The Principle of Inclusion and Exclusion

The Principle of Inclusion and Exclusion, hereafter called PIE, gives a formula for the size of the union of n finite sets. Usually the universe is finite too. It is a generalisation of the familiar formulas $|A \cup B| = |A| + |B| - |A \cap B|$ and $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.

Theorem PIE. If $P_1, P_2, \ldots, P_n$ be are finite sets, then $|P_1 \cup P_2 \cup \cdots \cup P_n| = \big(|P_1| + |P_2| + \cdots + |P_n|\big) - \big(|P_1 \cap P_2| + |P_1 \cap P_3| + \cdots |P_{k-1} \cap P_n|\big) + \big(|P_1 \cap P_2 \cap P_3| + |P_1 \cap P_2 \cap P_4| + \cdots + |P_{n-2} \cap P_{n-1} \cap P_n|\big) + \ldots + (-1)^n |P_1 \cap P_2 \cap \ldots \cap P_n|$

That is, the cardinality of the union $P_1 \cup P_2 \cup \ldots \cup P_k$ can be calculated by *including* (adding) the sizes of all of the sets together, then *excluding* (subtracting) the sizes of the intersections of all pairs of sets, then including the sizes of the intersections of all triples, excluding the sizes of the intersections of all quadruples, and so on until, finally, the size of the intersection of all of the sets has been included or excluded, as appropriate. If n is odd it is included, and if n is even it is excluded. The formula can be expressed more compactly as

$$|P_1 \cup P_2 \cup \cdots \cup P_n| = \sum_{i=1}^{k} (-1)^k \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} |P_{i_1} \cap P_{i_2} \cap \cdots \cap P_{i_k}|.$$

It is important to remember that all sets involved must be finite.

To prove PIE, both sides count only elements that belong to some positive number of the sets. Each of these is counted once on the LHS. To determine the number of times it is counted on the RHS, suppose it belongs to $t \ge 1$ of the sets, and calculate the contribution it makes to each intersection. You'll end up making use of the Binomial Theorem expansion of $(1 + (-1))^t$.

When to use PIE. Vaguely speaking, you should try PIE when you are trying to count something described by a bunch of conditions, any number of which might hold at the same time, and you can't see how to organise the counting by cases. Often PIE is used in conjunction with *counting the complement*. That is, you use it to count the number of objects in the universe that you don't want, and subtract this from the size of the universe (which needs to be finite!).

In applying PIE, the setup is of great importance. You need to be clear about what the sets are (what it means to belong to one or more of them), what the universe is, and how the principle gives you what you want. Once you've done this, things often reduce to more or less straightforward counting problems.

Example PIE 1. *Count the number of sequences of 10 distinct letters that contain none of THE, MATH, and QUIZ.*

Here we're are asked to count sequences of 10 distinct letters that have some special properties, so a good choice for the universe is the set of all sequences of 10 distinct letters. Thus, $|\mathcal{U}| = 26!/16!$.

Let's try to set up some sets whose union will contain the sequences we don't want, and then we can determine the number that we do want by subtraction. Let A, B and C be the set of all sequences in $\mathcal{U}$ that contain THE, MATH, and QUIZ, respectively. Then, by the definition of union, $A \cup B \cup C$ is the set of all sequences that contain at least one of THE, MATH, and QUIZ, so the number we want is $|\mathcal{U}| - |A \cup B \cup C| = 26!/16! - |A \cup B \cup C|$, where the last term can be determined using PIE. Each of the terms on the RHS of $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$ can be determined by a straightforward counting argument, so long as we are clear about what it means to belong to each intersection. For example, a sequence belongs to $B \cap C$ if and only if it contains both MATH and QUIZ. To determine the number of such sequences, use glue. After doing the counting, we find that $|A \cup B \cup C| = \binom{23}{7}8! + \binom{22}{6}7! + \binom{22}{6}7! - \binom{21}{5}6! - \binom{19}{3}5! - \binom{18}{2}4! + \binom{17}{1}3!$. Thus, the number of sequences of 10 distinct letters that contain none of THE, MATH, and QUIZ equals $26!/16! - \binom{23}{7}8! - \binom{22}{6}7! - \binom{22}{6}7! + \binom{21}{5}6! + \binom{19}{3}5! + \binom{18}{2}4! - \binom{17}{1}3!$.

Example PIE 2. *Determine the number of integer solutions to* $x_1 + x_2 + x_3 + x_4 \leq 70$ *such that* $1 \leq x_1 \leq 12,\ 0 \leq x_2 \leq 10,\ -3 \leq x_3 \leq 13,\ 5 \leq x_4 \leq 35$.

After making a change of variables (let $y_1 = x_1 - 1$, etc.) and adding a slack variable, this is equivalent to determining number of integer solutions to $y_1 + y_2 + y_3 + y_4 + y_5 = 67$ such that $0 \leq y_1 \leq 11,\ 0 \leq y_2 \leq 10,\ 0 \leq y_3 \leq 16,\ 0 \leq y_4 \leq 30$, and $y_5 \geq 0$.

