$p \to q$	$\neg p \vee q$	$\neg q \to \neg p$	$q \to p$
0	0	1	1	1	1	1	1
0	1	1	0	1	1	1	0
1	0	0	1	0	0	0	1
1	1	0	0	1	1	1	1

Notice that the implication $p \to q$ and its contrapositive $\neg q \to \neg p$ have exactly the same truth table, that is, they are each true for exactly the same truth values of p and q. It is reasonable to regard these statements as being "the same" (and we will) in a similar way as we regard 0.25 and 2/8 as being the same.

For completeness, we note that the *inverse* of $p \to q$ is the statement $\neg p \to \neg q$. It is the contrapositive the converse (or the other way around) of $p \to q$.

1.5 Quantifiers

When we make an assertions like "*if* $x^2+3x+2 = 0$ *then* $x = -1$ *or* $x = -2$", the intention is to convey that the assertion holds for every real number x. Similarly, an assertion like "*some rectangles are squares*" is intended to convey that at least one rectangle is a square.

If an assertion contains one or more variables, it isn't possible to know its truth value until something about the variables is known. There are two options:

1. If values are given to the variables, then the assertion will be either true or false for those particular values. Giving different values to the variables might result in a different truth value for the assertion.

2. Specify the quantity (that is, number) of allowed replacements for each variable that result in the assertion being true. This specification is an assertion that is either true or false, that is, it is a statement.

We will pursue the second option.

The *universe* of a variable is the collection of values it is allowed to take.

The *universal quantifier* $\forall$ asserts that the given assertion is true *for all* allowed replacements for a variable. Think of the upside-down "A" as representing "All". Synonyms for "*for all*", include "*all*", "*every*" and "*for each*".

An example of using a universal quantifier is: "*for all integers* n*, the integer* $n(n+1)$ *is even*". We could take a first step towards a symbolic representation of this statement by writing "$\forall n, n(n+1)$ *is even*", and specifying that the universe of n is the integers. (This statement is true.)

The *existential quantifier* $\exists$ asserts that *there exists* at least one allowed replacement for a variable for which the given assertion is true. Think of the backwards "E" as representing "exists". Synonyms for "*there exists*" include "*there is*", "*there are*", "*some*", and "*at least one*".

An example of using an existential quantifier is "*there exists an integer* n *such that* $n^2 - n + 1 = 0$". A symbolic representation of this statement is obtained by writing $\exists n, n^2 - n + 1 = 0$, and specifying that the universe of n is the integers. (This statement is false.)

We can completely write the statement "$\forall n, n(n+1)$ *is even*" in symbols by remembering the definition of an even integer. An integer k is *even* when there is an integer t such that $k = 2t$. Symbolically, k is even when $\exists t, k = 2t$, where the universe of t is the integers. With this in mind "$\forall n, n(n+1)$ *is even*" becomes "$\forall n, \exists t, n(n+1) = 2t$".

Let $s(x)$ denote a statement involving the variable x. Observe that if $\forall x, s(x)$ is true, then so is $\exists x, s(x)$, provided the universe contains a nonzero number of elements: if an assertion is true for all x is the universe, then it is true for at least one x (provided there is one). If the universe contains no elements, then $\forall x, s(x)$ is always true, and $\exists x, s(x)$ is never true (why?). Of course, the truth of $\exists x, s(x)$ tells us nothing about the truth of $\forall x, s(x)$.

Both universal and existential quantifiers can be (unintentionally) hidden, as in the example used to begin this section. Another example is the statement "if $(a \neq 0)$ and $(ax^2 + bx + c = 0)$ then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\text{''}$$

is meant to apply to all real numbers x. If the universal quantifier were made explicit, it would read "for all real numbers $x \ldots$". Similarly, "a real number can have more than one decimal expansion" is intended to assert the existence of one or more such numbers. If the existential quantifier were made explicit, it would read "there is a real number x such that x has more than one decimal expansion".

When quantifiers are nested, they are read in order from left to right. For example, if x and y are understood to be numbers, "$\forall x, \exists y, x + y = 0$" is read as follows: *for all x, the statement "$\exists y, x + y = 0$" is true.* No matter the value of x, the number y can be chosen to be its negative. Hence, $\exists y, x + y = 0$ is true for any x. Consequently, $\forall x, \exists y, x + y = 0$ is true.

