The Pigeonhole Principle

It is easy to see, and follows immediately from Proposition F2 in the Functions section, that no function from a set of size at least $k+1$ to a set of size k can be 1-1. Thus, some two elements of the domain have the same image. The Pigeonhole Principle, in its simplest form, is a rephrasing of this statement.

Proposition PHP1. (The Pigeonhole Principle, simple version.) If $k+1$ or more pigeons are distributed among k pigeonholes, then at least one pigeonhole contains two or more pigeons.

Proof.

The contrapositive of the statement is: If each pigeonhole contains at most one pigeon, then there are at most k pigeons. This is easily seen to be true. ∎

The same argument can be used to prove a variety of different statements. We prove the general version of the Pigeonhole Principle and leave the others as exercises.

Proposition PHP2. (The Pigeonhole Principle.) If n or more pigeons are distributed among $k > 0$ pigeonholes, then at least one pigeonhole contains at least $\lceil \frac{n}{k} \rceil$ pigeons.

Proof.

Suppose each pigeonhole contains at most $\lceil \frac{n}{k} \rceil - 1$ pigeons. Then, the total number of pigeons is at most $k(\lceil \frac{n}{k} \rceil - 1) < k(\frac{n}{k}) = n$ pigeons (because $\lceil \frac{n}{k} \rceil - 1 < \frac{n}{k} \leq \lceil \frac{n}{k} \rceil$). ∎

Exercises. Prove:

(a) If n objects are distributed among $k > 0$ boxes, then at least one box contains at most $\lfloor \frac{N}{k} \rfloor$ objects.

(b) Given $t > 0$ pigeonholes $h_1, h_2, \ldots, h_t$, and t integers $n_1, n_2, \ldots, n_t$, if $\left[\sum_{i=1}^{t}(n_i - 1)\right] + 1$ pigeons are distributed among these t pigeonholes, then there exists at least one i suct that pigeonhole h_i contains at least n_i pigeons.

(c) Given a collection of n numbers, at least one of the numbers is at least as large as the average.

(d) Given a collection of n numbers, at least one of the numbers is no larger than the average.

Example PHP1. Prove that if seven distinct numbers are selected from $\{1, 2, \ldots, 11\}$, then some two of these numbers sum to 12.

Let the pigeons be the numbers selected. Define six pigeonholes corresponding to the six sets: $\{1, 11\}, \{2, 10\}, \{3, 9\}, \{4, 8\}, \{5, 7\}, \{6\}$. (Notice that the numbers in each of the first five sets sum to 12, and that there is no pair of distinct numbers containing 6 that sum to 12.) When a number is selected, it gets placed into the pigeonhole corresponding to the set that contains it. (This means at most one number can go into the pigeonhole corresponding to $\{6\}$. This does not cause trouble.) Since seven numbers are selected and placed in six pigeonholes, some pigeonhole contains two numbers. By the way the pigeonholes were defined, these two numbers sum to 12.

Another way to write up the above proof is: *Since seven numbers are selected, the Pigeonhole Principle guarantees that two of them are selected from one of the six sets* $\{1, 11\}, \{2, 10\}, \{3, 9\}$, $\{4, 8\}$, $\{5, 7\}, \{6\}$. *These two numbers sum to 12.*

In Example PHP1, the quantity seven is the best possible in the sense that it is possible to select six numbers from $\{1, 2, \ldots, 11\}$ so that no two of the numbers selected sum to 12. One example of six such numbers is 1, 2, 3, 4, 5, 6.

Proving things with the Pigeonhole Principle. There are four steps involved. First, decide what the pigeons are. They will be the things that you'd like several of to have some special property. Second, set up the pigeonholes. You want to do this so that when you get two pigeons in the same pigeonhole, they have the property you want. To use the Pigeonhole Principle, it is necessary to set things up so that there are fewer pigeonholes than pigeons. Sometimes the way to do this relies on some astute observation. Third, give a rule for assigning the pigeons to the pigeonholes. It is important to note that the conclusion of the Pigeonhole Principle holds for any assignment of pigeons to pigeonholes, so it holds for any assignment you describe. Pick the rule so that when "enough" pigeons occupy the same pigeonhole, that collection has the property you want. Fourth, apply the Pigeonhole Principle to your setup and get the desired conclusion.

