Big-O, Big-Theta, and Big-Omega

Memorize: Suppose $f : \mathbf{Z} \to \mathbf{R}$ and $g : \mathbf{Z} \to \mathbf{R}$ are functions. We say f is $O(g)$ if there exists constants C and k so that $|f(n)| \leq C|g(n)|$ for *all* $n > k$.

In other words, f is $O(g)$ if it is never larger than a constant times g for all large values of n. The function $Cg(n)$ gives an upper bound on the size of $f(n)$ for all large values of n. Usually the expression for g is less complex for the expression for f, and that's one of the things that makes big-O notation useful. Notice that we don't care what happens for "small" values of n. Also, usually we don't worry too much about the absolute value signs since we usually compare functions that take positive values.

To prove f is $O(g)$ using the definition you need to find the constants C and k. Sometimes the proof involves mathematical induction (for instance in showing that n^2 is $O(2^n)$), but often it just involves manipulation of inequalities. What I recommend doing in the latter case is starting with $f(n)$ and writing a chain of inequalities that ends with $\leq Cg(n)$. Some of these inequalities will only be true when n is greater than some lower limit. The largest of these limits is the k you want.

Example 1. *Prove that* $5n^2 - 2n + 16$ *is* $O(n^3)$. Consider $5n^2 - 2n + 16 \leq 5n^2 + 16$ (if $n \geq 0$) $\leq 5n^2 + n^2$ (if $n \geq 4$) $= 6n^2 \leq 6n^3$ (if $n \geq 1$). Thus, if we take $C = 6$ and $k = 4$ in the definition, the above calculation demonstrates that $5n^2 - 2n + 16$ is $O(n^3)$.

To prove f is not $O(g)$ you need to argue that the C and k required by the defintion do not exist. Usually you go about this by *assuming they do exist* (i.e., assuming f is $O(g)$) and *arguing that this leads to a contradiction.*

Example 2. *Prove that* $5n^2 - 2n + 16$ *is not* $O(n)$. Assume $5n^2 - 2n + 16$ is $O(n)$. Then there exist constants C and k so that $5n^2 - 2n + 16 \leq Cn$ for all $n > k$. Dividing both sides by n (and assuming $n > 0$) we get $5n - 2 + 16/n \leq C$, or $n \leq C + 2 - 16/n \leq C + 2$. This inequality does not hold for $n > C + 2$, contrary to our assumption that it held for all large values of n. Therefore $5n^2 - 2n + 16$ is not $O(n)$.

Things you should know. There is no need to memorize the bounds on n; you can always work them if you have to.

- $1 \leq \log(n)$ if $n \geq 10$.
- $\log n \leq n^\alpha$ if $\alpha > 0$ (the bound on n depends on α).
- $n^\alpha \leq t^n$ for $\alpha > 0$ and $t > 1$ (the bound on n depends on α and t).
- $s^n \leq t^n$ if $1 \leq s \leq t$ and $n \geq 1$.
- $t^n \leq n!$ (the bound on n depends on t).
- $n! \leq n^n$ if $n \geq 1$.

Another importat fact is that the base of logarithms is not important. Any two log functions are related by multiplication by a constant. The formula is $\log_a(n) = \log_b(n)\log_a(b)$ (so the constant is $\log_a(b)$). To prove this, start with $a^{\log_a(n)} = b^{\log_b(n)}$ (both of these equal n), take logs to base a of both sides and simplify using the rules of logarithms.

Fact B1. If f is $O(g)$ and g is $O(h)$, then f is $O(h)$.

You should know how to prove this Fact. It implies that if f is $O(g)$, then it is also Big-O of any function "bigger" than g. This is why one is typically interested in finding a "best possible" big-O expression for f (i.e., one that can not be replaced by a "smaller" function).

Usually one can guess a "best possible" big-O estimate for a function by first throwing away all constants, and second keeping only the biggest term in the expression. (You still need to prove that your guess is correct.) For example applying these guidelines to $f(n) = 10 \cdot 2^n n^2 + 17n^3 \log(n) - 500$ suggests that a best possible big-O form is $O(2^n n^2)$.

A quick way to decide if f is $O(g)$ is to use limits. It turns out that f is $O(g)$ if

$$\lim_{n \to \infty} \frac{|f(n)|}{|g(n)|}$$

exists and is finite. The proof that this works involves the definition of the limit of a function, which we have not studied. Most often you will be asked to prove f is $O(g)$ using the definition, so this method will not be accepted.

Memorize: Suppose $f : \mathbf{Z} \to \mathbf{R}$ and $g : \mathbf{Z} \to \mathbf{R}$ are functions. We say f is $\Omega(g)$ if there exists constants C and k so that $|f(n)| \geq C|g(n)|$ for *all* $n > k$.

Big-Ω is just like big-O, except that $Cg(n)$ is now a lower bound for $f(n)$ for all large values of n. All of the same comments and proof techniques as above apply except the inequalities are in the other direction.

Fact B2. A function f is $\Omega(g)$ if and only if g is $O(f)$.

You should know how to prove this Fact, and should also be able to use it in arguments involving big-Ω.

Memorize: Suppose $f : \mathbf{Z} \to \mathbf{R}$ and $g : \mathbf{Z} \to \mathbf{R}$ are functions. We say f is $\Theta(g)$ if f is $O(g)$ and f is $\Omega(g)$.

In other words, a function f is $\Theta(g)$ if and only if there are constants C_1 and C_2 so that $C_1 g(n) \leq f(n) \leq C_2 g(n)$ for all large vales of n.

Fact B3. A function f is $\Theta(g)$ if and only if f is $O(g)$ and g is $O(f)$.

You should know how to prove this Fact, and should also be able to use it in arguments involving big-Θ.

It turns out that f is $\Theta(g)$ if

$$\lim_{n \to \infty} \frac{|f(n)|}{|g(n)|}$$

exists, is finite, and is not equal to zero. Most often, however, you will be asked to show f is $\Theta(g)$ directly, so this method will not apply.

Here are a couple of facts you should be able to prove:

- A polynomial is big-Θ of its largest term.
- For any integer k, $1^k + 2^k + \ldots + n^k$ is $\Theta(n^{k+1})$.