The order of quantifiers is important. The statement "$\exists x, \forall y, x + y = 0$" says that there is a real number x such that, for every real number y, the quantity $x + y = 0$, which is false.

1.6 Negating Statements Involving Quantifiers

The negation of a universally quantified statement is an an existentially quantified statement. If it is not the case that a statement is true for all allowed replacements in the universe, then it is false for at least one allowed replacement.

For example "$\neg \forall n \geq 0, n^2 - n + 41$ *is prime*" says "it is not the case that for every positive integer n the number $n^2 - n + 41$ is prime", or in other words "there exists a positive integer n such that $n^2 - n + 41$ is not prime".

In symbols, if $s(x)$ is an assertion involving the variable x (and maybe some other variables and quantifiers) $\neg\forall x, s(x)$ is the same is $\exists x, \neg s(x)$.

The negation of an existentially quantified statement is a universally quantified statement. If it is not the case that a statement is true for at least one allowed replacement in the universe, then it is false for all allowed replacements.

For example "$\neg\exists a, b, \frac{a}{b} = \sqrt{2}$" says "it is not the case that there exists (integers) a and b such that $\frac{a}{b} = \sqrt{2}$", or in other words "for all (integers) a and b, $\frac{a}{b} \neq \sqrt{2}$", that is, "$\sqrt{2}$ is irrational".

In symbols, if $s(x)$ is an assertion involving the variable x (and maybe some other variables and quantifiers) $\neg\exists x, s(x)$ is the same is $\forall x, \neg s(x)$.

Using what we've done above, the statement $\neg\exists x, \forall y, x + y = 0$ is the same as $\forall x, \neg\forall y, x+y = 0$ (take $s(x)$ to be $\forall y, x+y = 0$). In turn, this is the same as $\forall x, \exists y, \neg(x + y = 0)$, or equivalently $\forall x, \exists y, x + y \neq 0$. The latter statement is easily seen to be true. No matter number x is, we can choose y to be any number different than $-x$, and $x + y \neq 0$.

1.7 Logical Equivalence

In the process of making the truth table for $(p \to q) \leftrightarrow (\neg p \vee q)$, we see that the two bracketed statements have the same truth values for given truth value assignments to p and q. Hence the double implication is always true.

A statement which is always true is called a *tautology*. A statement which is always false is called a *contradiction*.

For example, $p \wedge (\neg p)$ is a contradiction, while $p \vee (\neg p)$ is a tautology. Most statements are neither tautologies nor contradictions.

One way to determine if a statement is a tautology is to make its truth table and see if it (the statement) is always true. Similarly, you can determine if a statement is a contradiction by making its truth table and seeing if it is always false.

Informally, two statements s_1 and s_2 are logically equivalent if they have the same truth table (up to the order of the rows). This happens exactly when the statement $s_1 \leftrightarrow s_2$ is a tautology.

Formally, two statements s_1 and s_2 are *logically equivalent* if $s_1 \leftrightarrow s_2$ is a tautology.

We use the notation $s_1 \Leftrightarrow s_2$ to denote the *fact* (theorem) that $s_1 \leftrightarrow s_2$ is a tautology, that is, that s_1 and s_2 are logically equivalent. Notice that $s_1 \leftrightarrow s_2$ is a statement and can in general be true or false, and $s_1 \Leftrightarrow s_2$ indicates the (higher level) fact that it is always true.

Logical equivalence plays the same role in logic that equals does in algebra: is specifies when two expressions are "the same". In the same way that equal expressions can be freely substituted for each other without changing the meaning of an expression, logically equivalent statements can be freely substituted for each other without changing the meaning of a compound statement. And, if two statements are each equivalent to the same statement, they are equivalent to each other.

Since logical equivalence is defined in terms of a statement being a tautology, a truth table can be used to check if (prove that) two statements are logically equivalent. Soon we will have other methods to do this as well.

1.8 The Laws of Logic

We now set out to develop an algebra of propositions. To do so, we need some basic operations (logical equivalences) that can be used. Each of the following can be verified (proved) with a truth table. It is a good idea to memorize them, so that they are at your fingertips when needed.

