Sets

Memorize: A *set* is a well-defined collection of objects called *elements* or *members* of the set. If x is a member of the set S, we write $x \in S$, and if x is a not member of the set S, we write $x \notin S$

Here, well-defined means that any given object must either be an element of the set, or not be an element of the set.

Memorize: We say sets A and B are *equal*, and write $A = B$ if $x \in A \Leftrightarrow x \in B$ (that is, have exactly the same elements).

Here are three ways of specifying a set:

1. **Explicit listing**: list its elements between brackets, as in $\{2, 3, 5, 7\}$.

2. **Implicit listing**: list enough of its elements to establish a pattern and use an elipsis $(\ldots)$. At least two elements must be listed to establish the pattern, sometimes more are needed. As examples, consider $\{\ldots - 3, -1, 1, 3, \ldots\}$ and $\{0, 2, 4, \ldots, 120\}$, the set of odd integers and the set of non-negative even integers less than or equal to 120, respectively.

3. **Set builder notation**: specify the set as the set of all x (say) that make some propositional function true, as in $\{x : (x \text{ is prime}) \ \wedge \ (x < 10)\}$.

Note that these are all ways of describing the set, but the set itself does not depend on the description. It just exists, how you describe it is a choice. In particular, for each object, what matters is whether or not it belongs to the set. This is why $\{1, 2, 2, 3\}, \{1, 2, 3, 3\}$ and $\{1, 2, 3\}$ all describe the same set.

Memorize: We say that a set A is a *subset* of a set B if every element of A is an element of B (i.e., $x \in A \Rightarrow x \in B$). If A is a subset of B we write $A \subseteq B$, and otherwise we write $A \not\subseteq B$.

Memorize: The *empty set* is the set that contains no elements. It is denoted by $\emptyset$ or $\{\}$.

For every set A, we have $A \subseteq A$ and $\emptyset \subseteq A$. Both statements follow from the definition of subset. The second statement is true because the condition $x \in \emptyset$ is never true. (You should be able to explain this if asked.)

Memorize: We say that A is a *proper subset* of B, and write $A \subset B$, if $A \subseteq B$ and $A \neq B$.

That is, A is a proper subset of B if $A \subseteq B$ and there is an element of B which is not an element of A. This is consistent with the general use of the word "proper" in mathematics - roughly spreaking it is used for "not equal to the whole thing".

Notice that two sets A and B are equal if $x \in A \Leftrightarrow x \in B$. This is the same as $x \in A \Rightarrow x \in B$ and $x \in B \Rightarrow x \in A$. That is $A = B$ is the same as $A \subseteq B$ and $B \subseteq A$.

How to prove two sets A and B are equal. Here are two ways.

1. Showing that each is a subset of the other. A proof like this has two parts. First you show $A \subseteq B$ by starting with "*Assume* $x \in A$" and then arguing that $x \in B$, and then you show $B \subseteq A$ by starting with "*Assume* $x \in B$" and then arguing that $x \in A$. The argument will usually have to make use of other information you know (and/or are given).

2. Using set buider notation to demonstrate that the sets can be described by logically equivalent propositional functions.

You must be able to distinguish between $\in$ and $\subseteq$. The first one makes the assertion that *a particular object belongs* to a set; the second one says that *every element of one set belongs* to another set. The confusion usually creeps in when the sets in question contain other sets as elements.

Memorize: The *power set* of a set X is the set $\mathcal{P}(X)$ whose elements are the subsets of X.

You need to keep the following facts straight:

- $\mathcal{P}(X)$ is a set.
- the elements of $\mathcal{P}(X)$ are sets (too).

- $A \in \mathcal{P}(X) \Leftrightarrow A \subseteq X$ (this is the definition).
- In particular, $\emptyset \in \mathcal{P}(X)$ and $X \in \mathcal{P}(X)$.

We always assume our sets are subsets of some (large) set called the *universe* (or universal set), and denoted by $\mathcal{U}$.

Memorize: Let A and B be sets:

- The *union* of A and B is the set $A \cup B = \{x : x \in A \vee x \in B\}$.
- The *intersection* of A and B is the set $A \cap B = \{x : x \in A \wedge x \in B\}$.
- The *difference* of A and B is the set $A - B = \{x : x \in A \wedge x \notin B\}$.
- The *complement* of A is the set $\overline{A} = \{x : x \notin A\} = \mathcal{U} - A$.
- The *symmetric difference* of A and B is the set $A \Delta B = (A - B) \cup (B - A)$.

Note that $A - B$ is, in general, not equal to $B - A$.

Set identities. These arise from using set builder notation and the logical equivalences from before (that is, they can all be proved that way). You should **memorize** them.

- $A \cap \mathcal{U} = A, \quad A \cap \emptyset = \emptyset$
- $A \cup \mathcal{U} = \mathcal{U}, \quad A \cup \emptyset = A$
- $A \cup A = A, \quad A \cap A = A$
- $A \cup B = B \cup A, \quad A \cap B = B \cap A$
- $(A \cap B) \cap C = A \cap (B \cap C), \quad (A \cup B) \cup C = A \cup (B \cup C)$
- Law of Double Complement: $\overline{\overline{A}} = A$
- Distributive Laws: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C), \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- DeMorgan's Laws: $\overline{A \cup B} = \overline{A} \cap \overline{B}, \quad \overline{A \cap B} = \overline{A} \cup \overline{B}$

You should be able to prove each of the above in two ways (set builder notation and showing that each side is a subset of the other).

Venn diagrams. These are a pictorial representation of sets and a good way to get intuition about (possible) set equalities. You should be able to use Venn diagrams to investigate whether two sets are equal. If they are equal, you should be able to prove this using one of the methods discussed before (a Venn diagram does not suffice as a proof). If the sets are not equal, you should be able to use the Venn diagram to get a particular example showing they are not equal.