Under this setup we're asked to count non-negative integer solutions to an equation that satisfy certain constraints. This suggests that the universe should be the set of all non-negative integer solutions to $y_1 + y_2 + y_3 + y_4 + y_5 = 67$. Thus, $|\mathcal{U}| = (67+4)!/(67!4!) = \binom{67+4}{4}$ (use bars and stars).

Let's set up some sets whose union will contain the solutions we don't want, that is, the ones that violate at least one of the upper bound constraints. Then, the number of solutions we do want can be obtained by subtraction. To do this, let

Y_1 be the set of solutions where $y_1 \geq 12$,

Y_2 be the set of solutions where $y_2 \geq 11$,

Y_3 be the set of solutions where $y_3 \geq 17$, and

Y_4 be the set of solutions where $y_4 \geq 31$.

(There are only four sets because there is no upper bound constraint on y_5.) Then, by definition of union, $Y_1 \cup Y_2 \cup Y_3 \cup Y_4$ is the set of solutions where one or more constraints are violated. Thus we want $|\mathcal{U}| - |Y_1 \cup Y_2 \cup Y_3 \cup Y_4|$, where the last term is determined by PIE.

Again, the counting is straightforward once it is clear what it means for a solution to belong to an intersection of several of the sets. For example, a solution belongs to

$Y_1 \cap Y_3 \cap Y_4$ if and only if $y_1 \geq 12,\ y_3 \geq 17$, and $y_4 \geq 31$.

After computing the sizes of the various intersections (using bars and stars), the answer is $\binom{67+4}{4} - \left[\binom{59}{4} + \binom{60}{4} + \binom{54}{4} + \binom{40}{4}\right] + \left[\binom{48}{4} + \binom{42}{4} + \binom{28}{4} + \binom{33}{4} + \binom{29}{4} + \binom{23}{4}\right] - \left[\binom{31}{4} + \binom{17}{4} + \binom{11}{4} + \binom{12}{4}\right] + 0$.

Example PIE 3. *Determine the number of ways for n couples to stand in a line so that no one stands beside her/his partner.*

Here we are counting arrangements of $2n$ people in a line satisfying certain conditions, which suggests that the universe should be the set of all such arrangements. Thus, $|\mathcal{U}| = (2n)!$.

On phrasing the conditions slightly differently, we see that there are n conditions that must be satisfied: couple 1 is not standing together, couple 2 is not standing together, ..., couple n is not standing together. This suggests we should set up n sets so that set i contains the arrangements in which the i-th condition is violated. That is, for $i = 1, 2, \ldots, n$, let X_i be the set of arrangements in which couple i is standing together. Then, $X_1 \cup X_2 \cup \cdots X_n$ is the set of arrangements where one or more couples are standing together, so the number we want is $|\mathcal{U}| - |X_1 \cup X_2 \cup \cdots X_n|$. The last term can be determined using PIE.

The counting is straightforward. An arrangement belongs to $X_{i_1} \cap X_{i_2} \cap \cdots \cap X_{i_k}$ if and only if couples $i_1, i_2, \ldots, i_k$ are each standing together. It is easy to see that the size of an intersection depends not on which couples are involved, but on *how many* couples are involved. Thus, $|X_{i_1} \cap X_{i_2} \cap \cdots \cap X_{i_k}| = 2^k(2n - 2k + k)!$ (use glue, and don't forget to include the glues couples among the objects to arrange).

Since for each k the size of the intersection of k of the sets depends only on k, and since there are $\binom{n}{k}$ ways of choosing k of the n sets to intersect, the answer is $(2n)! - \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} 2^k (2n-k)!$. Looking at the answer, it is possible to recognise that the first term, $(2n)!$ is of the same form as the terms in the summation but with $k = 0$. That is $(2n)! = (-1)^0 \binom{n}{0} 2^0 (2n - 0)!$. Hence it can be absorbed into the sum to get a single summation that describes the answer. We must also remember to look after the minus sign in front of the sum, i.e., put it together with the $(-1)^{k-1}$ to get $(-1)^k$. Afeter doing so, the answer is $\sum_{k=0}^{n} (-1)^k \binom{n}{k} 2^k (2n-k)!$.

Example PIE 3 brings out **some important points that should be remembered**:

• It might help in setting up the sets if the conditions were rephrased so it is clear how they apply to the objects under consideration. Statements like "none of these n objects satisfy condition C" can (and often should) be rephrased as n conditions: "object 1 does not satisfy condition C", "object 2 does not satify condition C" and so on.

• There are $\binom{n}{k}$ possible intersections of k out of n sets. If the size of the intersections in question depends only on the number of sets involved (not which sets are involved), then the term which is included or excluded is $\binom{n}{k}$ times the size of the intersection of k sets. Note that $|X_i|$ can be thought of as the size of an intersection of one set.