In what follows, **1** denotes a statement that is always true (i.e. a tautology) , and **0** denotes a statement that is always false (i.e. a contradiction).

- Idempotence: $p \vee p \Leftrightarrow p, \quad p \wedge p \Leftrightarrow p$
- Commutative: $p \wedge q \Leftrightarrow q \wedge p, \quad p \vee q \Leftrightarrow q \vee p$
- Associative: $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r), \quad (p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$
- Distributative: $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r), \quad p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$
- Double Negation: $\neg(\neg p) \Leftrightarrow p$
- DeMorgan's Laws: $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q, \quad \neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$

- Absorbtion: $p \wedge \mathbf{1} \Leftrightarrow p, \quad p \wedge \mathbf{0} \Leftrightarrow \mathbf{0}$
- Dominance: $p \vee \mathbf{1} \Leftrightarrow \mathbf{1}, \quad p \vee \mathbf{0} \Leftrightarrow p$

The following are some other useful logical equivalences.

- $p \rightarrow q \Leftrightarrow \neg p \vee q$
- $p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p) \Leftrightarrow (\neg p \vee q) \wedge (p \vee \neg q)$

It is apparent that the Laws of Logic come in pairs. The *dual* of a statement is obtained by replacing $\vee$ by $\wedge$; $\wedge$ by $\vee$; $\mathbf{0}$ by $\mathbf{1}$; and $\mathbf{1}$ by $\mathbf{0}$, wherever they occur. It is a theorem of logic that if s_1 is logically equivalent to s_2, then the dual of s_1 is logically equivalent to the dual of s_2.

For an example of using the Laws of Logic, we show that $p \leftrightarrow q \Leftrightarrow (p \wedge q) \vee (\neg p \wedge \neg q)$.

$$\begin{array}{lll}
p \leftrightarrow q & & \\
\Leftrightarrow & (\neg p \vee q) \wedge (p \vee \neg q) & \text{Known L.E.} \\
\Leftrightarrow & (\neg p \wedge (p \vee \neg q)) \vee (q \wedge (p \vee \neg q)) & \text{Distributive} \\
\Leftrightarrow & [(\neg p \wedge p) \vee (\neg p \wedge \neg q)] \vee [(q \wedge p) \vee (q \wedge \neg q)] & \text{Distributive (twice)} \\
\Leftrightarrow & [\mathbf{0} \vee (\neg p \wedge \neg q)] \vee [(q \wedge p) \vee \mathbf{0}] & \text{Known contradictions} \\
\Leftrightarrow & (\neg p \wedge \neg q) \vee (q \wedge p) & \text{Dominance} \\
\Leftrightarrow & (p \wedge q) \vee (\neg p \wedge \neg q) & \text{Commutative } (3 \times\ )
\end{array}$$

There are two other forms of the Distributive Laws. These can be derived from the versions given above:

$$\begin{array}{lll}
(q \wedge r) \vee p & & \\
\Leftrightarrow & p \vee (q \wedge r) & \text{Commutative} \\
\Leftrightarrow & (p \vee q) \wedge (p \vee r) & \text{Distributive} \\
\Leftrightarrow & (q \vee p) \wedge (r \vee p) & \text{Commutative (twice)}
\end{array}$$

Similarly $(q \vee r) \wedge p \Leftrightarrow (q \wedge p) \vee (r \wedge p)$.

The Laws of Logic can be used in several other ways. One of them is to prove that a statement is a tautology without resorting to a truth table. This amounts to showing it is logically equivalent to **1**. For example, $\neg q \vee (p \rightarrow q)$ is a tautology because:

$$\begin{array}{llr} \neg q \vee (p \rightarrow q) & & \\ \Leftrightarrow & \neg q \vee (\neg p \vee q) & \text{Known L.E.} \\ \Leftrightarrow & \neg q \vee (q \vee \neg p) & \text{Commutative} \\ \Leftrightarrow & (\neg q \vee q) \vee \neg p & \text{Associative} \\ \Leftrightarrow & \mathbf{1} \vee \neg p & \text{Known tautology} \\ \Leftrightarrow & \mathbf{1} & \text{Dominance} \end{array}$$

Similarly, a statement is proved to be a contradiction when it is shown to be logically equivalent to **0**.

