Counting

The best way to learn about counting is to do lots of problems. What I'll try to do here is outline the guiding principles and techniques. Only experience will help you decide what to do when.

Rule of Sum. The number of outcomes of a process which consists of a collection of mutually exclusive cases is the sum of the number of outcomes of the cases.

Rule of Product. The number of outcomes of a process which is a sequence of steps is the product of the number of outcomes of the steps, *provided that in each step the number of outcomes is not different depending on the outcome of a previous step.*

The italicized part is important. If you encounter a situation where the number of outcomes of the current step depends on the outcome of a previous step, then you probably need to either re-order the steps, or break some part of the process into several cases. As and example, try counting the number of even numbers between 1000 and 9999 inclusive that have distinct digits. (Answer: $9 \times 8 \times 7 \times 1 + 8 \times 8 \times 7 \times 4$.)

When I count the number of ways to do something, I imagine making it as a sequence of steps. In the example above, I imagine making the number by filling in each of its digits (4 steps). One usually must make a judicious choice of the ordering of the steps. Here, it is wise to "fill in" the ones digit first.

Forced choices. When the outcome of a step is determined in advance, the number of outcomes of that step is 1. For example, the number of numbers between 1000 and 9999 inclusive that end in a seven is $9 \times 10 \times 10 \times 1$.

Counting the complement. Sometimes it is easier to count the number of outcomes you don't want, and subtract this from the total number of outcomes. This technique is useful in problems with "at least" or "at most". As an example, try counting the number of numbers between 1000 and 9999 inclusive that have at least one zero. (Answer: $9 \times 10^3 - 9^4$.)

Another way to deal with problems involving "at least k" or "at most k" is to break into cases. In the previous example, one could the consider the three cases: exactly one 0, exactly two 0s, and exactly three 0s. (Answer: $3 \times 1 \times 9^3 + 3 \times 9 \times 1 \times 1 \times 9 + 9 \times 1 \times 1 \times 1$.)

Inclusion - Exclusion. The number of outcomes where A happens or B happens is the number where A happens, plus the number where B happens, minus the number where both A and B happen. For example, try counting the number of numbers between 1000 and 9999 inclusive that are divisible by 7 or 11. (Answer: $\left(\lfloor\frac{9999}{7}\rfloor - \lfloor\frac{999}{7}\rfloor\right) + \left(\lfloor\frac{9999}{11}\rfloor - \lfloor\frac{999}{11}\rfloor\right) + \left(\lfloor\frac{9999}{77}\rfloor - \lfloor\frac{999}{77}\rfloor\right)$.)

The above principle comes from the following fact about sets: For finite sets X and Y,

$$|X \cup Y| = |X| + |Y| - |X \cap Y|.$$

To see the correspondence, let X be the set of outcomes where A happens, and Y be the set of outcomes where B happens.

Glue. This is used in arrangement problems where you want several objects to be together. First, you glue them together (and count the number of ways). Then you arrange the glued object and the other objects (and count the ways). For example, suppose there are 3 boys and 4 girls lining up for a photo, and you want to count the number of arrangements in which the boys stand together. First, glue the boys together: $3 \times 2 \times 1$ possible orders. Then, arrange the five things you have (the 4 girls, and one object consisting of the 3 boys stuck together) in a line: $5 \times 4 \times 3 \times 2 \times 1$ ways. Thus, the total number of arrangements is $3 \times 2 \times 1 \times 5 \times 4 \times 3 \times 2 \times 1$.

Overcounting. If you can devise a procedure that counts every object you want the same number of times, say k, then the number of distinct objects equals the number of outcomes of the procedure divided by k. As an example, try counting the number of arrangements of the letters in PEPPER. Temporarily call the Ps P_1, P_2 and P_3, and the Es E_1 and E_2. Then we have 6 distinct ojects, so the number of arrangements is 6!. Now rub out the subscripts on the Ps and Es, and each arrangement of the letters appears 3!2! times (for each fixed arrangement of PEPPER there are 3!2! ways of calling the the Ps P_1, P_2 and P_3, and the Es E_1 and E_2). Thus the number of arrangements of the letters in PEPPER is 6!/3!2!.

n choose k. For integers $n \geq k \geq 0$, we define $\binom{n}{k}$ (read: n choose k) to be the number of ways of selecting a collection of k distinct objects from a collection of n distinct objects *without regard for the order in which they are chosen.*

It turns out that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. You should be able to derive this formula. You do this by counting the number of ways to line up k out of n people in two different ways: either choose the k first and then line them up, or construct the lineup from left to right by selecting the next person from among those not already lined up.

$\binom{n}{0} = \binom{n}{n} = 1$, $\binom{n}{1} = \binom{n}{n-1} = n$, and in general $\binom{n}{k} = \binom{n}{n-k}$. You should be able to explain each of these in English without resorting to the formula for $\binom{n}{k}$. The key to the last one is that deciding which objects to take from a collection is the same as deciding which objects to leave behind.

Pascal's Identity. If $n \geq k \geq 1$ then $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. You should be able to prove this without using the formula for $\binom{n}{k}$. To do this count the number of ways to select k children out of n to take on a trip in two ways. The first way is to just select the k children, and the second is to count the number of collections that include Gary, and the number that don't.