Exercise. Prove the general version of Example PHP1: If $n+1$ numbers are selected from $\{1, 2, \ldots, 2n-1\}$, then some two of these numbers sum to $2n$. Show that it is possible to select n numbers so that no two of them sum to $2n$. Formulate and prove similar statements for collections of numbers selected from $\{1, 2, \ldots, 2n\}$.

Example PHP2. Prove that if five points are selected from the interior of a 1×1 square, then there are two points whose distance is less than $\sqrt{2}/2$.

Let the pigeons be the five points selected. Define the four pigeonholes corresponding to the four $1/2 \times 1/2$ subsquares obtained by joining the midpoints of opposite sides of the square. When a point is selected is is placed into a pigeonhole according to the subsquare that contains it, and points on the boundary of these subsquares (and interior to the whole square) can be assigned arbitrarily. Since five points are selected and placed in four pigeonholes, some pigeonhole contains two points. Since these points are on the interior of the square, the distance between them is less than the length of a diagonal of a subsquare, which is $\sqrt{2}/2$.

Exercises. Prove that if four points are selected from the interior of a unit circle, then there are two points whose distance apart is less than $\sqrt{2}$. How many points must be selected from the interior of an equilateral triangle of side two in order to guarantee that there are two points whose distance apart is less than one?

Example PHP3. For a subset $X \subseteq \{1, 2, \ldots 9\}$, define $\sigma(X) = \sum_{x \in X} x$. (For example $\sigma(\{1, 6, 8\}) = 1 + 6 + 8 = 15$.) Prove that among any 26 subsets of $\{1, 2, \ldots 9\}$, each having size at most three, there are subsets A and B such that $\sigma(A) = \sigma(B)$.

For $X \subseteq \{1, 2, \ldots 9\}$ with $|X| \leq 3$, the possible values of $\sigma(X)$ lie between 0 (corresponding to $X = \emptyset$) and 24 (corresponding to $X = \{7, 8, 9\}$). Since there are 25 possible values for $\sigma(X)$ and 26 subsets are selected, we have by the Pigeonhole Principle that the selection contains subsets A and B such that $\sigma(A) = \sigma(B)$.

Exercise. Write up Example PHP3 to make the pigeons and pigeonholes explicit.

Exercise. How many subsets of $\{1, 2, \ldots 10\}$, each containing at least three and at most five elements, must be selected in order to grarantee that the selection contains subsets A, B and C such that $\sigma(A) = \sigma(B) = \sigma(C)$?

Example PHP4. Prove that if 10 integers are selected from $\{1, 2, \ldots, 18\}$, the selection includes integers a and b such that $a|b$ (that is, a *divides* b – there exists an integer k such that $ak = b$).

Let the pigeons be the 10 integers selected. Define nine pigeonholes corresponding to the odd integers 1, 3, 5, 7, 9, 11, 13, 15, and 17. Place each integer selected into the pigeonhole coresponding to its largest odd divisor (which must be one of $1, 3, 5, \ldots, 17$). Notice that if x gets placed in the pigeonhole corresponding to the odd integer m, then $x = 2^k m$ for some integer $k \geq 0$. Since 10 integers are selected and placed in nine pigeonholes, some pigeonhole contains two integers a and b, where $a < b$. Suppose this pigeonhole corresponds to the odd integer t. Then, $a = 2^r t$ and $b = 2^s t$, where $r < s$, so that $a2^{s-r} = b$. Since $s - r$ is a positive integer, it follows that $a|b$.

Here is an alternative write up. The largest odd divisor of an integer between 1 and 18 is one of the nine numbers 1, 3, 5, 7, 9, 11, 13, 15, 17. Since 10 integers are selected, the Pigeonhole Principle guarantees that some two of them have the same largest odd divisor, t. Let these two numbers be a and b, where $a < b$. Then, $a = 2^r t$ and $b = 2^s t$, where $r < s$, so that $a2^{s-r} = b$. Since $s - r$ is a positive integer, it follows that $a|b$.

Exercise. Prove that if $n + 1$ integers are selected from $\{1, 2, \ldots 2n\}$, then the selection includes integers a and b such that $a|b$.

Example PHP5. Prove that if 11 integers are selected from among $\{1, 2, \ldots, 20\}$, then the selection includes integer a and b such that $a - b = 2$.