Another use of the Laws of Logic is to "simplify" statements. While the term "simplify" needs to be explained (quantified somehow) to be meaningful, or so we know when we are done, sometimes it is clear that an equivalent expression found is simpler than the one that was started with. For example:

$$\begin{array}{llr} \neg(\neg p \rightarrow q) \vee (p \wedge \neg q) & & \\ \Leftrightarrow & \neg(\neg\neg p \vee q) \vee (p \wedge \neg q) & \text{Known L.E.} \\ \Leftrightarrow & \neg(p \vee q) \vee (p \wedge \neg q) & \text{Double Negation} \\ \Leftrightarrow & (\neg p \wedge \neg q) \vee (p \wedge \neg q) & \text{DeMorgan} \\ \Leftrightarrow & (p \wedge \neg p) \vee \neg q & \text{Dist've (from right to left)} \\ \Leftrightarrow & \mathbf{0} \vee \neg q & \text{Konown contradiction} \\ \Leftrightarrow & \neg q & \text{Absorbtion} \end{array}$$

This section concludes with one last example. Suppose we are asked to show that $(p \wedge q) \wedge [(q \wedge \neg r) \vee (p \wedge r)] \Leftrightarrow \neg(p \rightarrow \neg q)$. Use LHS to denote the expression on the left hand side. Then

$$\begin{array}{llr} LHS & & \\ \Leftrightarrow & [(p \wedge q) \wedge (q \wedge \neg r)] \vee (p \wedge q) \wedge (p \wedge r)] & \text{Distributive} \\ \Leftrightarrow & [((p \wedge q) \wedge q) \wedge \neg r) \vee [((p \wedge q) \wedge p) \wedge r)] & \text{Associative} \\ \Leftrightarrow & [(p \wedge (q \wedge q)) \wedge \neg r) \vee [((p \wedge p) \wedge q) \wedge r)] & \text{Commutative, Associative} \\ \Leftrightarrow & [(p \wedge q) \wedge \neg r) \vee [(p \wedge q) \wedge r)] & \text{Idempotence} \\ \Leftrightarrow & (p \wedge q) \wedge (\neg r \vee r) & \text{Distributive} \\ \Leftrightarrow & (p \wedge q) \wedge \mathbf{1} & \text{Known tautology} \\ \Leftrightarrow & (p \wedge q) & \text{Absorbtion} \\ \Leftrightarrow & \neg\neg(p \wedge q) & \text{Double Negation} \\ \Leftrightarrow & \neg(\neg p \vee \neg q) & \text{DeMorgan} \\ \Leftrightarrow & \neg(p \rightarrow q) & \text{Known L.E.} \end{array}$$

1.9 Using Only And, Or, and Not

It turns out that any statement is logically equivalent to one that uses only the connectives $\wedge, \vee$, and $\neg$. The logical equivalences above allow statements involving the logical connectives $\rightarrow$ and $\leftrightarrow$ to be replaced by equivalent statements that use only $\wedge, \vee$, and $\neg$.

It is also possible to do this directly from the truth table, as will now be demonstrated. Let s be the statement involving p and q for which the truth table is given below.

p	q	s
0	0	1
0	1	1
1	0	0
1	1	1

First, for each row of the truth table where the statement s is true, write a statement that's true only when p and q have the truth values in that row. This statement will involve the logical connective "and". For the truth table above:

- Row 1: $\neg p \wedge \neg q$
- Row 2: $\neg p \wedge q$
- Row 4: $p \wedge q$

Now, to get an expression that's logically equivalent to s, take the disjunction of these statements: it will be true exactly when the truth values of p and q correspond to a row of the truth table where s is true (row 1 or row 2 or row 4). Thus $s \Leftrightarrow (\neg p \wedge \neg q) \vee (\neg p \wedge q) \vee (p \wedge q)$. The process is exactly the same if there are more than two statements involved.