Arranging non-distinct objects: another way. First determine (for yourself) the number of objects of each type. Then, choose which places will contain the objects of the first type, which of the remaining places will contain objects of the second type, and so on until all places are filled. As an example, the number of arrangements of the letters in PEPPER is $\binom{6}{3}\binom{3}{2}\binom{1}{1}$. When expanded using the formula for $\binom{n}{k}$ and simplified this equals $\frac{6!}{3!2!1!}$.

In general, if you have n objects, of which n_1 are of type 1, n_2 are of type 2, ..., n_t are of type t, then counting as above by choosing the the number of places that hold objects of each type and simplifying, implies that the number of arrangements is $\frac{n!}{n_1!n_2!\cdots n_t!}$.

Separation. To count the number of outcomes where no two objects of a certain type are together, first arrange the other objects (and count the ways), and then choose which of the places "inbetween" (don't forget the beginning and end) will contain the objects you want separated. As an example, the number of arrangements of the letters in STATISTICS in which no two Ss are adjacent is $\frac{7!}{3!2!}\binom{8}{3}$.

Some things in a fixed order. To count the number of arrangements of objects in which some of them ("special ones") must appear in a particular order, first change the "special ones" to empty boxes and arrange the remaining objects and the boxes (and count the ways). Then, fill in the boxes with the special objects in the desired order (and count the ways). As an example, to count the number of arrangements of ABELIAN in which the vowels are in alphabetical order, first change the vowels to boxes, and arrange 4 boxes and B, L, N: 7!/4! ways. Now fill in the boxes with A, A, E, I in the desired order: 1 way. This the number of arrangements is 7!/4!.

Arranging objects in a circle. Here you want to know the number of ways to arrange a collection of objects around a circle subject to the condition that arrangements that differ by a rotation of the circle are considered the same. Determine the number of spaces that need to be filled, and the number of ways to fill them. This counts every arrangement a certain number of times, so divide (as in Overcounting, above). For example, the number of seatings of n people around a circular table is $n!/n = (n-1)!$. This method generalises to arranging objects around other shapes (for example, squares), and to allowing equivalence under other symmetries like flips. There is another method that involves rotating the circle until a fixed object is at the top – this then imposes an order on the remaining objects – but the method does not generalize to other shapes.

Selections of objects which are not all distinct. The basic premise is that the number of non-negative integer solutions to $x_1 + x_2 + x_3 + \cdots + x_k = n$ is equal to the number of ways to place n identical balls into k labelled boxes, which is equal to the number of sequences of n stars (*) and $k-1$ bars (|). The correspondence in the former case is to let x_i equal the number of balls in box $i, 1 \leq i \leq k$, and in the latter case it is to let the number of stars before the first bar equal the number of balls in box 1, the number of stars between the first and second bar equal the number of balls in box 2, the number of stars between the second and third bar equal the number of balls in box 3, and so on until, finally, the number of stars following the last bar equals the number of balls in box k. You can imagine the n stars as the balls, and the k spaces created by the $k-1$ bars (including at the beginning and end) as the k boxes. The number of solutions is then the number of arrangements of these $n+k-1$ non-distinct objects, and is $\frac{(n+k-1)!}{n!(k-1)!} = \binom{n+k-1}{n} = \binom{n+k-1}{k-1}$.

For example, consider counting the number of ways to order 12 soft drinks chosen from Coke, Pepsi, Mountain Dew, and Ginger Ale. All that matters here is how drinks many of each type are ordered. Let x_1 equal the number of Cokes ordered, x_2 equal the number of Pepsis ordered, x_3 equal the number of Mountain Dews ordered, and x_4 equal the number of Ginger Ales ordered. Since each of these numbers is a non-negative integer, we want the number of integers solutions to $x_1 + x_2 + x + 3 + x_4 = 12$, subject to $x_i \geq 0,\ 1 \leq 1 \leq 4$. There are $\frac{(12+3)!}{12!3!}$ of these.

To deal with counting the number of non-negative integer solutions to inequalities like $x_1 + x_2 + \cdots + x_k \leq n$, introduce a new variable x_{k+1} (called a *slack variable*) whose value will be $n - (x_1 + x_2 + \cdots + x_k)$ (i.e., the slack in the inequality). There is then a 1-1 correspondence between non-negative integer solutions to $x_1 + x_2 + \cdots + x_k \leq n$ and non-negative integer solutions to $x_1 + x_2 + \cdots + x_{k+1} = n$.

To deal with constraints on the variables which are different than non-negativity, convert the equation into one involving only non-negativity constraints. For example, the number of solutions in integers to $x_1 + x_2 + x_3 = 15$ subject to $x_1 \geq -1,\ x_2 > 4, x_3 \geq 2$ is equal to the number of solutions in integers to $(x_1+1)+(x_2-5)+(x_3-2) = 15+1-5-2$ subject to $x_1 + 1 \geq -1 + 1,\ x_2 - 5 > 4 - 5,\ x_3 - 2 \geq 2 - 2$. Let $y_1 = x_1 + 1$, $y_2 = x_2 - 5$, and $y_3 = x_3 - 2$. Then the problem becomes counting the number of integer solutions to $y_1 + y_2 + y_3 = 9$ subject to $y_i \geq 0,\ 1 \leq i \leq 3$.