Let the pigeons be the 11 integers selected. Define 10 pigeonholes corresponding to the sets $\{3, 1\}, \{4, 2\}, \{7, 5\}, \{8, 6\}, \{11, 9\}, \{12, 10\}, \{15, 13\}, \{16, 14\}, \{19, 17\}, \{20, 18\}$. Place each integer selected into the pigeonhole corresponding to the set that contains it. Since 11 integers are selected and placed into 10 pigeonholes, some pigeonhole contains two pigeons. By the way the pigeonholes were defined, these two integers differ by two.

Exercise. Prove an alternative write up (as above) of Example PHP5.

Exercise. Prove that if 11 integers are selected from among $\{1, 2, \ldots, 20\}$, then the selection includes integers a and b such that $b = a + 1$.

Exercise. Prove that if $n + 1$ integers are selected from among $\{1, 2, \ldots, 2n\}$, then the selection includes integers a and b such that $b = a + 1$. This implies that if $n + 1$ integers are selected from among $\{1, 2, \ldots, 2n\}$, then the selection includes integer a and b such that $gcd(a, b) = 1$. Why is that?

Example PHP6. Over a 44 day period, Gary will train for triathlons at least once per day, and a total of 70 times in all. Show that there is a period of consecutive days during which he trains exactly 17 times.

For $i = 1, 2, \ldots, 44$, let x_i be the number of times Gary trains up to the end of day i. Then $1 \leq x_1 < x_2 < x_3 < \cdots < x_{44} = 70$. We need to find subscripts i and j

such that $x_i + 17 = x_j$. This implies that Gary trains exactly 17 times in the period of days $i+1, i+2, \ldots, j$. Therefore, we want one of $x_1, x_2, \ldots, x_{44}$ to be equal to one of $x_1 + 17, x_2 + 17, \ldots, x_{44} + 17$. Using the inequality for the x_i s it follows that $18 \leq x_1 + 17 < x_2 + 17 < \cdots < x_{44} + 17 = 87$. Thus, the 88 numbers $x_1, x_2, \ldots, x_{44}$, $x_1 + 17, x_2 + 17, \ldots, x_{44} + 17$ can take on at most 87 different values. Hence, by the Pigeonhole Principle, some two of them must be equal. The inequalities imply that one of $x_1, x_2, \ldots, x_{44}$ must equal one of $x_1 + 17, x_2 + 17, \ldots, x_{44} + 17$, which is what we wanted.

Exercise. Over a 30 day period, Rick will walk the dog at least once per day, and a total of 45 times in all. Prove that there is a period of consecutive days in which he walks the dog exactly 14 times.

Example PHP7. A party is attended by $n \geq 2$ people. Prove that there will always be two people in attendance who have the same number of friends at the party. (Assume that the relation "is a friend of" is symmetric, that is, if x is a friend of y then y is a friend of x.)

Each person either is, or is not, a friend of each of the the other $n-1$ people in attendance. Thus, the possible values for the number of friends a person can have in attendance at the party are $0, 1, \ldots, n-1$. However, it can not be the case that there is someone at the party with 0 friends and someone else with $n-1$ friends: if a person is friends with everyone then (since "is a friend of" is symmetric) everyone at the party has at least one friend there. Thus, the possible values for the number of friends a person can have in attendance at the party are $0, 1, \ldots, n-2$ or $1, 2, \ldots, n-1$. In either case, there are n numbers (of friends among the people in attendance) that can take on at most $n-1$ different values. By the Pigeonhole Princple, two of the numbers are equal. Thus, some two people in attendance who have the same number of friends at the party.

Exercise. Ten baseball teams are entered in a round-robin touranment (meaning that every team plays every other team exactly once) in which ties are not allowed. Prove that if no team loses all of its games, then some two teams finish the tournament with the same number of wins.

Example PHP8. We prove that any collection of eight distinct integers contains distinct integers x and y such that $x - y$ is a multiple of 7,

By the Division Algorithm, every integer n can be written as $n = 7q + r$, where $0 \leq r \leq 6$. Since there are eight integers in the collection but only seven possible values for the remainder r on division by 7, the Pigeonhole Principle asserts that the collection contains integers x and y that leave the same remainder on division by 7, that is, there exists s with $0 \leq s \leq 6$ such that $x = 7q_1 + s$ and $y = 7q_2 + s$. For this x and y we have $x - y = 7q_1 + s - (7q_2 + s) = 7(q_1 - q_2)$. Since $(q_1 - q_2)$ is an integer, x is a multiple of 7.

Exercise. Prove that in and collection of $n+1$ distinct integers, there are distinct integers x and y such that $x - y$ is a multiple of n.