There is some terminology and an important fact (important in computer science) associated with what we have done. The expression associated with each row of the truth table – a conjunction of variables or their negations – is called a *minterm*. The compound statement derived using the process consists of the disjunction of a collection of minterms (that is, they are all joined together using "or"). It is called the *disjunctive normal form* of the statement s. Since every statement has a truth table, and every truth table leads to a statement constructed as above, a consequence of the procedure just described is the theorem that *every statement is logically equivalent to one that is in disjunctive normal form.*

It can be observed directly from the truth table that

$$\begin{aligned} s &\Leftrightarrow p \rightarrow q \\ &\Leftrightarrow \neg p \vee q \quad \text{Known L.E.} \end{aligned}$$

The principle that things that are logically equivalent to the same statement are logically equivalent to each other now implies it should be true that that $(\neg p \wedge \neg q) \vee (\neg p \wedge q) \vee (p \wedge q) \Leftrightarrow \neg p \vee q$. This can be shown with the Laws of Logic.

It is possible to go beyond writing statements so they involve only $\wedge, \vee$, and $\neg$. With careful use of DeMorgan's Laws, really only need $\vee$ and $\neg$, or $\wedge$ and $\neg$, are needed. For example, from before, $p \leftrightarrow q \Leftrightarrow (\neg p \vee q) \wedge (\neg q \vee p)$. By DeMorgan's Law, $(\neg p \vee q) \wedge (\neg q \vee p) \Leftrightarrow \neg\big(\neg(\neg p \vee q) \vee \neg(\neg q \vee p)\big)$, so $p \leftrightarrow q \Leftrightarrow \neg\big(\neg(\neg p \vee q) \vee \neg(\neg q \vee p)\big)$. The latter statement uses only $\vee$ and $\neg$. If you use DeMorgan's Law in a different way, then you can get an expression for $p \leftrightarrow q$ than involves only $\wedge$ and $\neg$.

One can go a bit farther and introduce the logical connective "nand" (not and), so that "p nand q" is the statement $\neg(p \wedge q)$. It transpires that any proposition can be expressed (in a possibly complicated way) using only "nand". The same thing applies to "nor", where "p nor q" is the statement $\neg(p \vee q)$.

1.10 Logical Implication

It is apparent from examining the truth table for $a \wedge b$ that, if this statement is true, then so is a. Since an implication is true by definition when the hypothesis is false, this means that the statement $(a \wedge b) \rightarrow a$ is a tautology. If, in the midst of an argument, we were to discover that $a \wedge b$ is true, we would be entitled to conclude (infer, or deduce) that a is true (and the same for b). In what follows we develop a collection of basic rules for making inferences.

Informally statement, a statement p logically implies a statement q if the truth or p guarantees the truth of q This happens exactly when $p \rightarrow q$ is a tautology. Note that we are not concerned about what happens if p is false. This is because of the truth table for implies: $p \rightarrow q$ is true (by definition) when p is false.

Formally, we say p *logically implies* q when $p \rightarrow q$ is a tautology.

We use the notation $p \Rightarrow q$ to denote the fact (theorem), that $p \rightarrow q$ is a tautology, that is, that p logically implies q. Notice that $p \rightarrow q$ is a statement and can in general be true or false, and $p \Rightarrow q$ indicates the (higher level) fact that it is always true.

Suppose $p \Leftrightarrow q$. Then $p \leftrightarrow q$ is a tautology. Since $p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$, the latter statement is also a tautology. Using the reasoning in the first paragraph of this section, this means that each of $(p \rightarrow q)$ and $(q \rightarrow p)$ is a tautology. Therefore $p \Rightarrow q$ and $p \Leftarrow q$ (which has the obvious intended meaning: $q \Rightarrow p$). In the same way, if both $p \Rightarrow q$ and $p \Leftarrow q$, then $p \Leftrightarrow q$.

1.11 Valid Arguments and Inference Rules

An *argument* is an implication $(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow q$. The statements $p_1, p_2, \ldots, p_n$ are called *premises*, and the statement q is called the *conclusion*. Put differently, an argument is an assertion. Since the truth table for implies says that an implication is true when its hypothesis is false, and since the hypothesis is the conjunction of all of the premises, the assertion being made is that *if the premises are all true, then so is the conclusion.*

An argument is called *valid* if the implication is a tautology (i.e., if the premises logically imply the conclusion, so that the conclusion is guaranteed to be true when all of the hypotheses are true), otherwise it is *invalid*.

