## Indexed Collections

Let $I$ be a set (finite of infinite). If for each element $i$ of $I$ there is an associated object $x_i$ (exactly one for each $i$) then we say the collection $x_i,\ i \in I$ is *indexed* by $I$. Each element $i \in I$ is called and *index* and $I$ is called the *index set* for this collection.

The definition is quite general. Usually the set $I$ is the set of natural numbers or positive integers, and the objects $x_i$ are numbers or sets. It is important to remember that there is no requirement for the objects associated with different elements of $I$ to be distinct. That is, it is possible to have $x_i = x_j$ when $i \neq j$. It is also important to remember that there is exactly one object associated with each index.

Technically, the above corresponds to having a set $X$ of objects and a function $i : I \to X$. Instead of writing $i(x)$ for element of $x$ associated with $i$, instead we write $x_i$ (just different notation).

**Examples of indexed collections**.

1. With each natural number $n$, associate a real number $a_n$. Then the indexed collection $a_0, a_1, a_2, \ldots$ is what we usually think of as a *sequence* of real numbers.

2. For each positive integer $i$, let $A_i$ be a set. Then we have the indexed collection of sets $A_1, A_2, \ldots$.

**Sigma notation**. Let $a_0, a_1, a_2, \ldots$ be a sequence of numbers. The notation $\sum_{i=k}^{n} a_i$ stands for the sum $a_k + a_{k+1} + a_{k+2} + \cdots a_n$.

If $k > n$, then $\sum_{i=k}^{n} a_i$ contains no terms. An empty sum (a sum containing no terms) is defined to be zero. One reason is because adding an empty sum to anything should not change it (the anything).

Here are some summation formulas you should know, and also should know how to prove. Sometimes the proof is by induction, sometimes it is by other means.

- $\sum_{i=1}^{n} i = n(n+1)/2$.
- $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$.
- $\sum_{i=1}^{n} i^3 = n^2(n+1)^2/4$.
- $\sum_{i=0}^{n} (a_{j+1} - a_j) = a_{n+1} - a_0$ (telescoping sum).
- $\sum_{i=0}^{n} ar^n = (ar^{n+1} - a)/(r-1)$ (sum of a geometric series).

**Pi notation**. Let $a_0, a_1, a_2, \ldots$ be a sequence of numbers. The notation $\prod_{i=k}^{n} a_i$ stands for the product $a_k \cdot a_{k+1} \cdot a_{k+2} \cdots a_n$.

If $k > n$, then $\prod_{i=k}^{n} a_i$ contains no terms. An empty product (a product containing no terms) is defined to be one. One reason is because multiplying any number by an empty product should not change the number.

**Unions and Intersections**. Let $A_1, A_2, \ldots$ be a collection of sets. The set $A_k \cup A_{k+1} \cup \cdots \cup A_n$ is the set of all $x$ which belong to at least one of $A_k, A_{k+1}, \ldots, A_n$ and $A_k \cap A_{k+1} \cap \cdots \cap A_n$

is the set of all $x$ which belong to all of $A_k, A_{k+1}, \ldots, A_n$. The notation $\cup_{i=k}^{n} A_i$ stands for the union $A_k \cup A_{k+1} \cup \cdots \cup A_n$, and $\cap_{i=k}^{n} A_i$ stands for the intersection $A_k \cap A_{k+1} \cup \cdots \cap A_n$.

The union of an empty collection of sets is defined to be $\emptyset$, while the intersection of an empty collection of sets is defined to be the universal set $\mathcal{U}$. You should be able to supply a justification for these definitions, as was done above.

Generalised DeMorgan's Laws. You should **memorize** these, and know how to prove them.

$$\overline{\cup_{i=1}^{n} A_i} = \cap_{i=1}^{n} \overline{A_i}$$

and

$$\overline{\cap_{i=1}^{n} A_i} = \cup_{i=1}^{n} \overline{A_i}$$

The index set does not have to be a set of integers. As an example, suppose for each real number $x$ we let $A_x$ be the open interval $(-x, 2x)$. Notice that this is a set of real numbers. In this case we might use notation like $\cup_{x<1.47} A_x$ to stand for the intersection of all sets $A_x$ such that the given condition (i.e., propositional function of $x$: here, $x < 1.47$) is true for $x$ (The union or intersection of an arbitrary collection of sets is defined simailarly to what we have above. You should be able to formulate this definition precisely.)

More generally, if $P(t)$ is a propositional function, the notation $\sum_{P(t)} a_t$ stands for the sum of all numbers $a_t$ for which $P(t)$ is true. Similarly, $\cup_{P(t)} A_t$ and $\cap_{P(t)} A_t$ stand for the union and intersection of all sets $A_t$ for which $P(t)$ is true, respectively.

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