• In cases where the size of the intersections in question depends only on the number of sets involved, and where you are "counting the complement", it is often possible to recognise that the first term, $|\mathcal{U}|$, is of the same form as the terms in the summation but with the index of summation equal to zero. In such a case, it can be absorbed into the sum to get a single summation that describes the answer.

Two classic examples of quantities that can be counted using PIE are onto functions and derangements. Since these are thoroughly discussed in most textbooks, (and in class) they will not be dealt with in detail here. Since you should be able to do these calculations, a very brief discussion of each topic is included.

Example PIE 4. *Find the number of functions from* $\{x_1, x_2, \dots x_m\}$ *onto* $\{y_1, y_2, \dots, y_n\}$.

Remember that a function f from $\{x_1, x_2, \dots x_m\}$ to $\{y_1, y_2, \dots, y_n\}$ is *onto* if every $y \in \{y_1, y_2, \dots, y_n\}$ is $f(x)$ for some $x \in \{x_1, x_2, \dots x_m\}$. (Notice how this corresponds to rephrasing the conditions, as discussed above.)

Now if $m < n$ the answer is zero, so assume $m \geq n$. We're counting functions, so the universe should be the set of all functions from $\{x_1, x_2, \dots x_m\}$ to $\{y_1, y_2, \dots, y_n\}$. To figure out $|\mathcal{U}|$, remember how to count functions.

For $i = 1, 2, \dots, n$ let P_i be the set of functions where y_i is not $f(x)$ for any $x \in \{x_1, x_2, \dots x_m\}$. (Notice how this uses the "new" description of the conditions to set up the sets.) Then $P_1 \cup P_2 \cup \cdots \cup P_n$ is the set of functions that are not onto, so we want $|\mathcal{U}| - |P_1 \cup P_2 \cup \cdots \cup P_n|$. After counting, and recognising that the second and third important points also apply here, the answer is $\sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m$.

Problems in which every thing in a first collection of *distinct* objects is assigned to some thing in a second collection of *distinct* objects can be thought of as problems involving functions. (The functions in question are from the first collection of objects to the second, and if x is an object in the first collection, then $f(x)$ should be the object in the second collection to which it is assigned.) If the assignments are such that no object in the second collection (the one "being assigned to") is left out, then the problem involves onto functions, and hence you can write the solution down without resorting to PIE. For example, the number of ways that Dr. M. can assign 7 different projects to his research team consisting of himself and 3 graduate students equals the number of functions from the set of 7 projects onto the 4 people, which is $\sum_{k=0}^{4} (-1)^k \binom{4}{k} (4-k)^7$. If he must keep 3 projects for himself and distribute the remainder to his students, then the number of ways is $\binom{7}{3} \sum_{k=0}^{3} (-1)^k \binom{3}{k} (3-k)^4$ (choose the 3 he keeps as the first step, then proceed as above).

A **derangement** of a set $X = \{x_1, x_2, \ldots, x_n\}$ is a permutation $y_1 y_2 \ldots y_n$ of the elements of X so that $y_i \neq x_i$ for $i = 1, 2, \ldots, n$.

Example PIE 5. *Determine $d(n)$, the number of derangements of $X = \{x_1, x_2, \ldots, x_n\}$.*

We're counting permutations, so the universe should be the set of all permutations of $x_1, x_2, \ldots, x_n$. For $i = 1, 2, \ldots, n$, let P_i be the set of permutations $y_1 y_2 \ldots y_n$ in which $y_i = x_i$. Then we want $|\mathcal{U}| - |P_1 \cup P_2 \cup \cdots \cup P_n|$. This turns out to equal $\sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)!$.

It is sometimes the case that a given problem can be recognised as one involving derangements, and thus solved without resorting to PIE. For example, suppose 8 people check their coat and their hat at an event. In how many ways can these be returned so that no one gets back either of his posessions? There are $d(8)$ ways to return the coats so that no one gets his own coat, and for each of these there are $d(8)$ ways to return the hats so that no one gets his own hat. Thus the answer is $d(8)^2 = \left[\sum_{k=0}^{8} (-1)^k \binom{8}{k} (8-k)!\right]^2$.

Curiosity. Notice that $d(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)! = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} (n-k)! = n! \sum_{k=0}^{n} (-1)^k \frac{1}{k!}$. ¿From calculus we know that $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. Thus, $e^{-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$. On comparing this with the final summation we have for $d(n)$, we notice that $d(n)$ equals $n!$ times the sum of the first $n+1$ terms (it starts at 0) in a series that converges (quickly!) to e^{-1}. Thus $d(n)/n! \approx e^{-1}$ or, equivalently, $d(n) \approx n!/e$. The approximation is quite good from $n = 5$ onwards. This says, for example, that if n people check their coat at a party and they are returned at random the probability that no one gets his own coat back is about e^{-1}, and thus (essentially) independent of n.