To show that an argument is invalid, it needs to be demonstrated that the implication is not a tautology. From the truth table for implies, this amounts to describing a single row of a truth table where each premise is true and the conclusion is false. Such a collection of truth values is called a *counterexample* to the argument.

Arguments are usually presented in the tabular format shown below for the example $[(p \rightarrow \neg q) \wedge (\neg r \rightarrow p) \wedge q] \rightarrow r$. The premises are listed first, and then the conclusion is listed below a separating line.

$$\begin{array}{rl} & p \rightarrow \neg q \\ & \neg r \rightarrow q \\ \hline \therefore & \neg r \rightarrow p \end{array}$$

The argument given above is invalid. To demonstrate that, we need to give a counterexample: a truth assignment for p, q and r such that the premises are all true and the conclusion is false. The best way to get started is to find the truth assignments that make the conclusion false. Here there is only one: the statement $\neg r \rightarrow p$ is false only when $\neg r$ is true (so r is false) and p is false. Now we want to choose a truth value for q, if necessary, so that all of the premises are true. When p is false, the implication $p \rightarrow \neg q$ is true. When r is false, the implication $\neg r \rightarrow q$ is true only when q is true. Thus, if p, q, r have the truth values $0, 1, 0$, respectively, the premises are all true and the conclusion is false. Therefore, the argument is not valid.

A truth table can, in principle, be used to show an argument is valid. But, if the number of premises involved is large, so is the table. A better way is to give a *proof*: a chain of logical equivalences and implications involving the premises (which are assumed to be true because an implication is true when its hypothesis is false). The idea is that every statement you write down is true, and is either a premise, or an allowed additional hypothesis, or is derived from statements known to be true via logical equivalences and implications.

Our ultimate goal is to write mathematical proofs in words. Proving logical implications using inference rules and logical equivalences is a step towards that goal.

The two following *inference rules* are each a logical implication. They are just common sense, but can be formally proved with a truth table. These will get used frequently in arguments and hence need to be at your fingertips, so they should be memorized.

- Modus Ponens: $(p \rightarrow q) \wedge p \Rightarrow q$
- Chain Rule (Law of Syllogism): $(p \rightarrow q) \wedge (q \rightarrow r) \Rightarrow p \rightarrow r$

We use these inference rules to prove some other rules. The rules above are worth memorizing. The rules below are easy consequences of them and need not be remembered.

Modus Tollens: $[(p \to q) \wedge \neg q] \Rightarrow \neg p$.

Proof.

1.	$p \to q$	Premise
2.	$\neg q \to \neg p$	L.E. to 1
3.	$\neg q$	Premise
4.	$\therefore\ \neg p$	$2, 3,$ M.P.

Disjunctive Syllogism: $[(p \vee q) \wedge \neg p] \Rightarrow q$.

Proof.

1.	$p \vee q$	Premise
2.	$\neg p \to q$	L.E. to 1
3.	$\neg p$	Premise
4.	$\therefore\ q$	$2, 3,$ M.P.

Resolution: $[(p \vee r) \wedge (q \vee \neg r)] \Rightarrow p \vee q$.

Proof.

1.	$p \vee r$	Premise
2.	$\neg p \to r$	L.E. to 1
3.	$q \vee \neg r$	Premise
4.	$\neg r \vee q$	3, Commutative
5.	$r \to q$	L.E. to 4
6.	$\neg p \to q$	$2, 5,$ Chain Rule
7.	$p \vee q$	L.E. to 6

Here are two more inference rules which are clearly true, and which can be formally proved with a truth table.

- Disjunctive Amplification: $p \Rightarrow p \vee q$
- Conjunctive Simplification: $p \wedge q \Rightarrow p$

Here is another example of using inference rules to prove an argument is valid.

$$\begin{array}{r} \neg p \leftrightarrow q \\ \neg q \rightarrow r \\ p \\ \hline \therefore\ r \end{array}$$

Proof.

1.	$\neg p \leftrightarrow q$	Premise
2.	$(\neg p \rightarrow q) \wedge (q \rightarrow \neg p)$	L.E. to 1
3.	$q \rightarrow \neg p$	2, Conjunctive Simplification
4.	$p \rightarrow \neg q$	3, Contrapositive
5.	$\neg q \rightarrow r$	Premise
6.	$p \rightarrow r$	$4, 5,$ Chain Rule
7.	p	Premise
8.	r	$6, 8,$ M.P.

In the next example, the argument is given in words. We can still check its validity using inference rules. First, we need to translate the argument into symbols.

If I run, then my ankle does not hurt
If I am not injured, then I run
My ankle hurts

$\therefore$ I am injured

Let $p, q,$ and r denote the statements "I run", "My ankle hurts", and "I am injured", respectively. Then the argument is:

$$\begin{array}{l} p \rightarrow \neg q \\ \neg r \rightarrow p \\ q \\ \hline \therefore\ r \end{array}$$

This argument is valid (unlike most stories about sports injuries). We can prove it using inference rules.

1.	$p \to \neg q$	Premise
2.	$\neg r \to p$	Premise
3.	$\neg r \to \neg q$	2, 1, Chain Rule
4.	$q \to r$	3, Contrapositive
5.	q	Premise
6.	$\therefore\ r$	4, 5, M.P.

We conclude this section with two more inference rules that can be proved with a truth table, and then some discussion about them.

- Proof by Contradiction: $(\neg p \to \mathbf{0}) \Rightarrow p$
- Proof by Cases: $(p \to r) \wedge (q \to r) \Rightarrow (p \vee q) \to r$

The rule "Proof by Contradiction" is best illustrated by a proof in words. An example will be given in the next section. The idea behind the rule is that one should only be able to obtain true statements when starting with true statements, and using logical equivalences and logical implications. Hence, if falsity of the desired conclusion leads to a statement that is never true (that is, a contradiction), then the conclusion can not be false. Here, we illustrate the use of this rule in a proof of the type above by giving a second proof of the rule "Resolution".

Alternate proof of the rule "Resolution".

1.	$\neg(p \vee q)$	Negation of conclusion, for proof by contradiction
2.	$\neg p \wedge \neg q$	1, DeMorgan
3.	$\neg p$	2, Conjunctive Simplification
4.	$\neg q$	2, Conjunctive Simplification
5.	$p \vee r$	Premise
6.	$\neg p \to r$	L.E. to 5
7.	r	3, 6, M.P.
8.	$q \vee \neg r$	Premise
9	$\neg q \to \neg r$	L.E. to 8
10.	$\neg r$	4, 9, M.P.
11.	$r \wedge \neg r \quad (\Leftrightarrow \mathbf{0})$	Known contradiction from 7, 10
12.	$p \vee q$	1, 11, Proof by Contradiction

The rule "Proof by Cases" is also best illustrated by a proof in words. It was mentioned in this section (as with proof by contradiction) to illustrate that the proof methods we will use have a basis in logic. The intuition for proof by cases is simple enough. If the truth of p guarantees the truth of a conclusion, r, and the truth of q guarantees the truth of r, and one of p and q must be true, then r must be true. The way this rule is applied is that *if one of several cases must arise, and the desired conclusion holds in each case, then the premises logically imply the conclusion.* An example is given in the next section.

1.12 Proofs in Words

Suppose you want to write a proof in words for a statement of the form "if p then q". That is, you wish to establish the theorem $p \Rightarrow q$. There are many techniques (methods) that can be tried. There is no guarantee of which method will work best in any given situation. Experience is a good guide, however. Once a person has written a few proofs, s/he gets a sense of the best thing to try first in any given situation.

To use the method of *direct proof* to show p logically implies q, *assume p is true* and then *argue using definitons, known implications and equivalences that q must be true.* The reason for assuming p is true comes from the definition of logical implication. In this case the first line of the proof is "*Assume p.*" and the last says, essentially, "*q is true*". What comes in between depends on p and q.

In the following example of a direct proof, we use the definition of an even integer: An integer n is *even* if there exists an integer k so that $n = 2k$.

Proposition 1.12.1 *If the integer n is even, then n^2 is even.*

Proof. Suppose that the integer n is even. Hence, there exists an an integer k so that $n = 2k$. Then, $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$. Since $2k^2$ is an integer, n^2 is even. □

It is customary in mathematics to use a box to indicate the end (or absence) or an argument.

Another proof technique is to *prove the contrapositive*. That is, assume q is false, and argue using the same things as above that p must also be false. This works since $p \to q \Leftrightarrow \neg q \to \neg p$. In this case the first line of the proof is "*Assume* $\neg q$." and the last is, essentially, "$\neg p$ *is true*". This method is sometimes called giving an *indirect proof*. The motivation for the name comes from the fact that the logical implication is proved indirectly, by its contrapositive.

In the following example of proving the contrapositive, we use the definition of an odd integer: An integer n is *odd* if there exists an integer k so that $n = 2k + 1$.

Proposition 1.12.2 *If the integer n^2 is even, then n is even.*

Proof. Suppose that the integer n is not even, that is, it is odd. We want to show that n^2 is odd. Since n is odd, there exists an an integer k so that $n = 2k + 1$. Then, $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. Since $2k^2 + 2k$ is an integer, n^2 is odd. □

Yet another technique is *proof by contradiction*. Such a proof begins by assuming q is false and, again proceeding as above, until deriving a statement which is a (logical) contradiction. This enables you to conclude that q is true. In such a situation, the first line of the proof is "Suppose $\neg q$." and the proof ends with " We have obtained a contradiction. Therefore q."

Here is a classic example of proof by contradiction. It uses the definition of a rational number: a number x is *rational* if there exist integers a and b so that $x = a/b$. A number is *irrational* if it is not rational.

Put slightly differently, x is rational if it is a ratio of two integers. There are many ratios of integers that equal a given number. In particular, there is always one where the fraction a/b is in *lowest terms*, meaning that a and b have no common factors other than one.

Proposition 1.12.3 $\sqrt{2}$ *is not irrational.*

Proof. Suppose $\sqrt{2}$ is rational. Then there exist integers a and b so that $\sqrt{2} = a/b$. The integers a and b can be chosen so that the fraction a/b is in lowest terms, so that a and b have no common factor other than 1. In particular, a and b are not both even.

Since $\sqrt{2} = a/b$, we have that $2 = (a/b)^2 = a^2/b^2$. By algebra, $2b^2 = a^2$. Therefore a^2 is even. By Proposition 1.12.2, a is even. Thus there exists an integer k so that $a = 2k$. It now follows that $2b^2 = a^2 = (2k)^2 = 4k^2$, so that $b^2 = 2k^2$. Therefore b^2 is even. By Proposition 1.12.2, b is even.

We have now derived the contradiction (a and b are not both even) and (a and b are both even). Therefore, $\sqrt{2}$ is not rational. □

Sometimes the hypotheses lead to a number of possible situations, and it is easier to consider each possibility in turn. In the method of *proof by cases*, one lists the cases that could arise (being careful to argue that all possibilities are taken into account), and then shows that the desired result holds in each case. It could be that different cases are treated with different proof methods. For example, one could be handled directly, and another by contradiction.

In the following example we make use the fact that every integer n can be uniquely written in the form $3k + r$, where k is an integer and r equals 0, 1, or 2. The integer r is the *remainder on dividing n by 3*. When the remainder equals 0 we have $n = 3k$, so that n is a multiple of 3.

Proposition 1.12.4 *If the integer n^2 is a multiple of 3, then n is a multiple of 3.*

Proof. We prove the contrapositive: if n is not a multiple of 3, then n^2 is not a multiple of 3. Suppose n is not a multiple of 3. Then the remainder when n is divided by 3 equals 1 or 2. This leads to two cases:

Case 1. The remainder on dividing n by 3 equals 1.

Then, there exists an integer k so that $n = 3k+1$. Hence $n^2 = (3k+1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$. Since $(3k^2 + 2k)$ is an integer, the remainder on dividing n^2 by 3 equals 1. Therefore n^2 is not a multiple of 3.

Case 2. The remainder on dividing n by 3 equals 2.

Then, there exists an integer k so that $n = 3k+2$. Hence $n^2 = (3k+2)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$. Since $(3k^2 + 4k + 1)$ is an integer, the remainder on dividing n^2 by 3 equals 1. Therefore n^2 is not a multiple of 3.

Both cases have now been considered. In each of them, we have shown that n^2 is not a multiple of 3. It now follows that if n is not a multiple of 3, then n^2 is not a multiple of 3. This completes the proof. □